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Mehmet Zeki Sarikaya and Hüseyin Yildirim

ON HARDY TYPE INEQUALITY WITH NON-ISOTROPIC KERNELS

(submitted by F. G. Avkhadiev)

Abstract. In the present paper we establish a Stein-Weiss type generalization of the Hardy type inequality with non-isotropic kernels depending on λ-distance for the spaces $L_{p(.)}(\Omega)$ with variable exponent $p(x)$ in the case of bounded domains Ω in R^n.

The λ-distance between points $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ is defined by the following formula given in [1,7-9,11];

$$|x-y|_\lambda := (|x_1-y_1|^{\frac{1}{\lambda_1}} + |x_2-y_2|^{\frac{1}{\lambda_2}} + ... + |x_n-y_n|^{\frac{1}{\lambda_n}})^{\frac{|\lambda|}{n}}.$$

where $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$, $\lambda_k \geq \frac{1}{2}$, $k = 1, 2, ..., n$, $|\lambda| = \lambda_1 + \lambda_2 + ... + \lambda_n$. Note that this distance has the following properties of homogeneity for any positive t,

$$\left(\left|t^{\lambda_1}x_1\right|^{\frac{1}{\lambda_1}} + ... + \left|t^{\lambda_n}x_n\right|^{\frac{1}{\lambda_n}}\right)^{\frac{|\lambda|}{n}} = t^{\frac{|\lambda|}{n}} \, |x|_\lambda, \quad t > 0.$$

From this relation it follows that the λ-distance is the $a-$homogeneous function [1,7-11] where $a = \frac{|\lambda|}{n}$. So the non-isotropic λ-distance has the following properties:

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1 $|x|_\lambda = 0 \Leftrightarrow x = \theta, \ \ \theta = (0,0,...,0)$

2. $\left|t^\lambda x\right|_\lambda = |t|^{\frac{|\lambda|}{n}} |x|_\lambda$

3. $|x+y|_\lambda \leq k(|x|_\lambda + |y|_\lambda)$

where $k = 2^{\left(1+\frac{1}{\lambda_{\min}}\right)\frac{|\lambda|}{n}}, \ \lambda_{\min} = \min\{\lambda_1, \lambda_2, ..., \lambda_n\}$.

Here we consider λ-spherical coordinates by the following formulas :

$$x_1 = (\rho \cos\varphi_1)^{2\lambda_1}, ..., x_n = (\rho \sin\varphi_1 \sin\varphi_2 ... \sin\varphi_{n-1})^{2\lambda_n}.$$

We obtained that $|x|_\lambda = \rho^{\frac{2|\lambda|}{n}}$. It can be seen that the Jacobian $J_\lambda(\rho,\varphi)$ of this transformation is $J_\lambda(\rho,\varphi) = \rho^{2|\lambda|-1}\Omega_\lambda(\varphi)$, where $\Omega_\lambda(\varphi)$ is the bounded function, which only depend on angles $\varphi_1, \varphi_2, ..., \varphi_{n-1}$. It is clear that if $\lambda_i = \frac{1}{2}, \ i = 1, ..., n$, then the λ-distance is Euclidean distance.

In [3], the classical Hardy inequality for fractional integrals states that

$$\left\| x^{\beta-\alpha} \int_0^x \frac{f(y)dy}{y^\beta (x-y)^{1-\alpha}} \right\|_{L_p(0,b)} \leq c \left\| f \right\|_{L_p(0,b)}, \quad 0 < \alpha < 1$$

where $\alpha - \frac{1}{p} < \beta < \frac{1}{q}, \ \frac{1}{p} + \frac{1}{q} = 1$ and $0 < b \leq \infty$. Its generalization

$$\int_{\mathbb{R}^n} |x|_\lambda^\mu \left| I_{\alpha,\lambda} f(x) \right|^p dx \leq c \int_{\mathbb{R}^n} |x|_\lambda^\gamma \left| f(x) \right|^p dx$$

for the following generalized Riesz potential with the non-isotropic kernel depending on λ-distance,

$$I_{\alpha,\lambda} f(x) = \int_{\mathbb{R}^n} |x-y|_\lambda^{\alpha-n} f(y) dy, \ \ 0 < \alpha < n. \tag{1}$$

where $x \in \mathbb{R}^n$. (1) equality is well-known the classical Riesz potential for $\lambda_i = \frac{1}{2}, \ i = 1, ..., n$. For classical Riesz potentials the Hardy type inequality was investigated by [6]. Here particular importance of the non-isotropic kernel is that it doesn't have the classical triangle inequality.

In this paper we consider the case $\lambda_i \geq \frac{1}{2}, \ i = 1, ..., n$.

For a positive r and any $x \in \mathbb{R}^n$ we denote the open $\lambda-$ball $B_\lambda(x,r)$ with radius r and a center x as

$$B_\lambda(x,r) = \{y \in \mathbb{R}^n : |y-x|_\lambda < r \ \}.$$

Let Ω be an open bounded set in $\mathbb{R}^n$, $n \geq 1$ and $p(x)$ a function on $\overline{\Omega}$ satisfying the conditions

$$1 < p_0 \leq p(x) \leq P < \infty, \ \ x \in \overline{\Omega} \tag{2}$$

and

$$|p(x)-p(y)| \leq \frac{A}{\ln \frac{1}{|x-y|_\lambda}}, \quad |x-y|_\lambda \leq \frac{1}{2}, \ x,y \in \overline{\Omega}. \tag{3}$$

Let the weighted maximal function

$$M_{\beta,\lambda} f(x) \ = \ |x-x_0|_\lambda^\beta \sup_{r>0} \frac{1}{|B_\lambda(x,r)|} \int\limits_{B_\lambda(x,r)\cap\Omega} \frac{|f(y)|}{|y-x_0|_\lambda^\beta} dy \tag{4}$$

where $x_0 \in \overline{\Omega}$. We write $M = M_{0,\lambda}$ in the case where $\beta = 0$.

By $L_{p(.)}(\Omega)$ we denote the space of measurable functions $f(x)$ on Ω such that

$$I_p(f) \ := \ \int\limits_\Omega |f(x)|^{p(x)}\, dx < \infty.$$

This is a Banach space with respect to the norm

$$\|f\|_{p(.)} \ = \ \inf\{\tau > 0 : \ I_p(\frac{f}{\tau}) \leq 1\}.$$

The Hölder inequality holds in the form

$$\int\limits_\Omega |f(x)g(x)|\, dx \ \leq \ K \|f\|_{p(.)} \|g\|_{q(.)}$$

with $K = \frac{1}{p_0} + \frac{1}{q_0}$. The functional $I_p(f)$ and the norm $\|f\|_{p(.)}$ are simultaneously greater than one and simultaneously less than 1 :

$$\|f\|_{p(.)}^P \leq \ I_p(f) \ \leq \|f\|_{p(.)}^{p_0} \quad \text{if} \quad \|f\|_{p(.)} \leq 1$$

and

$$\|f\|_{p(.)}^{p_0} \leq \ I_p(f) \ \leq \|f\|_{p(.)}^P \quad \text{if} \quad \|f\|_{p(.)} \geq 1.$$

The imbedding

$$L_{p(x)} \subseteq L_{r(x)}, \ 1 \leq r(x) < p(x) \leq P < \infty$$

is valid if $|\Omega| < \infty$. In that case

$$\|f\|_{r(.)} \leq m \|f\|_{p(.)}, \ m = a_2 + (1-a_1)|\Omega| \tag{5}$$

where $a_1 = \inf\limits_{x\in\Omega} \frac{r(x)}{p(x)}$ and $a_2 = \sup\limits_{x\in\Omega} \frac{r(x)}{p(x)}$.

Lemma 1: Let $0 < \alpha < n$. Then there is the following inequality.

$$\left| |x-z|_\lambda^{\alpha-n} - |z-y|_\lambda^{\alpha-n} \right| \leq M |x-y|_\lambda |x-z|_\lambda^{\alpha-n-1}, \text{ for } \ x,y,z \in \mathbb{R}^n$$

where $|x-z|_\lambda \geq 2|x-y|_\lambda$, and M is a constant which does not depend on x, y and z.

Lemma 1 is proved in [7].

Lemma 2: Let $0 < \alpha < n$. There is the following inequality

$$\sup_{r>0} r^{-2|\lambda|} \int\limits_{|y|_\lambda < r} \frac{dy}{|y-x|_\lambda^{n-\alpha}} \leq C |x|_\lambda^{\alpha-n}$$

where $x, y \in \mathbb{R}^n$,$\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$, $\lambda_k \geq \frac{1}{2}$, $k = 1, 2, ..., n$, $|\lambda| = \lambda_1 + \lambda_2 + ... + \lambda_n$ and the constant C is independent of x, y and r.

İspat: Passing to the λ-spherical coordinates we obtain

$$\int\limits_{|y-x|_\lambda < \frac{|x|_\lambda}{2}} |y-x|_\lambda^{\alpha-n} dy = \Omega_\lambda(\varphi) \int\limits_0^{\frac{|x|_\lambda}{2}} \rho^{\alpha-n+2|\lambda|-1} d\rho = C\,|x|_\lambda^{\alpha-n+2|\lambda|}.$$

In case $\frac{|x|_\lambda}{2} \geq r$, from Lemma 1 and the $\lambda-$spherical coordinates we have

$$\begin{aligned} \int\limits_{|y|_\lambda<r} |y-x|_\lambda^{\alpha-n} dy &\leq C_1 \int\limits_{|y|_\lambda<r} \left||x|_\lambda - |y|_\lambda\right|^{\alpha-n} dy \\ &\leq C_1 2^{n-\alpha} |x|_\lambda^{\alpha-n} \frac{r^{2|\lambda|}}{2|\lambda|} \Omega_\lambda(\varphi) \qquad (6) \\ &= C_2 r^{2|\lambda|} |x|_\lambda^{\alpha-n}. \end{aligned}$$

In case $\frac{|x|_\lambda}{2} < r$, we can write the following inequality

$$\begin{aligned} &\int\limits_{|y|_\lambda<r} |y-x|_\lambda^{\alpha-n} dy \\ &\leq \int\limits_{|y|_\lambda<r} \frac{dy}{\left(\frac{|x|_\lambda}{2}\right)^{n-\alpha}} + \int\limits_{|y|_\lambda<\frac{|x|_\lambda}{2}} \frac{dy}{|y-x|_\lambda^{n-\alpha}} \\ &\leq 2^{n-\alpha} |x|_\lambda^{\alpha-n} \int\limits_{|y|_\lambda<r} dy + \int\limits_{|y|_\lambda<\frac{|x|_\lambda}{2}} 2^{n-\alpha} |x|_\lambda^{\alpha-n} dy \\ &= C_3 r^{2|\lambda|} |x|_\lambda^{\alpha-n} + C_4 |x|_\lambda^{\alpha-n+2|\lambda|}. \qquad (7) \end{aligned}$$

Thus, by (6), (7) we get

$$\begin{aligned} r^{-2|\lambda|} \int\limits_{|y|_\lambda<r} |y-x|_\lambda^{\alpha-n} dy &\leq \begin{cases} C_2 |x|_\lambda^{\alpha-n}, & \frac{|x|_\lambda}{2} \geq r \\ \left(C_3 |x|_\lambda^{\alpha-n} + C_4 \frac{|x|_\lambda^{\alpha-n+2|\lambda|}}{r^{2|\lambda|}}\right), & \frac{|x|_\lambda}{2} < r \end{cases} \\ &\leq \begin{cases} C_2 |x|_\lambda^{\alpha-n}, & \frac{|x|_\lambda}{2} \geq r \\ C_5 |x|_\lambda^{\alpha-n}, & \frac{|x|_\lambda}{2} < r \end{cases} \end{aligned}$$

Now, for $C = \max\{C_2, C_5\}$ we obtain

$$\sup_{r>0} r^{-2|\lambda|} \int\limits_{|y|_\lambda<r} |y-x|_\lambda^{\alpha-n} dy \leq C\,|x|_\lambda^{\alpha-n}.$$

Theorem 1: Let $p(x)$ satisfy conditions (2), (3). If

$$0 < \beta < \frac{n}{q(x_0)}, \tag{8}$$

then there is a following inequality

$$[M_{\beta,\lambda} f]^{p(x)} \leq C \left(1 + \frac{1}{|B_\lambda(x,r)|} \int_{B_\lambda(x,r)} |f(y)| \, dy \right) \tag{9}$$

for all $f \in L_{p(.)}(\Omega)$ such that $\|f\|_{p(.)} \leq 1$, where $C = C(p, \beta, \lambda)$ is a constant not depending on x, r and x_0.

Proof. We will adapt to our paper the proof given by Kokilashvili and Samko [4] for classical Maximal operator. From (8) and the continuity of $p(x)$ we conclude that there exists a $d > 0$ such that

$$\beta q(x) < n \text{ for all } |x - x_0|_\lambda \leq d \tag{10}$$

without loss of generality we assume that $d \leq 1$. Let

$$p_r(x) = \min_{|x-y|_\lambda \leq r} p(y)$$

and $\frac{1}{q_r(x)} = 1 - \frac{1}{p_r(x)}$. From (8) it is easily seen that

$$\beta q_r(x) < n \text{ if } |x - x_0|_\lambda \leq \frac{d}{2} \text{ and } 0 < r \leq \frac{d}{4}.$$

In case $|x - x_0|_\lambda \leq \frac{d}{2}$ and $0 < r \leq \frac{d}{4}$, applying the Hölder inequality with the exponents $p_r(x)$ and $q_r(x)$ to the integral on the right-hand side of the equality

$$\left| M_{\beta,\lambda} \left(\frac{f(y)}{|y - x_0|_\lambda^\beta} \right) \right|^{p(x)} \leq \frac{C}{r^{2|\lambda| p(x)}} \left(\int_{B_\lambda(x,r)} \frac{f(y)}{|y - x_0|_\lambda^\beta} dy \right)^{p(x)}$$

and taking into account (10), we get

$$\left| M_{\beta,\lambda} \left(\frac{f(y)}{|y - x_0|_\lambda^\beta} \right) \right|^{p(x)}$$
$$\leq \frac{C}{r^{2|\lambda| p(x)}} \left(\int_{B_\lambda(x,r)} |f(y)|^{p_r(x)} \, dy \right)^{\frac{p(x)}{p_r(x)}} \left(\int_{B_\lambda(x,r)} \frac{dy}{|y - x_0|_\lambda^{\beta q_r(x)}} \right)^{\frac{p(x)}{q_r(x)}} \tag{11}$$

From Lemma 2, we obtain

$$\left| M_{\beta,\lambda} \left(\frac{f(y)}{|y-x_0|_\lambda^\beta} \right) \right|^{p(x)} \leq \frac{C |x-x_0|_\lambda^{-\beta p(x)}}{r^{\frac{2|\lambda| p(x)}{p_r(x)}}} \left(\int_{B_\lambda(x,r)} |f(y)|^{p_r(x)} \, dy \right)^{\frac{p(x)}{p_r(x)}}.$$

Hence

$$\int\limits_{B_\lambda(x,r)} |f(y)|^{p_r(x)}\,dy \;\leq\; \int\limits_{B_\lambda(x,r)} dy + \int\limits_{\substack{B_\lambda(x,r)\\ \{y:\,|f(y)|\geq 1\}}} |f(y)|^{p(y)}\,dy$$

since $p_r(x) \leq p(y)$ for $y \in B_\lambda(x,r)$. Since $p(x)$ is bounded, we see that

$$\left|M_{\beta,\lambda}\left(\frac{f(y)}{|y-x_0|_\lambda^\beta}\right)\right|^{p(x)} \;\leq\; \frac{C_1|x-x_0|_\lambda^{-\beta p(x)}}{r^{\frac{2|\lambda|p(x)}{p_r(x)}}}\left(r^{2|\lambda|}+\frac{1}{2}\int\limits_{B_\lambda(x,r)}|f(y)|^{p(y)}\,dy\right)^{\frac{p(x)}{p_r(x)}}.$$

Since $r \leq \frac{d}{2} \leq \frac{1}{2}$ and the second term in the brackets is also less than or equal to $\frac{1}{2}$, we arrive at the estimate

$$\begin{aligned}|M_{\beta,\lambda}f|^{p(x)} &\leq \frac{C}{r^{\frac{2|\lambda|p(x)}{p_r(x)}}}\left(r^{2|\lambda|}+\int\limits_{B_\lambda(x,r)}|f(y)|^{p(y)}\,dy\right)\\ &\leq Cr^{2|\lambda|\frac{p_r(x)-p(x)}{p_r(x)}}\left(1+\frac{1}{r^{2|\lambda|}}\int\limits_{B_\lambda(x,r)}|f(y)|^{p(y)}\,dy\right).\end{aligned}$$

From here (10) follows, since $r^{2|\lambda|\frac{p_r(x)-p(x)}{p_r(x)}} \leq C$.

In case $|x-x_0|_\lambda \geq \frac{d}{2}$ and $0<r\leq \frac{d}{4}$. Then we have

$$|y-x_0|_\lambda \geq K^{-1}|x-x_0|_\lambda - |x-y|_\lambda \geq K^{-1}\frac{d}{2}-\frac{d}{4}=\frac{d}{2}(2K^{-1}-1). \quad (12)$$

Thus $|y-x_0|_\lambda^\beta \geq \left(\frac{d}{2}(2K^{-1}-1)\right)^\beta$. Since $|x-x_0|_\lambda^\beta \leq (diam\ \Omega)^\beta$, it follows that $M_{\beta,\lambda}f(x) \leq CM_\lambda f$, and one may proceed as above for the case $\beta=0$ (the condition $|x-x_0|_\lambda \leq \frac{d}{2}$ is not need in this case).

In case $r \geq \frac{d}{4}$. It suffices to show that the left-hand side of (9) is bounded. We have have

$$M_{\beta,\lambda}f(x) \;\leq\; \frac{C(diam\ \Omega)}{(\frac{d}{4})^{2|\lambda|}}\left(\int\limits_{|y-x_0|_\lambda\leq\frac{d}{8}}\frac{f(y)dy}{|y-x_0|_\lambda^\beta}+\int\limits_{|y-x_0|_\lambda\geq\frac{d}{8}}\frac{f(y)dy}{|y-x_0|_\lambda^\beta}\right).$$

Here the first integral is estimated via the Hölder inequality with exponents

$$p_{\frac{d}{8}}=\min_{|y-x_0|_\lambda\leq\frac{d}{8}}p(y) \quad \text{and} \quad q_{\frac{d}{8}}=p'_{\frac{d}{8}}$$

as in (11), which is possible since $\alpha q_{\frac{d}{8}}<n$. The estimate of the second integral is same as (12) since $|y-x_0|_\lambda \geq \frac{d}{8}$.

Corollary: Let $0<\beta<\frac{n}{q(x_0)}$. If conditions (2), (3) are satisfied, then

$$|M_{\beta,\lambda}f|^{p(x)} \;\leq\; C\left(1+M\left[|f(.)|^{p(.)}\right](x)\right) \quad (13)$$

for all $f \in L_{p(.)}(\Omega)$ such that $\|f\|_{p(.)} \leq 1$.

Theorem 2: Let $p(x)$ satisfy conditions (2), (3). The operator $M_{\beta,\lambda}$ with $x_0 \in \Omega$ is bounded in $L_{p(x)}(\Omega)$ if

$$-\frac{n}{p(x_0)} < \beta < \frac{n}{q(x_0)}.$$

Proof. We have to show that

$$\|M_{\beta,\lambda}f\|_{p(.)} \leq c$$

in some ball $\|f\|_{p(.)} \leq R$, which is equivalent to the inequality

$$I_p(M_{\beta,\lambda}f) \leq c \text{ for } \|f\|_{p(.)} \leq R.$$

We observe that

$$|x-x_0|_\lambda^{\beta p(x)} \sim |x-x_0|_\lambda^{\beta p(x_0)} \tag{14}$$

in case $p(x)$ satisfies the condition (3). Following the idea in [2] and so from (14) we have the following inequality

$$\begin{aligned} I_p(M_{\beta,\lambda}f) &\leq c\int\limits_\Omega |x-x_0|_\lambda^{\beta p(x)} \left|M(\tfrac{f(y)}{|y-x_0|_\lambda^\beta})\right|^{p(x)} dx \\ &\leq c\int\limits_\Omega |x-x_0|_\lambda^{\beta p(x_0)} \left|M(\tfrac{f(y)}{|y-x_0|_\lambda^\beta})\right|^{p(x)} dx. \end{aligned}$$

For $r(x) = \frac{p(x)}{p_0}$, we have the following inequality

$$I_p(M_{\beta,\lambda}f) \leq c\int\limits_\Omega \left(|x-x_0|_\lambda^{\beta r(x_0)} \left|M(\tfrac{f(y)}{|y-x_0|_\lambda^\beta})\right|^{r(x)}\right)^{p_0} dx.$$

We will proof the theorem breaks up into two case $\beta \leq 0$ and $\beta \geq 0$.

Case 1. Let $-\frac{n}{p(x_0)} < \beta \leq 0$. Estimate (13) with $\beta = 0$ says that

$$|M_\lambda \phi(x)|^{r(x)} \leq C\left(1 + M\left[\phi^{r(.)}\right](x)\right) \tag{15}$$

for all $\phi \in L_{r(.)}(\Omega)$ with $\|\phi\|_{r(.)} \leq 1$. For $\phi(x) = \frac{|f(x)|}{|x-x_0|_\lambda^\beta}$, we have

$$\|\phi\|_{r(.)} \leq a_0 \|f\|_{r(.)}, \quad a_0 = (diam\ \Omega)^{|\beta|},$$

where we took into account that $\beta \leq 0$. From imbedding (5) we obtain

$$\|\phi\|_{r(.)} \leq a_0.k \|f\|_{p(.)} \leq a_0 kR.$$

Therefore we choose $R = \frac{1}{a_0 k}$. Then $\|\phi\|_{r(.)} \leq 1$, so that (15) is applicable. From (15), we obtain

$$I_p(M_{\beta,\lambda}f) \leq C\int\limits_\Omega \left(|x-x_0|_\lambda^{\beta r(x_0)}\left[1 + M\left(\left|\tfrac{f(y)}{|y-x_0|_\lambda^\beta}\right|^{r(y)}\right)\right]\right)^{p_0} dx.$$

Thus we have

$$\begin{aligned}
&I_p(M_{\beta,\lambda}f)\\
&\leq C\int\limits_{\Omega}\left\{|x-x_0|_{\lambda}^{\beta p(x_0)}+\left(|x-x_0|_{\lambda}^{\beta r(x_0)}M\left(\frac{|f(y)|^{r(y)}}{|y-x_0|_{\lambda}^{\beta r(x_0)}}\right)\right)^{p_0}\right\}dx\\
&\leq C+C\int\limits_{\Omega}M^{\gamma}\left(|f(.)|^{r(.)}\right)^{p_0}dx
\end{aligned}$$

where $\gamma=\beta r(x_0)=\frac{\beta p(x_0)}{p_0}$. As is know [5], the weighted maximal operator M^{γ} is bounded in L_{p_0} with a constant p_0 if $-\frac{n}{p_0}<\gamma<\frac{n}{p_0'}$, which is satisfied since $-\frac{n}{p(x_0)}<\beta\leq 0$. Therefore, we obtain

$$\begin{aligned}
I_p(M_{\beta,\lambda}f) &\leq C+C\int\limits_{\Omega}|f(y)|^{r(y)p_0}dy\\
&\leq c+c\int\limits_{\Omega}|f(y)|^{p(y)}dy<\infty.
\end{aligned}$$

Case 2. Let $0\leq\beta\leq\frac{n}{q(x_0)}$. We represent the functional $I_p(M_{\beta,\lambda}f)$ in the form

$$I_p(M_{\beta,\lambda}f)=\int\limits_{\Omega}\left(|M_{\beta,\lambda}f(x)|^{r(x)}\right)^{\tau}dx$$

with $r(x)=\frac{p(x)}{\tau}>1,\ \tau>1$, where τ will be chosen in the interval $1<\tau<p_0$. From above similar estimate we have

$$|M_{\beta,\lambda}f(x)|^{r(x)}\leq c\left(1+M\left(f^{r(.)}\right)(x)\right)$$

if $\|f\|_{r(.)}\leq c$ and

$$\beta<\frac{n}{[r(x_0)]'}. \tag{16}$$

The condition $\|f\|_{r(.)}\leq c$ is satisfied since $r(x)\leq p(x)$. Condition (16) is fulfilled if $\tau<\frac{n-\beta}{n}p(x_0)$. Thus, under the choice

$$1<\tau<\min\left(p_0,\ \frac{n-\beta}{n}p(x_0)\right)$$

we have

$$\begin{aligned}
I_p(M_{\beta,\lambda}f) &\leq c+c\int\limits_{\Omega}\left|M\left(\left|f^{r(.)}\right|\right)\right|^{\tau}dx\\
&\leq c+c\int\limits_{\Omega}\left(|f(x)|^{r(x)}\right)^{\tau}dx
\end{aligned}$$

by the boundedness of the maximal operator M in $L_{\tau}(\Omega),\ \tau>1$. Hence

$$I_p(M_{\beta,\lambda}f)\leq c+c\int\limits_{\Omega}|f(x)|^{p(x)}dx.$$

This proves the theorem.

Theorem 3: Let $p(x)$ satisfy conditions (2), (3) and Ω be a bounded domain in $\mathbb{R}^n$. Then the Hardy-type inequality is valid.

$$\left\| |x-x_0|_\lambda^{\beta-\alpha} \int\limits_\Omega \frac{|f(y)|}{|y-x_0|_\lambda^\beta |x-y|_\lambda^{n-\alpha}} dy \right\|_{L_{p(.)}} \leq c \|f\|_{L_{p(.)}}, \quad 0<\alpha<n \tag{17}$$

for all β in the interval

$$\alpha - \frac{n}{p(x_0)} < \beta < \frac{n}{q(x_0)}. \tag{18}$$

Proof. For simplicity we take $x_0 = 0 \in \overline{\Omega}$. We may consider non-negative functions f and assume that f is continued as zero outside the domain Ω.

We take

$$I_{\alpha,\lambda}^\beta f(x) = |x|_\lambda^{\beta-\alpha} \int\limits_\Omega \frac{|f(y)|}{|y|_\lambda^\beta |x-y|_\lambda^{n-\alpha}} dy.$$

Hence we can split $I_{\alpha,\lambda}^\beta f$ as follow

$$\begin{aligned} I_{\alpha,\lambda}^\beta f(x) &= |x|_\lambda^{\beta-\alpha} \int\limits_{|x-y|_\lambda < 2k|x|_\lambda} \frac{|f(y)|}{|y|_\lambda^\beta |x-y|_\lambda^{n-\alpha}} dy \\ &\quad + |x|_\lambda^{\beta-\alpha} \int\limits_{|x-y|_\lambda \geq 2k|x|_\lambda} \frac{|f(y)|}{|y|_\lambda^\beta |x-y|_\lambda^{n-\alpha}} dy \\ &= J^1 + J^2. \end{aligned}$$

Since $\alpha + 2|\lambda| > n$ with $\lambda_i \geq \frac{1}{2}$we obtain

$$\begin{aligned} J^1 &= |x|_\lambda^{\beta-\alpha} \sum_{m=1}^\infty \int\limits_{2^{-m}k|x|_\lambda < |x-y|_\lambda < 2^{-m+1}k|x|_\lambda} \frac{|f(y)|}{|y|_\lambda^\beta |x-y|_\lambda^{n-\alpha}} dy \\ &\leq 2^{2|\lambda|} |x|_\lambda^{\beta+2|\lambda|-n} k^{\alpha+2|\lambda|-n} \\ &\quad \times \sum_{m=1}^\infty 2^{-m(\alpha+2|\lambda|-n)} \frac{1}{(2^{-m}k|x|_\lambda)^{2|\lambda|}} \int\limits_{|x-y|_\lambda < 2^{-m+1}k|x|_\lambda} \frac{|f(y)|}{|y|_\lambda^\beta} dy \\ &= 2^{2|\lambda|} |x|_\lambda^{2|\lambda|-n} k^{\alpha+2|\lambda|-n} \sum_{m=1}^\infty 2^{-m(\alpha+2|\lambda|-n)} M_{\beta,\lambda} f(x) \end{aligned}$$

Therefore

$$J^1 \leq c|x|_\lambda^{2|\lambda|-n} M_{\beta,\lambda} f(x) \tag{19}$$

where $c = 2^{n-\alpha} k^{\alpha+2|\lambda|-n}$.

On the other hand, it remains to prove the boundedness of the operator J^2. Obviously, $|x-y|_\lambda \geq 2k\,|x|_\lambda$ implies that

$$|x-y|_\lambda \leq k(|x|_\lambda + |y|_\lambda)$$
$$|y|_\lambda \geq k^{-1}\,|x-y|_\lambda - |x|_\lambda$$
$$|x-y|_\lambda \leq 2k\,|y|_\lambda \,.$$

Therefore we have

$$J^2 \;=\; |x|_\lambda^{\beta-\alpha} \int\limits_{|x-y|_\lambda \leq 2k|y|_\lambda} \tfrac{|f(y)|}{|y|_\lambda^{\beta}|x-y|_\lambda^{n-\alpha}} dy := J_1^2$$

The operator conjugate to J_1^2 has the form

$$J_1^{2*} \;=\; |x|_\lambda^{\beta} \int\limits_{|x-y|_\lambda \leq 2k|x|_\lambda} \tfrac{|g(y)|}{|y|_\lambda^{\alpha-\beta}|x-y|_\lambda^{n-\alpha}} dy$$

which is nothing else but the operator of the familiar type J^1.

According to (19) and Theorem 2 the operator J_1^{2*} is bounded in conjugate space $L_{q(.)}(\Omega)$ if and only if $-\frac{n}{q(0)} < \alpha - \beta < \frac{n}{p(0)}$, that is $\alpha - \frac{n}{p(0)} < \beta < \alpha + \frac{n}{q(0)}$. Therefore, the operator J_1^2 is bounded in $L_{p(.)}(\Omega)$ and J^2 is bounded in this space.

Remark. Analysis of the proof of Theorem 3 shows that it is also valid in the case when order α is variable as well, in the form

$$\left\| |x-x_0|_\lambda^{\beta-\alpha(x_0)} \int\limits_{\Omega} \tfrac{|f(y)|}{|y-x_0|_\lambda^{\beta}|x-y|_\lambda^{n-\alpha(x_0)}} dy \right\|_{L_{p(.)}} \;\leq\; c\,\|f\|_{L_{p(.)}}$$

for all β in the interval

$$\alpha(x_0) - \frac{n}{p(x_0)} < \beta < \frac{n}{q(x_0)}$$

if $\inf\limits_{x\in\Omega} \alpha(x) > 0$ and $\alpha(x)$ satisfies the same logarithmic condition as $p(x)$ in (3)

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Department of Mathematics, Faculty of Science and Arts, Kocatepe University, Afyon-TURKEY

E-mail address: sarikaya@aku.edu.tr

E-mail address: hyildir@aku.edu.tr

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