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ALMOST PRODUCT STRUCTURES AND MONGE-AMPÈRE EQUATIONS

(submitted by V.V. Lychagin)

Abstract. Tensor invariants of an almost product structure are constructed. We apply them to solving the problem of contact equivalence and the problem of contact linearization for Monge-Ampère equations.

In this paper we study almost product structures. By this structure we mean an ordered set $\mathcal{P}$ of real or complex distributions on a smooth manifold N such that the tangent space T_aN (or its complexification $T_aN^{\mathbb{C}}$) splits into the direct sum of the subspaces from $\mathcal{P}$ at each point $a \in N$.

Almost product structures in above sense arise in various forms: as a fields of semi-simple endomorphisms, as non-holonomic webs, and (what is most important for us) as Monge-Ampère equations.

An interpretation of a Monge-Ampère equation as an almost product structure allows us to solve the problems of contact linearization and the problem of contact equivalence for Monge-Ampère equations.

The solution of the first problem for non-degenerated Monge-Ampère equations was annonced by the author in [9]. Here we give a complete proof.

In the series of papers (see [3], [4], [5], [6], [7], [8]) for generic symplectic Monge-Ampère equations was constructed an e-structure (absolute parallelism). In this paper we solve similar problem in contact case. After this result the problem of contact equivalence of Monge-Ampère equations becomes trivial.

An history (not only!) of classification problem for Monge-Ampère equations can be founded by reader in [10].

Moreover, we suppose that the results obtained in the paper for general almost product structure are interesting without their applications to Monge-Ampère equations. For example, Monge-Ampère structure considered in the first section is no else than a non-integrable CR-structure (Cauchy-Riemann structure) in an elliptic case or a non-integrable para-CR-structure in a hyperbolic case on a 5-dimensional contact smooth manifold.

A few words about the structure of the paper.

In the first section we recall a notion of an almost product structure and give some important examples.

The tensor invariants of an almost product structure are constructed in the second section. To this end we consider a decomposition of the de Rham complex to the generated by an almost product structure direct sum. The main result of this section is based on differential's structure of differential graded algebra. We explain a geometrical sense of the constructed in the previous section tensors and prove an analog of the Frobenius theorem for subdistributions of an almost product structure.

Our main results in theory of Monge-Ampère equations are presented in the last section.

In the first and second subsections of the fourth section we introduce a reader to V. Lychagin's theory of Monge-Ampère equations and recall some definitions and notations.

In the third one we construct contact tensor invariants of hyperbolic and elliptic Monge-Ampère equations and calculate (as an example) the tensors for the non-linear wave equation.

A solution of the linearization problem for non-degenerate equations we present in the fourth subsection. As an example of applications of the result we consider the generalized Hunter-Saxton equation. This equation arises in the theory of a director field of a liquid crystal and in the geometry of Einstein-Weil spaces.

At last, in the fifth section we construct an e-structure for generic Monge-Ampère equations. We introduce the non-holonomic de Rham complex and construct the set of relative and absolute contact invariants of equations.

1. Almost product structures: definition and examples

Let N be a (real) smooth manifold and let $\mathcal{P} = (P_1, \dots, P_r)$ be an ordered set of real or complex distributions on N, i.e.

$$P_i : N \ni a \mapsto P_i\,(a) \subset T_a N\ ,$$

or

$$P_i : N \ni a \mapsto P_i\,(a) \subset T_a N^{\mathbb{C}} = T_a N \otimes \mathbb{C},$$

$i = 1, \dots, r$.

The set $\mathcal{P}$ is called an *almost product structure* (*$\mathcal{AP}$-structure*) on N if at each point $a \in N$ the tangent space $T_a N$ (for real distributions) or its complexification $T_a N^{\mathbb{C}}$ (for complex ones) splits in the direct sum of the subspaces $P_1\,(a), \dots, P_r\,(a)$, i.e.

$$\bigoplus_{i=1}^{r} P_i\,(a) = T_a N \text{ (or } T_a N^{\mathbb{C}}).$$

Let $D(N)$ and $\Omega^s\,(N)$ be the modules of vector fields and differential s-forms on N respectively. A submodule of vector fields from the distribution P_i we denote by $D(P_i)$:

$$D(P_i) \stackrel{\text{def}}{=} \{X \in D(P) | X_a \in P(a) \forall a \in M\}\,.$$

For the distribution P_i we define a submodule of vanishing on the distributions P_j $(j = 1, \dots, r;\ j \neq i)$ differential s-forms:

$$\Omega^s(P_i) \stackrel{\text{def}}{=} \{\alpha \in \Omega^s\,(N) |\, X \rfloor \alpha = 0\ \forall X \in D(P_j),\ j = 1, \dots, r;\ j \neq i\}\,.$$

Let us consider some examples.

Example 1 (Field of semi-simple endomorphisms)**.** Let A be a field of endomorphisms on a smooth n-dimensional manifold N. Suppose that at each point $a \in N$ the linear operator $A_a : T_a N \to T_a N$ is semi-simple.

Let $\lambda_1, \dots, \lambda_r$ be eigenvalues of A and let p_i is a multiplicity of the eigenvalue λ_i $(i = 1, \dots, r;\ p_1 + \dots + p_r = n)$. Then $T_a N^{\mathbb{C}}$ splits into the direct sum of eigensubspaces $P_1\,(a), \dots, P_r\,(a)$[1] of the operator A_a: $T_a N^{\mathbb{C}} = \oplus_{i=1}^r P_i\,(a)$. Here $\dim P_i\,(a) = p_i$ $(i = 1, \dots, r)$. Suppose also that $p_1, \dots, p_r$ are constant. Then the maps $P_i : a \mapsto P_i\,(a)$ $(i = 1, \dots, r)$ are complex distributions on N and the set $\mathcal{P} = (P_1, \dots, P_r)$ is a complex $\mathcal{AP}$-structure.

Suppose $n = 2k$. If $A^2 = -1$ or $A^2 = 1$, then A is a classical almost complex structure ($\mathcal{AC}$-structure) or classical almost product structure respectively.

[1] P_i corresponds to λ_i.

Example 2 ($\mathfrak{f}$-structure)**.** A field of endomorphisms $\mathfrak{f}$ on a smooth manifold N is called $\mathfrak{f}$-structure if $\mathfrak{f}^3+\varepsilon\mathfrak{f}=0$, where $\varepsilon=\pm1$ (see [16], [2]). An $\mathfrak{f}$-structure is called *hyperbolic* or *elliptic* if $\varepsilon=-1$ or $\varepsilon=1$ respectively. At each point $a\in N$ the tangent space T_aN splits into the direct sum

$$T_aN=\mathfrak{M}_a\oplus\mathfrak{L}_a,$$

where $\mathfrak{M}_a=\ker\mathfrak{f}_a$ and $\mathfrak{L}_a=\operatorname{Im}\mathfrak{f}_a$, $\dim\mathfrak{M}_a=m$, $\dim\mathfrak{L}_a=2n$.

For hyperbolic $\mathfrak{f}$-structure the tangent space T_aN at each point $a\in N$ splits into the direct sum of real eigensubspaces of $\mathfrak{f}_a$ that are corresponding to eigenvalues 0, $+1$ and -1:

$$T_aN=\mathfrak{M}_a\oplus\mathfrak{L}_a^+\oplus\mathfrak{L}_a^-,$$

For elliptic $\mathfrak{f}$-structure the complexification of T_aN splits into the direct sum of complex eigensubspaces of $\mathfrak{f}_a$ that are corresponding to eigenvalues 0, $\iota=\sqrt{-1}$ and $-\iota$:

$$T_aN^{\mathbb{C}}=\mathfrak{M}_a^{\mathbb{C}}\oplus\mathfrak{L}_a^+\oplus\mathfrak{L}_a^-.$$

The splitting generates an almost product structures on N.

Example 3 (Almost contact structures)**.** The triplet (η,ξ,Φ), where η is a contact differential 1-form, ξ is a vector field, and Φ is a field of endomorphism on a smooth manifold N is called an *almost contact structure* if the following conditions hold:

(1) $\eta(\xi)=1$,
(2) $\eta\circ\Phi=0$,
(3) $\Phi(\xi)=0$,
(4) $\Phi^2=-\varepsilon+\eta\otimes\xi$,

where $\varepsilon=\pm1$ (see [2]). Similar to the previous example the almost contact structure is called *hyperbolic* or *elliptic* if $\varepsilon=-1$ or $\varepsilon=1$ respectively. An almost contact structure generates an almost product structure with three distributions: one of them (the one-dimensional distribution) is generated by the vector field ξ and other two are generated by eigensubspaces of the restriction $\Phi_a|_{\ker\eta_a}$, $a\in N$.

The following example is main for us.

Example 4 (Monge-Ampère structure on a 5-dimensional manifold)**.** Let N be a 5-dimensional smooth manifold which is endowed with a contact structure $C:a\longmapsto C(a)\subset T_aN$ and let J be a "non-holonomic" field of endomorphisms[2] J, $J_a:C(a)\to C(a)$, $J^2=\varepsilon$, where $\varepsilon=\pm1$.

[2] i.e. J is a section of the vector bundle $\pi:N\ni a\to\alpha_a\otimes X_a\in\Lambda^1(C(a)^*)\otimes C(a)$.

Suppose that the distribution C is generated by the differential 1-form U on N: $C(a) = \ker U_a$ for each $a \in N$[3]. Let Ω_a is a restriction of dU to $C(a)$: $\Omega_a = dU|_{C(a)}$. Then Ω_a is a symplectic structure on $C(a)$. Assume that J_a is symmetric with respect to Ω_a,i.e.

$$\Omega_a\left(J_aX, Y\right) = \Omega_a\left(X, J_aY\right)$$

$\forall X, Y \in C(a)$, $\forall a \in N$.

The pair (C, J) is called a *Monge-Ampère structure* (*MA-structure*) on N. This structure is called *hyperbolic* or *elliptic* if $\varepsilon = 1$ or $\varepsilon = -1$ respectively.

A Monge-Ampère structure generates an almost-product structure $\mathcal{P} = (C_{+,}l, C_-)$ on N, where the 2-dimensional distributions[4] $C_\pm : a \mapsto C_\pm\left(a\right)$ are generated by the eigensubspaces $C_\pm(a)$ of J and 1-dimensional distribution l is generated by the intersection of the first derivatives[5] $C_+^{(1)}$, $C_-^{(1)}$ of the distributions C_+ and C_-: $l(a) = C_+^{(1)}(a) \cap C_-^{(1)}(a)$.

Indeed (see [12]), $C_+(a)$ and $C_-(a)$ are skew-orthogonal with respect to Ω_a or its complexification $\Omega_a^{\mathbb{C}}$. Moreover, Ω_a is non-degenerate on $C_+(a)$ and $C_-(a)$. Then for hyperbolic MA-structure

$$U\left([X_\pm, Y_\pm]\right) = -dU\left(X_\pm, Y_\pm\right) \neq 0$$

for any $X_\pm, Y_\pm \in D(C_\pm)$. Similarly, $U^{\mathbb{C}}\left([X_\pm, Y_\pm]\right) \neq 0$ for elliptic one. This means that the tangent space T_aN (for a hyperbolic MA-structure) or its complexification $T_aN^{\mathbb{C}}$ (for an elliptic one) splits into the direct sum

$$T_aN \text{ (or } T_aN^{\mathbb{C}}) = C_+(a) \oplus l(a) \oplus C_-(a).$$

Note that in the elliptic case the complex distribution l is generated by a real vector field. Indeed, since the operator J_a is real, the subspaces $C_+\left(a\right)$ and $C_-\left(a\right)$ are complex conjugate: $C_+\left(a\right) = \overline{C_-\left(a\right)}$. Then the subspaces $C_+^{(1)}\left(a\right)$ and $C_-^{(1)}\left(a\right)$ are complex conjugated also, and its intersection is complex conjugate to itself: $l\left(a\right) = \overline{l\left(a\right)}$. Therefore this complex line is generated by a real vector Z: $l\left(a\right) = \mathbb{C}Z_a$, $Z \in T_a\left(N\right)$ (see [10]).

The previous example is a partial case of the non-integrable (=non-holonomic) Cauchy-Riemann or para-Cauchy-Riemann structure.

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[3]The differential 1-form is defined up to multiplication by non-vanishing function.

[4]$\dim_{\mathbb{R}} C_\pm = 2$ for hyperbolic MA-structures and $\dim_{\mathbb{C}} C_\pm = 2$ for elliptic ones.

[5]The first derivative $P^{(1)}$ of a distribution P is the distribution which is generated by the vector fields from P and by its all possible sorts of commutators.

Example 5 (CR- and para-CR-structures)**.** A smooth manifold N is called a *Cauchy-Riemann manifold* or *CR-manifold* if there exists a distribution P on N such that at each point $a \in N$ the vector space $P(a) \subset T_aN$ endowed with a complex structure J_a, $J_a^2 = -1$ and J depends on a smoothly. Analogously N is called *para-CR-manifold* if $J_a^2 = 1$.

In this case $P(a)^{\mathbb{C}}$ (or $P(a)$ in case of para-CR-structure) splits into two eigensubspaces of the operator J_a and we obtain two (complex for the CR-structure and real for the para-CR-structure) subdistributions P_1 and P_2 of the distribution P. A (para)-CR-structure is called *integrable* if the distributions P_1 and P_2 are completely integrable, otherwise it is called *non-integrable* or *non-holonomic*.

Let us consider a non-holonomic (para)-CR-structure. Let $P_3(a) \stackrel{\text{def}}{=} P_1^{(1)}(a) \cap P_2^{(2)}(a)$ be an intersection of the first derivatives of the distributions P_1 and P_2 at a point $a \in N$. Suppose that $P_3 : a \mapsto P_3(a)$ is a distribution. If by some reason $P_3(a)$ is a compliment of $P(a)$ to the tangent space T_aN, the we obtain an almost product structure $\mathcal{P} = (P_1, P_2, P_3)$on N.

2. Algebra and geometry of almost product structures

2.1. **A structure of a differential graded algebra.** Let $A = \oplus_{\mathbf{k}} A^{\mathbf{k}}$ be a differential r-graded algebra over a field of characteristic 0 with a differential d, i.e.

$$d\,(a \cdot b) = da \cdot b + (-1)^a\, a \cdot db.$$

Here $\mathbf{k}$ is a multi-index, $\mathbf{k} = (k_1, \ldots, k_r)$, $k_i \in \{0, 1, \ldots, n_i\}$, $n_i \in \mathbb{N}$ are some numbers, $|\mathbf{k}| = k_1 + \cdots + k_r$, and $(-1)^a = (-1)^{\deg a}$ where $\deg a \stackrel{\text{def}}{=} s$ if $a \in A_s \stackrel{\text{def}}{=} \oplus_{|\mathbf{k}|=s} A^{\mathbf{k}}$. We assume also that A is a super-commutative algebra, i.e.

$$a \cdot b = (-1)^{ab} b \cdot a,$$

and the differential and multiplication are compatible with grad:

$$A^{\mathbf{k}} \cdot A^{\mathbf{t}} \subset A^{\mathbf{k}+\mathbf{t}}$$

$$dA_s \subset A_{s+1}$$

The differential d splits in the following direct sum:

$$d = \bigoplus_{|\sigma|=1} d_\sigma = \left(\bigoplus_{i=1}^{r} d_i \right) \oplus \left(\bigoplus_{|\mathbf{t}|=1} d_{\mathbf{t}} \right),$$

where $\sigma = (\sigma_1, \ldots, \sigma_r)$, $\sigma_j \in I_j = \{z \in \mathbb{Z} | \, |z| \le n_j\}$, $d_\sigma : A^{\mathbf{k}} \to A^{\mathbf{k}+\sigma}$ ($|\sigma| = 1$), $d_i = d_{(0\ldots 1_i \ldots 0)}$[6] and $\mathbf{t}$ has negative components.

Lemma 1. *The operators d_i and $d_{\mathbf{t}}$ satisfy the Leibniz rule:*

$$d_i\,(a \cdot b) = d_i a \cdot b + (-1)^a\, a \cdot d_i b$$

and

$$d_{\mathbf{t}}\,(a \cdot b) = d_{\mathbf{t}} a \cdot b + (-1)^a\, a \cdot d_{\mathbf{t}} b$$

In particular, $d_{\mathbf{t}}$ is an A_0-homomorphism, i.e.

$$d_{\mathbf{t}}\,(a \cdot b) = a \cdot d_{\mathbf{t}} b$$

for any $a_0 \in A_0$ and for any $a, b \in A$.

Proof. We prove the second part of the Lemma only. For $a \in A_0 = A^{(0,\ldots,0)}$ and $b \in A^{\mathbf{k}}$ we get:

$$d(a \cdot b) = \sum_{|\sigma|=1} d_\sigma(a \cdot b) = \sum_{i=1}^{r} d_i(a \cdot b) + \sum_{|\mathbf{t}|=1} d_{\mathbf{t}}(a \cdot b),$$

On the other hand,

$$\begin{aligned} d(a \cdot b) &= da \cdot b + (-1)^a\, a \cdot db \\ &= \left(\sum_{i=1}^{r} d_i a + \sum_{|\mathbf{t}|=1} d_{\mathbf{t}} a \right) \cdot b + (-1)^a\, a \cdot \left(\sum_{i=1}^{r} d_i b + \sum_{|\mathbf{t}|=1} d_{\mathbf{t}} b \right). \end{aligned}$$

Therefore $d_i a \cdot b \notin A^{\mathbf{k}+\mathbf{t}}$ for each $i = 1, \ldots, r$ and one gets that $d_{\mathbf{t}}(a \cdot b) = a \cdot d_{\mathbf{t}} b$. □

Example 6. Let us consider the case $r = 2$ and $n_{1,2} = 2$, i.e. $\mathbf{k} = (k_1, k_2)$, where $k_i \in \{0, 1, 2\}$. Then

$$A^s\,(N) = \bigoplus_{p+q=s} A^{p,q}\,(N)\,,$$

$s = 0, 1, 2, 4$ and

$$d = d_{1,0} \oplus d_{0,1} \oplus d_{2,-1} \oplus d_{-1,2},$$

where $d_{2,-1}$, $d_{-1,2}$ are A_0-homomorphisms and $d_{1,0}$, $d_{0,1}$ are differentiations (see diagram below).

[6] 1 is only on ith place.

$$
\begin{tikzcd}
 & & A^{2,0} \arrow[dr, "d_{0,1}"] \arrow[dddr, "d_{-1,2}"] & & \\
 & A^{1,0} \arrow[ur, "d_{1,0}"] \arrow[dr, "d_{0,1}"] \arrow[dddr, "d_{-1,2}"] & & A^{2,1} \arrow[dr, "d_{1,0}"] & \\
A^{0,0} \arrow[ur, "d_{1,0}"] \arrow[dr, "d_{0,1}"'] & & A^{1,1} \arrow[ur, "d_{1,0}"] \arrow[dr, "d_{0,1}"] & & A^{2,2} \\
 & A^{0,1} \arrow[ur, "d_{1,0}"] \arrow[uuur, "d_{2,-1}"] \arrow[dr, "d_{0,1}"'] & & A^{1,2} \arrow[ur, "d_{1,0}"'] & \\
 & & A^{0,2} \arrow[ur, "d_{1,0}"'] \arrow[uuur, "d_{2,-1}"] & &
\end{tikzcd}
$$

2.2. **Tensor invariants for almost product structures.** Using Lemma 1 one can construct tensor invariants of almost product structures.

First, we consider a real almost product structures on N. The vector space $\Lambda^s(T_a^*N)$ of exterior s-forms on T_aN falls into direct sum

$$\Lambda^s(T_a^*N) = \bigoplus_{|\mathbf{k}|=s} \Lambda^{\mathbf{k}}(T_a^*N),$$

where $\mathbf{k}$ is a multi-index, $\mathbf{k} = (k_1, \dots, k_r)$, $k_i \in \{0, 1, \dots, \dim P_i\}$ and

$$\Lambda^{\mathbf{k}}(T_a^*N) = \bigotimes_{i=1}^{r} \Lambda^{k_i}(P_i(a)).$$

Here

$$\Lambda^{k_i}(P_i(a)) \stackrel{\mathrm{def}}{=} \Big\{ \alpha \in \Lambda^{k_i}(T_a^*N) \Big| \, X \rfloor \alpha = 0 \quad \forall X \in P_j(a),\ j = 1, \dots, r;\ j \neq i \Big\}.$$

This gives us a decomposition of the de Rham complex: the $C^\infty(N)$-modules of differential s-forms $\Omega^s(N)$ split in the direct sum

$$\Omega^s(N) = \bigoplus_{|\mathbf{k}|=s} \Omega^{\mathbf{k}}, \tag{1}$$

where

$$\Omega^{\mathbf{k}} \stackrel{\mathrm{def}}{=} \bigotimes_{i=1}^{r} \Omega^{k_i}(P_i).$$

In the case of complex almost product structures we have to consider the complexification $\Omega^s(N)^{\mathbb{C}}$ of the module $\Omega^s(N)$.

Then de Rham differential d splits into the following direct sum:

$$d = \bigoplus_{|\sigma|=1} d_\sigma,$$

where

$$\sigma_j \in I_j \overset{\text{def}}{=} \{z \in \mathbb{Z} |\, |z| \leq \dim P_j\}$$

and

$$d_\sigma : \Omega^{\mathbf{k}} \to \Omega^{\mathbf{k}+\sigma}.$$

From Lemma 1 it follows that if one of the component t_i of a multi-index $\mathbf{t}$ is negative, then operator $d_{\mathbf{t}}$ is a tensor which acts from $\Omega^{\mathbf{k}}$ to $\Omega^{\mathbf{k}+\mathbf{t}}$.

The tensor $d_{\mathbf{t}}$ is a sum of the tensors of the type $\theta \otimes X$, where θ is a differential s-form and X is a $(s-1)$-vector field on N.

Recall that the tensor $\theta \otimes X$ acts on a differential $(s-1)$-form α and on an s-vector field Y as

$$(\theta \otimes X)(\alpha) = (X \rfloor \alpha)\,\theta,$$
$$(\theta \otimes X)(Y) = (Y \rfloor \theta)\,X$$

respectively. Therefore a tensor $\theta \otimes X$ for $\theta \in \Omega^s(P_i)$ and $X \in D^{s-1}(P_j)$ can be regarded as a map from $D^s(P_i)$ to $D^{s-1}(P_j)$. Here $D^s(P)$ is a module of s-vector fields from the distribution P.

Example 7 (Classical $\mathcal{AP}$- and $\mathcal{AC}$-structures on $\mathbb{R}^4$)**.** Let us consider a classical $\mathcal{AP}$-structure (or a classical $\mathcal{AC}$-structure) J on $\mathbb{R}^4$. The tangent space $T_a\mathbb{R}^4$ (or the complexification $(T_a\mathbb{R}^4)^{\mathbb{C}}$ for $\mathcal{AC}$-structure) splits into the direct sum of eigensubspaces $P_1(a)$ and $P_2(a)$ of the operator J_a Here $\dim_{\mathbb{R}} P_i(a) = 2$ (or $\dim_{\mathbb{C}} P_i(a) = 2$ for $\mathcal{AC}$-structure), $i = 1, 2$. The module $\Omega^s(\mathbb{R}^4)$ (or its complexification for $\mathcal{AC}$-structure) falls into the direct sum of $\Omega^{p,q}$, where $p + q = s$, and

$$\Omega^{p,q} \overset{\text{def}}{=} \Omega^p(P_1) \otimes \Omega^q(P_2).$$

(see Example 6). Moreover, we get the following decomposition of the exterior differential $d : \Omega^s(\mathbb{R}^4) \to \Omega^{s+1}(\mathbb{R}^4)$:

$$d = d_{1,0} \oplus d_{0,1} \oplus d_{2,-1} \oplus d_{-1,2}.$$

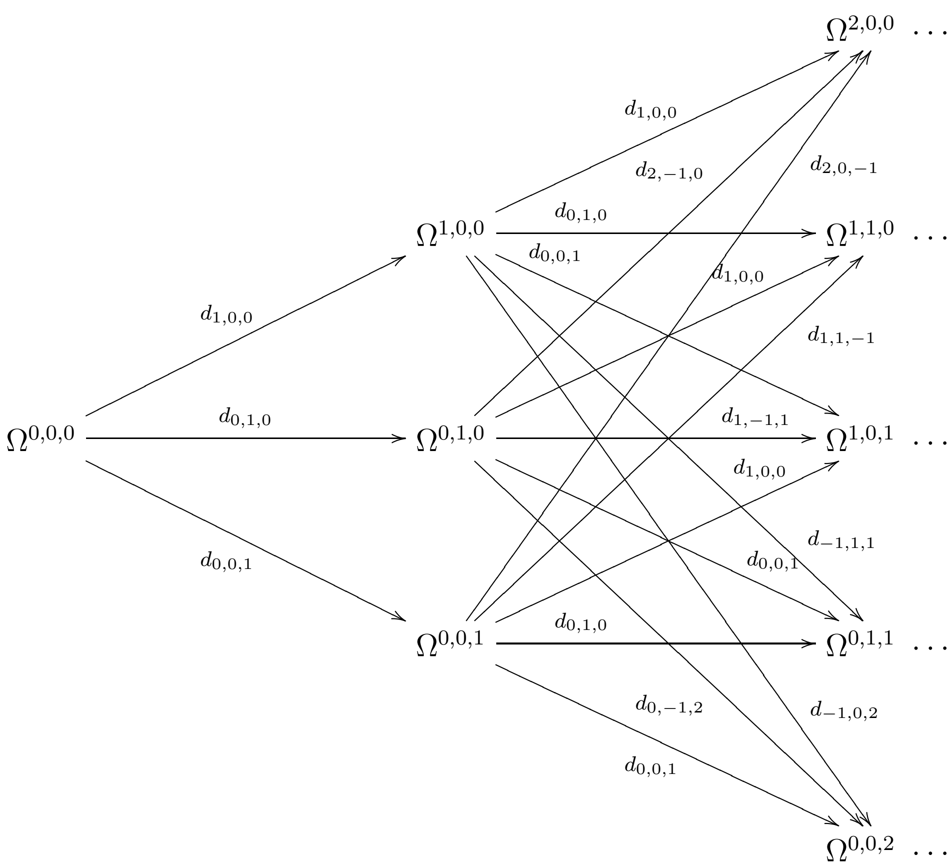
The components $d_{2,-1}$ and $d_{-1,2}$ in this sum are tensors.

Note that for $\mathcal{AC}$-structure the tensors $d_{-1,2}$ and $d_{2,-1}$ are complex conjugated: $\overline{d_{-1,2}} = d_{2,-1}$. In case $d_{-1,2} = d_{2,-1} = 0$ we obtain the usual Dolbeault complex.

Example 8 (Monge-Ampère structure on $\mathbb{R}^5$)**.** Let (C, J) be a Monge-Ampère structure on $\mathbb{R}^5$. We suppose that this structure is hyperbolic, i.e. $J^2 = 1$. The corresponding $\mathcal{AP}$-structure is $\mathcal{P} = (C_{1,0}, l, C_{0,1})$, where $C_{1,0} = C_+$ and $C_{0,1} = C_-$. We get (see diagram below) the decompositions of the modules of exterior differential forms:

$$
\begin{aligned}
&\Omega^0\left(\mathbb{R}^5\right) = C^\infty\left(\mathbb{R}^5\right),\\
&\Omega^1\left(\mathbb{R}^5\right) = \Omega^{1,0,0} \oplus \Omega^{0,1,0} \oplus \Omega^{0,0,1},\\
&\Omega^2\left(\mathbb{R}^5\right) = \Omega^{2,0,0} \oplus \Omega^{1,1,0} \oplus \Omega^{1,0,1} \oplus \Omega^{0,1,1} \oplus \Omega^{0,0,2},\\
&\Omega^3\left(\mathbb{R}^5\right) = \Omega^{2,1,0} \oplus \Omega^{2,0,1} \oplus \Omega^{1,1,1} \oplus \Omega^{1,0,2} \oplus \Omega^{0,1,2},\\
&\Omega^4\left(\mathbb{R}^5\right) = \Omega^{2,1,1} \oplus \Omega^{2,0,2} \oplus \Omega^{1,1,2},\\
&\Omega^5\left(\mathbb{R}^5\right) = \Omega^{2,1,2},
\end{aligned}
$$

and the decomposition of exterior differential:



We have the following tensors: $d_{-1,1,1}$, $d_{1,1,-1}$, $d_{1,-1,1}$, $d_{0,-1,2}$, $d_{2,-1,0}$, $d_{2,1,-2}$ and $d_{-2,1,2}$.

Since the distributions $C_{1,0}$ and $C_{0,1}$ are skew-orthogonal, we get $d_{1,-1,1} = 0$. It is not hard to prove that the tensors $d_{2,1,-2}$, $d_{-2,1,2}$, $d_{2,0,-1}$, and $d_{-1,0,2}$ are zero also. We have four two-covariant and one-contravariant tensors: $d_{-1,1,1}$, $d_{1,1,-1}$, $d_{0,-1,2}$, and $d_{2,-1,0}$.

Any tensor $q^s_{j,k} \overset{\text{def}}{=} d_{\mathbf{1}_j+\mathbf{1}_k-\mathbf{1}_s} : \Omega^{\mathbf{1}_s} \to \Omega^{\mathbf{1}_j+\mathbf{1}_k}$ $(s \neq k, j)$ can be regarded as a map

$$q^s_{j,k} : D\left(P_j\right) \times D(P_k) \to D(P_s).$$

Extend an action of $q^s_{j,k}$ to the module $D(N) \times D(N)$ by the formula:

$$q^s_{j,k}\left(X,Y\right) \overset{\text{def}}{=} q^s_{j,k}\left(\mathbf{P}_j X, \mathbf{P}_k Y\right),$$

where $\mathbf{P}_s$ is the projector to the distribution P_s.

For any almost product structure $\mathcal{P}$ we can define a tensor field $Q_{\mathcal{P}}$ on N:

$$Q_{\mathcal{P}} \overset{\text{def}}{=} \sum_{s,j,k\ (s\neq j,k)} q^s_{j,k}.$$

It is not hard to see that

$$Q_{\mathcal{P}}\left(X,Y\right) = -\sum_{s,j,k\ (s\neq j,k)} \mathbf{P}_s\left[\mathbf{P}_j X, \mathbf{P}_k Y\right].$$

2.3. **Subdistributions.** Let

$$P_I \overset{\text{def}}{=} \bigoplus_{i\in I} P_i, \tag{2}$$

where $I \subset \{1, 2, \ldots, r\}$. In this Section we study the following problem: *when the distribution P_I is completely integrable?*

It is convenient to formulate answer to this question in terms of multi-indices. Let $\operatorname{Ann}(P)$ be an annihilator of a distribution P:

$$\operatorname{Ann}(P) \overset{\text{def}}{=} \left\{\alpha \in \Omega^1(N) \middle| \alpha(X) = 0\, \forall X \in D(P)\right\}.$$

Let us introduce multi-indices of length r: $\mathbf{1} \overset{\text{def}}{=} (1, \ldots, 1)$, $\mathbf{1}_i \overset{\text{def}}{=} (0, \ldots, 0, 1_i, 0, \ldots, 0)$[7], $\mathbf{k} \overset{\text{def}}{=} \sum_{i\in I} \mathbf{1}_i$, and $\overline{\mathbf{k}}$, where $\overline{\mathbf{k}} + \mathbf{k} = \mathbf{1}$, and put $(\mathbf{a}, \mathbf{b}) \overset{\text{def}}{=} a_1 b_1 + \cdots a_r b_r$ for $\mathbf{a} = (a_1, \ldots, a_r)$, $\mathbf{b} = (b_1, \ldots, b_r)$.

[7] 1 is in the ith place only.

The distribution P_I describes by the index $\mathbf{k}$ uniquely, therefore we will denote P_I by $P_{\mathbf{k}}$ also. Then

$$\operatorname{Ann}(P_{\mathbf{k}}) = \bigoplus_{\left(\mathbf{1}_i,\overline{\mathbf{k}}\right)=1} \Omega^{\mathbf{1}_i},$$

Theorem 1. *The distribution $P_{\mathbf{k}}$ is completely integrable if and only if the tensors $d_{\mathbf{t}} = 0$ for all multi-indices $\mathbf{t}$ such that $\left(\mathbf{t},\overline{\mathbf{k}}\right) = -1$.*

Proof. Without loss of generality we can suppose that $I = \{l+1, \dots, r\}$. Then

$$\overline{\mathbf{k}} = (\underbrace{1, \dots, 1}_{l}, \underbrace{0 \dots 0}_{r-l}).$$

Then

$$\operatorname{Ann}(P_{\mathbf{k}}) = \left\langle \alpha_1, \dots, \alpha_{p_1}, \alpha_{p_1+1}, \dots, \alpha_{p_1+p_2}, \dots, \alpha_{p_1+\dots+p_l} \right\rangle,$$

where $p_i = \dim P_i$, $(\alpha_1, \dots, \alpha_{p_1})$ is a free basis of $\Omega^{\mathbf{1}_1}$, $(\alpha_{p_1+1}, \dots, \alpha_{p_1+p_2})$ is a free basis of $\Omega^{\mathbf{1}_2}$, etc.

Let $\alpha_j \in \Omega^{\mathbf{1}_j} \subset \operatorname{Ann}(P_{\mathbf{k}})$ and $\mathbf{t} = (t_1, \dots, t_r)$, where $t_j = -1$. If $t_s = 0$ for $s \in \left\{1, \dots, \widehat{j}, \dots, l\right\}$, then

$$\alpha_1 \wedge \dots \wedge \alpha_{p_1+\dots+p_l} \wedge d_{\mathbf{t}}\alpha_j = 0 \Leftrightarrow d_{\mathbf{t}} = 0.$$

If $t_s \neq 0$ for $s \in \left\{1, \dots, \widehat{j}, \dots, l\right\}$, then $\alpha_1 \wedge \dots \wedge \alpha_{p_1+\dots+p_l} \wedge d_{\mathbf{t}}\alpha_j = 0$. So, $\alpha_1 \wedge \dots \wedge \alpha_{p_1+\dots+p_l} \wedge d\alpha_j = 0 \Leftrightarrow d_{\mathbf{t}} = 0$ for all $\mathbf{t}$ such that $\left(\mathbf{t},\overline{\mathbf{k}}\right) = -1$. □

3. Monge-Ampère equations

3.1. **A geometric point of view.** A differential-geometric structures that generated by Monge-Ampère equations was described by V. Lychagin. In this section we recall his ideas and some his results [11], [12]. We restrict our consideration by Monge-Ampère equations with two independent variables only.

Let M be a 2-dimensional smooth manifold and let J^1M be the manifold of 1-jets of smooth functions on M. The manifold J^1M is endowed by the natural contact structure (*Cartan's distribution*)

$$C : a \in J^1M \to C(a) \subset T_a(J^1M)$$

given by the universal differential one-form $U \in \Omega^1(J^1M)$ (*Cartan's form*): $C(a) \overset{\text{def}}{=} \ker U_a$. Naturally, the form U is defined up to multiplication by non-vanishing smooth function on J^1M.

At each point $a \in J^1M$ the 2-form

$$\Omega_a \overset{\text{def}}{=} dU|_{C(a)} \in \Lambda^2(C^*(a))$$

is the standard symplectic structure on $C(a)$. We get a so-called *non-holonomic symplectic structure*

$$\Omega : J^1M \ni a \longmapsto \Omega_a \in \Lambda^2\left(C^*(a)\right)$$

on J^1M.

With any differential 2-form ω on J^1M we can associate a differential operator $\Delta_\omega : C^\infty(M) \to \Omega^2(M)$, which acts as

$$\Delta_\omega(v) = \omega|_{j^1(v)(M)}. \tag{3}$$

Here $v \in C^\infty(M)$ is a smooth function and $j_1(v)(M) \subset J^1M$ is the graph of 1-jet $j_1(v)$, and $\omega|_{j^1(v)(M)}$ is the restriction of ω to $j_1(v)(M)$.

The equation

$$E_\omega \overset{\text{def}}{=} \{\Delta_\omega(v) = 0\} \subset J^2M$$

is called a *Monge-Ampère equation*.

But constructed map "differential 2-forms" $\to$ "differential operators" is not a one-to-one map. In order to set one-to-one map we should restrict a class of differential 2-forms and consider only *effective* differential 2-forms.

Recall the notion of an effective form.

Differential forms on J^1M vanishing on any integral manifold of the Cartan distribution, and therefore producing zero differential operators, form a graded ideal of the exterior algebra $\Omega^*(J^1M)$. We denote this ideal by

$$I^* = \bigoplus_{s\geq 0} I^s,$$

$I^s \subset \Omega^s(J^1M)$. The ideal I^* is generated by forms

$$U \wedge \alpha + dU \wedge \beta, \tag{4}$$

where α and β are some differential forms. Note also, that $I^0 = 0$ and $I^s = \Omega^s(J^1M)$ for $s \geq 3$.

Elements of the quotient module $\Omega^s(J^1M)/I^s$ we call *effective* s-forms ($s \leq 2$):

$$\Omega^s_\varepsilon\left(J^1M\right) \cong \Omega^s\left(J^1M\right)/I^s. \tag{5}$$

For each element of $\Omega^2_\varepsilon(J^1M)$ one can choice a unique representative $\omega \in \Omega^2(J^1M)$ such that $X_1 \rfloor \omega = 0$ and $\omega \wedge dU = 0$. Here X_1 is the Reeb vector field – a contact vector field with generating function 1.

Let h be a nonvanishing smooth function on J^1M. It is clear that the forms ω and $h\omega$ generate the same equation.

Let ω and $\widetilde{\omega}$ be effective differential 2-forms. Two Monge-Ampère equations E_ω and $E_{\widetilde{\omega}}$ are (local) contact equivalent at a point $a \in J^1M$

if there exists a contact diffeomorphism $\varphi : J^1M \to J^1M$, $\varphi(a) = a$ and some function $h_\varphi \in C^\infty(J^1M)$, $h_\varphi(a) \neq 0$, such that

$$\varphi^*(\omega)_\varepsilon = h_\varphi \widetilde{\omega}.$$

Here $\varphi^*(\omega)_\varepsilon$ is the effective part of $\varphi^*(\omega)$.

Any effective differential form ω generates the *non-holonomic field of endomorphisms*

$$A_\omega : J^1M \ni a \longmapsto A_{\omega_a} \in \operatorname{End}(C(a)). \tag{6}$$

by the formula

$$X_a \rfloor \omega_a = A_{\omega,a} X_a \rfloor \Omega_a$$

for any tangent vector $X_a \in C(a)$.

The operator A_ω is symmetric with respect to Ω, i.e.,

$$\Omega(A_\omega X, Y) = \Omega(X, A_\omega Y)$$

for any vector field $X, Y \in D(C)$.

A function $\operatorname{Pf}(\omega) \in C^\infty(J^1M)$ is called a *Pfaffian* of the form ω if

$$\operatorname{Pf}(\omega)\,\Omega \wedge \Omega = \omega \wedge \omega.$$

Note that

$$\operatorname{Pf}(h\omega) = h^2 \operatorname{Pf}(\omega)$$

for a function $h \in C^\infty(J^1M)$.

For an effective 2-form ω the square of A_ω is scalar and

$$A_\omega^2 + \operatorname{Pf}(\omega) = 0. \tag{7}$$

We say that a Monge-Ampère equation E_ω (a form ω, an operator Δ_ω) are *hyperbolic, elliptic* or *parabolic* at a point $a \in J^1M$ if $\operatorname{Pf}(\omega)$ is negative, positive or zero at this point respectively. Hyperbolic and elliptic equations are called *non-degenerate*.

If $\operatorname{Pf}(\omega)(a) \neq 0$, we can normalize the form ω in some neighborhood of the point a:

$$\omega \mapsto \frac{1}{\sqrt{|\operatorname{Pf}(\omega)|}}\omega.$$

If $|\operatorname{Pf}(\omega)| = 1$, then the form ω is called *normed*. The Pfaffian of a normed form is equal to -1 for a hyperbolic form and $+1$ for an elliptic one.

The operator A_ω corresponding to the normed form ω is denoted by A. It is clear that for hyperbolic and elliptic equations we have $A^2 = 1$ and $A^2 = -1$ respectively.

So, for a non-degenerated operators we obtain a Monge-Ampère structure (C, A) on J^1M which generates an almost product structure $\mathcal{P} = (C_{+,}l, C_{-})$ (see Example 4).

Note that non-degenerate Monge-Ampère equations, in contrast Monge-Ampère operators, generate $\mathcal{AP}$-structures (C_+, l, C_-) up to the change C_+ and C_-. Indeed, effective 2-forms $\omega_1 = \omega$ and $\omega_2 = -\omega$ generate the same equation, but $C_-^1 = C_+^2$ and $C_+^1 = C_-^2$. Here $C_\pm^i$ are eigensubspaces of the operators A_{ω_i} $(i = 1, 2)$.

On the other hand, any pair of arbitrary real distributions $C_{1,0}$ and $C_{0,1}$ on J^1M such that

(1) $\dim C_{1,0} = \dim C_{0,1} = 2$;
(2) at each point $a \in J^1M$ $C(a) = C_{1,0}(a) \oplus C_{0,1}(a)$;
(3) at each point $a \in J^1M$ the subspaces $C_{1,0}(a)$ and $C_{0,1}(a)$ are skew-orthogonal with respect to the symplectic structure Ω_a;

determines the operator A up to the signum. Therefore a hyperbolic Monge-Ampère equation can be regarded as such unordered pair $\{C_{1,0}, C_{0,1}\}$.

Analogously, an elliptic Monge-Ampère equation can be regarded as such unordered pair $\{C_{1,0}, C_{0,1}\}$ of complex conjugate distributions on J^1M that

(1) $\dim_{\mathbb{C}} C_{1,0} = \dim_{\mathbb{C}} C_{0,1} = 2$;
(2) at each point $a \in J^1M$ $C(a)^{\mathbb{C}} = C_{1,0}(a) \oplus C_{0,1}(a)$;
(3) at each point $a \in J^1M$ the subspaces $C_{1,0}(a)$ and $C_{0,1}(a)$ are skew-orthogonal with respect to the complexification $\Omega_a^{\mathbb{C}}$ of the symplectic structure Ω_a.

3.2. **Coordinate representations.** We have the following representations of main objects in the canonical local coordinates $(q,u,p) = (q_1, q_2, u, p_1, p_2)$ on the manifold J^1M:

- The Cartan form
$$U = du - p_1 dq_1 - p_2 dq_2;$$
- The Cartan distribution C is generated by the vector fields
$$\frac{d}{dq_1} = \frac{\partial}{\partial q_1} + p_1 \frac{\partial}{\partial u}, \quad \frac{d}{dq_2} = \frac{\partial}{\partial q_2} + p_2 \frac{\partial}{\partial u}, \quad \frac{\partial}{\partial p_1}, \quad \frac{\partial}{\partial p_2}. \tag{8}$$
- The Reeb vector field $X_1 = \partial/\partial u$;
- An effective 2-form
$$\begin{aligned} \omega = {} & E dq_1 \wedge dq_2 + B\,(dq_1 \wedge dp_1 - dq_2 \wedge dp_2) + \\ & + C dq_1 \wedge dp_2 - A dq_2 \wedge dp_1 + D dp_1 \wedge dp_2, \end{aligned} \tag{9}$$

where A, B, C, D, E are some smooth functions on J^1M;

- The Monge-Ampère equation E_ω

$$Av_{xx} + 2Bv_{xy} + Cv_{yy} + D\left(v_{xx}v_{yy} - v_{xy}^2\right) + E = 0; \tag{10}$$

- The Pfaffian

$$\mathrm{Pf}\,(\omega) = AC - DE - B^2.$$

3.3. **Tensor invariants of non-degenerate Monge-Ampère equations.** In Example 8 we have four contact tensor invariants of a Monge-Ampère equation: $d_{-1,1,1}$, $d_{1,1,-1}$, $d_{0,-1,2}$, and $d_{2,-1,0}$.

In order to unify hyperbolic and elliptic types, we consider a complex tangent bundle $T^{\mathbb{C}}(J^1M)$. Let us describe there structures:

$$\begin{aligned} d_{1,1,-1} &= U^{\mathbb{C}} \wedge \alpha_1 \otimes Q_1 + U^{\mathbb{C}} \wedge \alpha_2 \otimes Q_2, \\ d_{-1,1,1} &= U^{\mathbb{C}} \wedge \beta_1 \otimes P_1 + U^{\mathbb{C}} \wedge \beta_2 \otimes P_2, \\ d_{2,-1,0} &= \alpha_3 \wedge \alpha_4 \otimes Z, \\ d_{0,-1,2} &= \beta_3 \wedge \beta_4 \otimes Z. \end{aligned} \tag{11}$$

Here $\alpha_i \in \Omega^1(C_{1,0}),\ \beta_i \in \Omega^1(C_{0,1}),\ P_j \in D\left(C_{1,0}\right),\ Q_j \in D\left(C_{0,1}\right), Z \in D(l)\ (i = 1, \ldots 3; j = 1, 2)$.

These tensors can be regarded as maps

$$\begin{aligned} &d_{1,1,-1} : C_{1,0}^{(1)} \times C_{1,0}^{(1)} \to C_{0,1}, \\ &d_{-1,1,1} : C_{0,1}^{(1)} \times C_{0,1}^{(1)} \to C_{1,0}, \\ &d_{2,-1,0} : C_{1,0} \times C_{1,0} \to l, \\ &d_{0,-1,2} : C_{0,1} \times C_{0,1} \to l. \end{aligned}$$

Let us explain a geometrical meaning of the tensors. Due to Theorem 1 the distribution $C_{1,0}^{(1)}$ is completely integrable if and only if $d_{1,1,-1} = 0$ and the distribution $C_{0,1}^{(1)}$ is completely integrable if and only if $d_{-1,1,1} = 0$.

Therefore, due to [15] we see that a non-degenerate Monge-Ampère equation is locally contact equivalent to the equation $v_{xx} + \varepsilon v_{yy} = 0$ with $\varepsilon = \pm 1$ if and only if $d_{1,1,-1} = d_{-1,1,1} = 0$.

Example 9 (Non-linear Wave Equation)**.** Consider the following non-linear wave equation:

$$v_{xy} = f\left(x, y, v, v_x, v_y\right).$$

Vector fields

$$P_1 = \frac{d}{dq_1} + f\frac{\partial}{\partial p_2},$$
$$P_2 = \frac{\partial}{\partial p_1}$$

constitute a basis for the distribution $C_{1,0}$ and

$$Q_1 = \frac{d}{dq_2} + f\frac{\partial}{\partial p_1},$$
$$Q_2 = \frac{\partial}{\partial p_2}$$

for the distribution $C_{0,1}$.

The distribution l is generated by the following vector field

$$Z = \frac{\partial}{\partial u} + f_{p_2}\frac{\partial}{\partial p_1} + f_{p_1}\frac{\partial}{\partial p_2}.$$

The vector fields P_1, P_2, Z, Q_1, Q_2 form a basis in vector fields on J^1M. The dual basis is

$$\begin{aligned}
\alpha_1 &= dq_1,\\
\alpha_2 &= dp_1 + p_1 f_{p_2} dq_1 + (p_2 f_{p_2} - f)\, dq_2 - f_{p_2} du,\\
U &= du - p_1 dq_1 - p_2 dq_2,\\
\beta_1 &= dq_2,\\
\beta_2 &= dp_2 + (p_1 f_{p_1} - f)\, dq_1 + p_2 f_{p_1} dq_2 - f_{p_1} du.
\end{aligned}$$

For this case we have the following representation of the four constructed tensor invariants:

$$\begin{aligned}
d_{-1,1,1} = {} & (f f_{p_2p_2} dq_1 \wedge du - f_{p_2p_2} dp_2 \wedge du - p_1 f_{p_2p_2} dq_1 \wedge dp_2\\
& - p_2 f_{p_2p_2} dq_2 \wedge dp_2 + (f_u - p_2 f_{p_2u} + f_{p_1} f_{p_2} - f f_{p_1p_2} - f_{q_2p_2})\, dq_2 \wedge du\\
& + (p_1 f_u - p_1 p_2 f_{p_2u} - p_2 f f_{p_2p_2} + p_1 f_{p_1} f_{p_2} - p_1 f f_{p_1p_2} - p_1 f_{q_2p_2})\, dq_1 \wedge dq_2)\\
& \otimes \frac{\partial}{\partial p_1},
\end{aligned}$$

$$\begin{aligned}
d_{1,1,-1} = {} & (f f_{p_1p_1} dq_2 \wedge du - f_{p_1p_1} dp_1 \wedge du - p_1 f_{p_1p_1} dq_1 \wedge dp_1\\
& - p_2 f_{p_1p_1} dq_2 \wedge dp_1 + (f_u + f_{p_1} f_{p_2} - p_1 f_{p_1u} - f f_{p_1p_2} - f_{q_1p_1})\, dq_1 \wedge du\\
& + (-p_2 f_u - p_2 f_{p_1} f_{p_2} + p_1 p_2 f_{p_1u} + p_2 f f_{p_1p_2} + p_1 f f_{p_1p_1} + p_2 f_{q_1p_1})\, dq_1 \wedge dq_2)\\
& \otimes \frac{\partial}{\partial p_2},
\end{aligned}$$

$$d_{2,-1,0} = (dq_1 \wedge dp_1 - f_{p_2} dq_1 \wedge du + (p_2 f_{p_2} - f)\, dq_1 \wedge dq_2) \otimes \left(\frac{\partial}{\partial u} + f_{p_2} \frac{\partial}{\partial p_1} + f_{p_1} \frac{\partial}{\partial p_2} \right),$$

$$d_{0,-1,2} = (dq_2 \wedge dp_2 - f_{p_1} dq_2 \wedge du - (p_1 f_{p_1} - f)\, dq_1 \wedge dq_2) \otimes \left(\frac{\partial}{\partial u} + f_{p_2} \frac{\partial}{\partial p_1} + f_{p_1} \frac{\partial}{\partial p_2} \right).$$

3.4. **Contact linearization of Monge-Ampère equations.** In this section we use the constructed tensors to solve the problem of contact linearization of non-degenerate Monge-Ampère equations. This problem is the following.

Find a class of Monge-Ampère equations that are locally contact equivalent to nonhomogeneous linear equations

$$v_{xx} + \varepsilon v_{yy} = r\,(x,y)\, v_x + s\,(x,y)\, v_y + c\,(x,y)\, v + d\,(x,y)\,. \tag{12}$$

We assume that all possible derivatives of the distributions $C_\pm$ are distributions also.

Note that for equation (12) the distributions $C_\pm^{(2)}$ are completely integrable and $\dim C_\pm^{(2)} \leq 4$. This means that the tensors $d_{-1,1,1}$ and $d_{1,1,-1}$ have the forms

$$d_{1,1,-1} = U^{\mathbb{C}} \wedge \alpha \otimes Q, \quad d_{-1,1,1} = U^{\mathbb{C}} \wedge \beta \otimes P$$

for some $\alpha \in \Omega^1(C_{1,0})$, $\beta \in \Omega^1(C_{0,1})$, $P \in D\,(C_{1,0})$, $Q \in D\,(C_{0,1})$.

Let us define two complex differential 2-forms $\xi_1, \xi_2 \in \Omega^{1,0,1}$:

$$\xi_1 \stackrel{\text{def}}{=} P \rfloor d_{2,-1,0} \left(U^{\mathbb{C}} \wedge \beta \right),$$

$$\xi_2 \stackrel{\text{def}}{=} Q \rfloor d_{0,-1,2} \left(U^{\mathbb{C}} \wedge \alpha \right).$$

Since $d_{0,-1,2}$ and $d_{2,-1,0}$ are tensors, we see that the forms ξ_1 and ξ_2 are contact invariant of the Monge-Ampère equation. Using representations (11) for $d_{2,-1,0}$ and $d_{0,-1,2}$, we get the following forms of ξ_1 and ξ_2:

$$\xi_1 = P \rfloor (\alpha_3 \wedge \alpha_4 \wedge \beta) = (\alpha_3\,(P)\,\alpha_4 - \alpha_4\,(P)\,\alpha_3) \wedge \beta,$$

$$\xi_2 = Q \rfloor (\beta_3 \wedge \beta_4 \wedge \alpha) = (\beta_3\,(Q)\,\beta_4 - \beta_4\,(Q)\,\beta_3) \wedge \alpha.$$

Note that for elliptic equations the forms ξ_1 and ξ_2 are complex conjugate: $\xi_1 = \overline{\xi_2}$.

Define differential 1-forms

$$\nu_1 \stackrel{\text{def}}{=} Q \rfloor \xi_1,$$

$$\nu_2 \stackrel{\text{def}}{=} P \rfloor \xi_2.$$

The vector fields Q and P are defined up to the multiplication by non-vanishing smooth functions and therefore the forms ν_1 and ν_2 are so. Therefore the unordered pair $\{\nu_1, \nu_2\}$ is a relative contact invariant of a Monge-Ampère equation.

Theorem 2 (Linearization of MAE)**.** *Assume that in some neighborhood of a point $a \in J^1M$ derivatives $C_{\pm}^{(k)}$ $(k = 1, 2)$ of the distributions $C_{\pm}$ are distributions also and the distributions $C_{\pm}^{(2)}$ are completely integrable and* $\dim C_{\pm}^{(2)} = 4$.

A Monge-Ampère equation E_ω is locally equivalent to a Monge-Ampère equation (12) *if and only if $\nu_1 = \nu_2 = 0$ and the differential two-forms ξ_1 and ξ_2 are closed.*

Proof. The proof of the necessity it trivial: it is not hard to check that for equation (12) conditions 1–3 hold. Let us prove the sufficiency.

From the first condition it follows that the equation E_ω is locally equivalent to a Monge-Ampère equation

$$v_{xx} + \varepsilon v_{yy} = f(x, y, v, v_x, v_y) \tag{13}$$

for some function $f \in C^\infty\left(J^1M\right)$ (see [15]). For this equation we have:

$$\begin{aligned}
&\nu_1 = h_1 \left(\left(2\sqrt{\varepsilon} f - \iota p_1 f_{p_2} - \sqrt{\varepsilon} f_{p_1}\right) dq_1 - p_2 \left(\iota f_{p_2} + \sqrt{\varepsilon} f_{p_1}\right) dq_2 + \right.\\
&\left.\left(\iota f_{p_2} + \sqrt{\varepsilon} f_{p_1}\right) du - 2\sqrt{\varepsilon} dp_1 - 2\iota\varepsilon dp_2\right),\\
&\nu_2 = h_2 \left(\varepsilon p_1 \left(\sqrt{\varepsilon} f_{p_1} - \iota f_{p_2}\right) dq_1 + \varepsilon \left(2\iota f + \sqrt{\varepsilon} p_2 f_{p_1} - \iota p_2 f_{p_2}\right) dq_2 + \right.\\
&\left.\varepsilon \left(\iota f_{p_2} - \sqrt{\varepsilon} f_{p_1}\right) du + 2\sqrt{\varepsilon} dp_1 - 2\iota dp_2\right),
\end{aligned}$$

where $h_1 = \varepsilon f_{p_1p_1} + 2\iota f_{p_1p_2} - f_{p_2p_2}$ and $h_2 = \varepsilon f_{p_1p_1} - 2\iota f_{p_1p_2} - f_{p_2p_2}$. We see that $\nu_1 = \nu_2 = 0$ if and only if $h_1 = h_2 = 0$. This means that

$$\begin{cases} f_{p_1p_2} = 0, \\ \varepsilon f_{p_1p_1} - f_{p_2p_2}. \end{cases}$$

Therefore the function f is linear with respect to p_1 and p_2:

$$f\left(q, u, p\right) = r\left(q, u\right) p_1 + s\left(q, u\right) p_2 + g\left(q, u\right)$$

for some functions $r, s, g \in C^\infty\left(J^0M\right)$.

From this place the hyperbolic and elliptic cases we consider separately.

Hyperbolic case. We see that the equation E_ω is contact equivalent to the equation

$$v_{xx} - v_{yy} = r\left(x, y, v\right) v_x + s\left(x, y, v\right) v_y + g\left(x, y, v\right).$$

The corresponding effective differential two-form is

$$\omega = dq_1 \wedge dp_2 + dq_2 \wedge dp_1 + (rp_1 + sp_2 + g) dq_1 \wedge dq_2.$$

The contact transformation

$$\varphi : q_1 \to \frac{q_1 - q_2}{\sqrt{2}},\ q_2 \to \frac{q_1 + q_2}{\sqrt{2}},\ u \to u,\ p_1 \to \frac{p_1 - p_2}{\sqrt{2}},\ p_2 \to \frac{p_1 + p_2}{\sqrt{2}}$$

takes it to the form

$$\omega_1 \overset{\text{def}}{=} \varphi^* (\omega) = dq_1 \wedge dp_1 - dq_2 \wedge dp_2 + (Rp_1 + Sp_2 + G) dq_1 \wedge dq_2, \quad (14)$$

where

$$R(q_1, q_2, u) = \frac{1}{\sqrt{2}} \left(s\left(\frac{q_1 - q_2}{\sqrt{2}}, \frac{q_1 + q_2}{\sqrt{2}}, u \right) + r\left(\frac{q_1 - q_2}{\sqrt{2}}, \frac{q_1 + q_2}{\sqrt{2}}, u \right) \right),$$
$$S(q_1, q_2, u) = \frac{1}{\sqrt{2}} \left(s\left(\frac{q_1 - q_2}{\sqrt{2}}, \frac{q_1 + q_2}{\sqrt{2}}, u \right) - r\left(\frac{q_1 - q_2}{\sqrt{2}}, \frac{q_1 + q_2}{\sqrt{2}}, u \right) \right),$$
$$G(q_1, q_2, u) = g\left(\frac{q_1 - q_2}{\sqrt{2}}, \frac{q_1 + q_2}{\sqrt{2}}, u \right).$$

Then we get the following coordinate representation of the constructed tensors:

$$d_{1,1,-1} = (Rs + G_u + p_2 S_u - R_{q_1}) (dq_1 \wedge du - p_2 dq_1 \wedge dq_2) \otimes \frac{\partial}{\partial p_2},$$
$$d_{-1,1,1} = (RS + G_u + p_1 R_u - S_{q_2}) (dq_2 \wedge du + p_1 dq_1 \wedge dq_2) \otimes \frac{\partial}{\partial p_1},$$
$$d_{2,-1,0} = (dq_1 \wedge dp_1 - S dq_1 \wedge du - (G + p_1 R) dq_1 \wedge dq_2) \otimes \left(\frac{\partial}{\partial u} + S \frac{\partial}{\partial p_1} + R \frac{\partial}{\partial p_2} \right),$$
$$d_{0,-1,2} = (dq_2 \wedge dp_2 - R dq_2 \wedge du + (G + p_2 S) dq_1 \wedge dq_2) \otimes \left(\frac{\partial}{\partial u} + S \frac{\partial}{\partial p_1} + R \frac{\partial}{\partial p_2} \right).$$

We can write the invariant 2-forms ξ_1 and ξ_2:

$$\xi_1 = (RS + G_u + p_1 R_u - S_{q_2}) dq_1 \wedge dq_2,$$
$$\xi_2 = -(RS + G_u + p_2 S_u - R_{q_1}) dq_1 \wedge dq_2.$$

Then

$$d\xi_1 = R_u dq_1 \wedge dq_2 \wedge dp_1 + (SR_u + RS_u + G_{uu} + p_1 R_{uu} - S_{uq_2}) dq_1 \wedge dq_2 \wedge du,$$

$$d\xi_2 = -S_u dq_1 \wedge dq_2 \wedge dp_2 - (SR_u + RS_u + G_{uu} + p_2 S_{uu} - R_{uq_1}) dq_1 \wedge dq_2 \wedge du.$$

We see that $R_u = S_u = G_{uu} = 0$ if and only if $d\xi_1 = d\xi_2 = 0$. In this case

$$G(q_1, q_2, u) = c(q_1, q_2)u + z(q_1, q_2)$$

and we get:

$$\omega_1 = 2\left(R(q)\,p_1 + S(q)\,p_2 + c(q)u + z(q)\right) dq_1 \wedge dq_2 + dq_1 \wedge dp_1 - dq_2 \wedge dp_2.$$

Elliptic case. In this case the equation E_ω is contact equivalent to the equation

$$v_{xx} + v_{yy} = r(x, y, v)\, v_x + s(x, y, v)\, v_y + g(x, y, v).$$

The invariant 2-form ξ_1 is

$$\xi_1 = \Bigg(\frac{1}{4}\left(\frac{ds}{dq_1} - \frac{dr}{dq_2}\right) \\ + \frac{\iota}{8}\left(r^2 + s^2 + 4g_u + 2(p_1 r_u - r_{q_1} + p_2 s_u - s_{q_2})\right)\Bigg)\, dq_1 \wedge dq_2$$

and $\xi_2 = \overline{\xi_1}$. Then

$$d\xi_1 = \frac{1}{4}\left(-r_u + \iota s_u\right) dq_1 \wedge dq_2 \wedge dp_2 + \frac{1}{4}\left(\iota r_u + s_u\right) dq_1 \wedge dq_2 \wedge dp_1 + \\ \frac{1}{4}\left((p_1 s_{uu} - p_2 r_{uu} - r_{uq_2} + s_{uq_1}) + \right. \\ \left. \iota\left(r r_u + s s_u + 2g_{uu} + p_1 r_{uu} + p_2 s_{uu} - s_{q_2 u} - r_{q_1 u}\right)\right) dq_1 \wedge dq_2 \wedge du.$$

We see that $r_u = s_u = g_{uu} = 0$ if and only if $d\xi_1 = 0$. In this case

$$g(q_1, q_2, u) = c(q_1, q_2)u + z(q_1, q_2)$$

and we get the following effective 2-form:

$$\omega = -\left(r(q)\,p_1 + s(q)\,p_2 + c(q)u + z(q)\right) dq_1 \wedge dq_2 + dq_1 \wedge dp_2 - dq_2 \wedge dp_1.$$

□

Example 10 (The Hunter-Saxton equation)**.** Let us consider the Hunter-Saxton equation

$$v_{tx} = v v_{xx} + \kappa u_x^2,$$

where κ is a constant. This equation is hyperbolic and it has applications in the theory of a director field of a liquid crystal [1] and in geometry of Einstein-Weil spaces. For this equation the corresponding effective differential 2-form is

$$\omega = 2u dq_2 \wedge dp_1 + dq_1 \wedge dp_1 - dq_2 \wedge dp_2 - 2\,\kappa p_1^2\, dq_1 \wedge dq_2$$

and the corresponding operator

$$A_\omega = \left\| \begin{array}{cccc} 1 & 2u & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -2\kappa p_1^2 & 1 & 0 \\ 2\kappa p_1^2 & 0 & 2u & -1 \end{array} \right\|.$$

in the free basis

$$\frac{d}{dq_1}, \frac{d}{dq_2}, \frac{\partial}{\partial p_1}, \frac{\partial}{\partial p_2}$$

of the module $D(C)$. Let us choice the following free basis of the module $D(J^1M)$:

$$\begin{aligned}
P_1 &= \frac{\partial}{\partial q_1} + p_1 \frac{\partial}{\partial u} + \kappa p_1^2 \frac{\partial}{\partial p_2}, \\
P_2 &= \frac{\partial}{\partial p_1} + u \frac{\partial}{\partial p_2}, \\
Q_1 &= \frac{\partial}{\partial q_2} + \kappa p_1^2 \frac{\partial}{\partial p_1} - u \frac{\partial}{\partial q_1} + (p_2 - u p_1) \frac{\partial}{\partial u} \\
Q_2 &= \frac{\partial}{\partial p_2}, \\
Z &= \frac{\partial}{\partial u} + (2\,\kappa - 1)\, p_1 \frac{\partial}{\partial p_2},
\end{aligned}$$

and the following dual free basis of the module $\Omega^1\left(J^1M\right)$

$$\begin{aligned}
\alpha_1 &= dq_1 + u dq_2, \\
\alpha_2 &= dp_1 - \kappa p_1^2 dq_2, \\
\beta_1 &= dq_2, \\
\beta_2 &= dp_2 + (1 - 2\kappa)\, p_1 du + (\kappa - 1)\, p_1^2 dq_1 + (2\kappa - 1)\, p_1 p_2 dq_2 - u dp_1\,, \\
U &= du - p_1 dq_1 - p_2 dq_2.
\end{aligned}$$

Here $P_1, P_2 \in D\left(C_{1,0}\right)$, $Q_1, Q_2 \in D\left(C_{0,1}\right)$, $Z \in D\left(l\right)$, $\alpha_1, \alpha_2 \in \Omega^{1,0,0}\left(J^1M\right)$, $\beta_1, \beta_2 \in \Omega^{0,0,1}\left(J^1M\right)$.We get the coordinate representation of the tensors:

$$d_{-1,1,1} = -\left(p_1 dq_1 \wedge dq_2 + dq_2 \wedge du\right) \otimes \left(\frac{\partial}{\partial q_1} + p_1 \frac{\partial}{\partial u} + \kappa p_1^2 \frac{\partial}{\partial p_2} \right),$$

$$\begin{aligned}
d_{1,1,-1} = 2(\kappa - 1)\, \big(&\kappa p_1^3 dq_1 \wedge dq_2 + \kappa p_1^2 dq_2 \wedge du \\
&- dp_1 \wedge du - p_1 dq_1 \wedge dp_1 - p_2 dq_2 \wedge dp_1\big) \otimes \frac{\partial}{\partial p_2},
\end{aligned}$$

$$d_{2,-1,0}=\left(dq_1\wedge dp_1-\kappa p_1^2dq_1\wedge dq_2+udq_2\wedge dp_1\right)\otimes\left(\frac{\partial}{\partial u}+(2\ \kappa-1)\,p_1\frac{\partial}{\partial p_2}\right)$$

$$\begin{aligned}d_{0,-1,2}=(&dq_2\wedge dp_2+(1-2\kappa)\,p_1dq_2\wedge du\\&+(1-\kappa)\,p_1^2dq_1\wedge dq_2-udq_2\wedge dp_1)\\&\otimes\left(\frac{\partial}{\partial u}+(2\ \kappa-1)\,p_1\frac{\partial}{\partial p_2}\right)\end{aligned}$$

and the differential 2-forms $\xi_1$and ξ_2:

$$\begin{aligned}\xi_1&=-dq_2\wedge dp_1,\\ \xi_2&=2\,(1-\kappa)\,dq_2\wedge dp_1.\end{aligned}$$

Due to the theorem, the Hunter-Saxton equation is linearized. The corresponding linear equation is the Euler-Poisson equation [14]

$$v_{tx}=\frac{1}{\kappa\,(t+x)}v_t+\frac{2\,(1-\kappa)}{\kappa\,(t+x)}v_x-\frac{2\,(1-\kappa)}{(\kappa\,(t+x))^2}u.$$

3.5. **The equivalence problem.**

3.5.1. *Relative invariants.* We consider a non-degenerate Monge-Ampère equation $E\overset{\text{def}}{=}E_\omega$. Let $\mathcal{P}=(C_{1,0},l,C_{0,1})$ be the corresponding almost-product structure. Let Z be a real vector field which is generates the complex distribution l and that is normed by the condition $U\,(Z)=1$.

Define a function k by the formula:

$$k\overset{\text{def}}{=}\langle\langle Z,d_{1,1,-1}\rangle\,,\langle Z,d_{-1,1,1}\rangle\rangle\,,$$

where $\langle,\rangle$ is a contraction.

Let $\widetilde{E}\overset{\text{def}}{=}E_{\widetilde{\omega}}$ be an another non-degenerate Monge-Ampère equation with the $\mathcal{AP}$-structure $\widetilde{\mathcal{P}}=\left(\widetilde{C}_{1,0},\widetilde{l},\widetilde{C}_{0,1},\right)$. If $\varphi:J^1M\to J^1M$, $\varphi^*\,(U)=\lambda U$ is a contact transformation such that $\varphi^*\,(\omega)_\varepsilon=h\widetilde{\omega}$ for some non-vanishing function h, then (up to permutation of 1st and 3rd members) $\varphi_*\left(\widetilde{\mathcal{P}}\right)=\mathcal{P}$ and

$$\begin{aligned}\varphi_*^{-1}\,(Z)&=\frac{1}{\lambda}\widetilde{Z},\\ \varphi^*\,(k)&=\frac{1}{\lambda^2}\widetilde{k}\end{aligned}$$

Here $\widetilde{Z}$ is a vector field that generates the distribution $\widetilde{l}$, $U\left(\widetilde{Z}\right)=1$. This means that Z and k are relative invariants of a Monge-Ampère equation.

3.5.2. *Non-holonomic de Rham complex.* Let us introduce submodules $\underline{\Omega}^s \subset \Omega^s\left(J^1M\right)$ of vanishing on Z differential s-forms:

$$\underline{\Omega}^s \stackrel{\text{def}}{=} \left\{\alpha \in \Omega^s\left(J^1M\right) \mid Z\rfloor\,\alpha = 0\right\}.$$

Elements of the submodules $\underline{\Omega}^s$ we call *l-horizontal* forms. The set of all l-horizontal forms form the algebra $\underline{\Omega}^*$ with respect to the operation of exterior multiplication.

Define a projection $\Pi : \Omega^s\left(J^1M\right) \to \underline{\Omega}^s$ and an operator $\partial : \Omega^s\left(J^1M\right) \to \underline{\Omega}^{s+1}$ by the following formulas:

$$\Pi\left(\alpha\right) \stackrel{\text{def}}{=} \alpha - U \wedge \left(Z\rfloor\,\alpha\right),$$
$$\partial \stackrel{\text{def}}{=} \Pi \circ d.$$

The kernel of the operator Π is

$$\ker \Pi = \left\{U \wedge \alpha \,\middle|\, \alpha \in \Omega^*\left(J^1M\right)\right\}.$$

Lemma 2. *Operators Π and ∂ are natural with respect to contact diffeomorphisms, i.e.*

$$\varphi^* \circ \Pi = \widetilde{\Pi} \circ \varphi^* \tag{15}$$

and

$$\varphi^* \circ \partial = \widetilde{\partial} \circ \varphi^*. \tag{16}$$

Proof. For an s-form $\alpha \in \Omega^s\left(J^1M\right)$ we have:

$$\begin{aligned}\varphi^*\left(\Pi\left(\alpha\right)\right) &= \varphi^*\left(\alpha\right) - \varphi^*\left(U\right) \wedge \varphi^*\left(Z\rfloor\,\alpha\right)\\ &= \varphi^*\left(\alpha\right) - U \wedge \left(\varphi_*^{-1}\left(Z\right)\rfloor\,\varphi^*\left(\alpha\right)\right)\\ &= \varphi^*\left(\alpha\right) - \lambda U \wedge \left(\frac{1}{\lambda}\widetilde{Z}\rfloor\,\varphi^*\left(\alpha\right)\right)\\ &= \widetilde{\Pi}\left(\varphi^*\left(\alpha\right)\right).\end{aligned}$$

Let us prove the second formula. For an arbitrary differential l-horizontal form α we have:

$$\begin{aligned}\varphi^*\left(\partial\alpha\right) = \varphi^* \circ \Pi \circ d\left(\alpha\right) = \widetilde{\Pi} \circ \varphi^* \circ d\left(\alpha\right) =\\ = \widetilde{\Pi} \circ d \circ \varphi^*\left(\alpha\right) = \widetilde{\partial}\left(\varphi^*\left(\alpha\right)\right).\end{aligned}$$

□

The restriction of ∂ to the algebra $\underline{\Omega}^*$ we denote by δ:

$$\delta : \underline{\Omega}^s \to \underline{\Omega}^{s+1},$$
$$\delta \stackrel{\text{def}}{=} \partial|_{\underline{\Omega}^*}.$$

The sequence

$$0 \xrightarrow{\delta} \underline{\Omega}^0 \xrightarrow{\delta} \underline{\Omega}^1 \xrightarrow{\delta} \underline{\Omega}^2 \xrightarrow{\delta} \underline{\Omega}^3 \xrightarrow{\delta} \underline{\Omega}^4 \xrightarrow{\delta} 0\,, \tag{17}$$

where $\underline{\Omega}^0 \overset{\mathrm{def}}{=} C^\infty(J^1M)$, is a complex, i.e. $\delta^2 = 0$. This complex we will call the *non-holonomic de Rham complex*.

Note that

$$\delta\,(\alpha \wedge \beta) = \delta\alpha \wedge \beta + (-1)^{\deg\alpha}\,\alpha \wedge \delta\beta$$

for any differential forms $\alpha, \beta \in \underline{\Omega}^s$.

A differential l-horizontal 2-form ω is called *l-effective* if $\omega \wedge \partial U = 0$. Note that

$$\partial\,(gU) = g\partial U$$

for any function $g \in C^\infty\,(J^1M)$ and therefore this definition is correct.

Let ω be an l-effective differential 2-form and let $\varphi : J^1M \to J^1M$ be a contact diffeomorphism such that $\varphi^*\,(\omega) = \widehat{\omega}$. Then the form $\widehat{\omega}$ is $\widehat{l}$-effective also.

3.5.3. *e-structures.* For any differential l-horizontal 2-form α we construct an l-effective part

$$\alpha_l \overset{\mathrm{def}}{=} \alpha - s_\alpha \partial U \wedge \partial U,$$

where the function s_α is defined by the formula

$$\alpha \wedge \partial U = s_\alpha \partial U \wedge \partial U.$$

The effective 2-form ω and the l-effective part ω_l are generating the same Monge-Ampère equation.

Now we can formulate the condition of contact equivalence of Monge-Ampère equations E_ω and $E_{\widetilde{\omega}}$ in terms of l-effective forms: $\varphi^*\,(\omega_l) = h\widetilde{\omega}_{\widetilde{l}}$.

From this place we suppose that ω and $\widetilde{\omega}$ are l-effective and $\widetilde{l}$-effective differential 2-forms respectively and $\varphi^*(U) = \lambda U$, $\varphi^*\,(\omega) = h\widetilde{\omega}$.

Define a function F and an operator $A : D(C) \to D(C)$ ω by the following formulas:

$$\begin{aligned} F\partial U \wedge \partial U &= \omega \wedge \omega, \\ AX \rfloor \partial U &= X \rfloor \omega, \end{aligned} \tag{18}$$

where the vector field $X \in D(C)$.

Then

$$\varphi^*(F) = \frac{h^2}{\lambda^2}\widetilde{F}.$$

Indeed,

$$h^2\widetilde{\omega}\wedge\widetilde{\omega}=\varphi^*\left(F\partial U\wedge\partial U\right)=\varphi^*\left(F\right)\varphi^*\left(\partial U\right)\wedge\varphi^*\left(\partial U\right)$$
$$=\varphi^*\left(F\right)\widetilde{\partial}\varphi^*\left(U\right)\wedge\widetilde{\partial}\varphi^*\left(U\right)=\lambda^2\varphi^*\left(F\right)\widetilde{\partial}U\wedge\widetilde{\partial}U.$$

On the other hand $\widetilde{\omega}\wedge\widetilde{\omega}=\widetilde{F}\widetilde{\partial}U\wedge\widetilde{\partial}U$. Then, $\lambda^2\varphi^*\left(F\right)=h^2\widetilde{F}$.

The square of the operator A is scalar and $A^2+F=0$.

The differential 2-form ∂U is non-degenerate on the module of vector fields from the Cartan distribution. This means that if $X\rfloor\partial U=0$ for $X\in D(C)$, then $X=0$.

For a function $H\in C^\infty(J^1M)$ the formula

$$X_H\rfloor\partial U=-\partial H \tag{19}$$

uniquely defines a vector field $X_H\in D\left(C\right)$.

Note that

$$\varphi_*^{-1}\left(X_H\right)=\frac{1}{\lambda}\widetilde{X}_{\varphi^*(H)}.$$

We need two technical lemmas.

Lemma 3. *We have:*

$$\widetilde{A}=\frac{\lambda}{h}\varphi_*^{-1}\circ A\circ\varphi_*.$$

Proof. Applying φ to the both parts of $(18)_2$ we have:

$$\lambda\varphi_*^{-1}\left(AX\right)\rfloor\widetilde{\partial}U=h\varphi_*^{-1}\left(X\right)\rfloor\widetilde{\omega}.$$

Moreover,

$$\varphi_*^{-1}\left(X\right)\rfloor\widetilde{\omega}=\widetilde{A}\varphi_*^{-1}\left(X\right)\rfloor\widetilde{\partial}U.$$

Since ∂U is non-degenerate, we get: $\lambda\varphi_*^{-1}\circ A=h\widetilde{A}\circ\varphi_*^{-1}$. Therefore $\widetilde{A}=\frac{\lambda}{h}\varphi_*^{-1}\circ A\circ\varphi_*$. □

Lemma 4. *For any function $h\in C^\infty(J^1M)$ we have:*

$$\partial h\wedge\omega=\frac{1}{2}AX_h\rfloor\left(\partial U\wedge\partial U\right).$$

Proof. The form ω is l-effective, i.e., $\omega\wedge\partial U=0$. Then

$$0=X_h\rfloor\left(\omega\wedge\partial U\right)=\left(AX_h\rfloor\partial U\right)\wedge\partial U-\partial h\wedge\omega.$$

□

Now we can construct an e-structure for the equation E. Define a vector field W which lies in the Cartan distribution. This vector fields is uniquely determined by the following relations:

$$W\rfloor\left(\partial U\wedge\partial U\right)=2\partial\omega,$$
$$U\left(W\right)=0.$$

Suppose that $k\neq 0$. Let us introduce the function

$$F_0\overset{\text{def}}{=}\frac{F}{k}.$$

and the vector field

$$V\overset{\text{def}}{=}\frac{1}{k}AW$$

Since Lemma 3, and the facts that

$$\varphi_*^{-1}\left(W\right)=\frac{1}{\lambda^2}\left(h\widetilde{\widetilde{W}}+\widetilde{A}\widetilde{X}_h\right),$$
$$\varphi^*\left(F_0\right)=h^2\widetilde{F}_0,$$

we obtain:

$$\varphi_*^{-1}\left(V\right)=\frac{1}{\varphi^*\left(k\right)}\left(\varphi_*^{-1}\circ A\circ\varphi_*\right)\circ\varphi_*^{-1}\left(W\right)=$$
$$=\frac{h^2}{\lambda}\widetilde{V}-\frac{h}{\lambda}\widetilde{F}_0\widetilde{X}_h$$

For the vector fields X_{F_0} we have:

$$\varphi_*^{-1}\left(X_{F_0}\right)=\frac{h^2}{\lambda}\widetilde{X}_{\widetilde{F}_0}+\frac{2h}{\lambda}\widetilde{F}_0\widetilde{X}_h.$$

Define vector fields $\mathcal{Z}\in D\left(l\right)$ and $\mathcal{Y}$ by the formulas

$$U\left(\mathcal{Z}\right)=\frac{1}{\sqrt{|k|}}$$

and

$$\mathcal{Y}\overset{\text{def}}{=}\frac{\sqrt{|k|}}{F_0}\left(X_{F_0}+2V\right).$$

The vector fields $\mathcal{Z}$ and $\mathcal{Y}$ are invariant (up to multiplication by -1) of E:

$$\varphi_*^{-1}\left(\mathcal{Y}\right)=\widetilde{\mathcal{Y}}.$$

The vector field $\mathcal{Y}$ splits into the sum

$$\mathcal{Y}=\mathcal{Y}_{1,0}+\mathcal{Y}_{0,1},$$

where $\mathcal{Y}_{1,0}\in D\left(C_{1,0}\right)$ and $\mathcal{Y}_{0,1}\in D\left(C_{0,1}\right)$.

Applying tensors $d_{-1,1,1}$ and $d_{1,1,-1}$ to $\mathcal{Z}, \mathcal{Y}_{0,1}, \mathcal{Y}_{1,0}$ we get two invariant vector fields from the distributions $C_{1,0}$ and $C_{0,1}$ respectively:

$$\mathcal{X}_{1,0} \stackrel{\text{def}}{=} d_{-1,1,1}\left(\mathcal{Z}, \mathcal{Y}_{0,1}\right),$$
$$\mathcal{X}_{0,1} \stackrel{\text{def}}{=} d_{1,1,-1}\left(\mathcal{Z}, \mathcal{Y}_{1,0}\right),$$

For the case of general Monge-Ampère equation the vector fields $\mathcal{Z}$, $\mathcal{X}_{1,0}$, $\mathcal{Y}_{1,0}$, $\mathcal{X}_{0,1}$, $\mathcal{Y}_{0,1}$ form an e-structure on J^1M. Denote the constructed e-structure by

$$e_E = \left(\mathcal{Z}, \left\{\mathcal{X}_{1,0}, \mathcal{X}_{0,1}\right\}, \left\{\mathcal{Y}_{1,0}, \mathcal{Y}_{0,1}\right\}\right).$$

The e-structure is real for a hyperbolic equation and complex for elliptic one. In the last case we can construct a real e-structure using and operation of complex conjugate.

Theorem 3. *Two non-degenerate Monge-Ampère equations E and $\widetilde{E}$ are contact equivalent if their constructed e-structures e_E and $e_{\widetilde{E}}$ are equivalent.*

So, the problem of contact equivalence of hyperbolic Monge-Ampère equations is a problem of equivalence of e-structures.

Example 11 (Non-linear wave equation)**.** Construct an e-structure for a non-linear wave equation

$$v_{xy} = f\left(x, y, v, v_x, v_y\right).$$

For this equation

$$Z = \frac{\partial}{\partial u} + f_{p_2}\frac{\partial}{\partial p_1} + f_{p_1}\frac{\partial}{\partial p_2}$$

and

$$\begin{aligned}\Pi\left(\omega\right) &= \left(p_1 f_{p_1} + p_2 f_{p_2} - 2f\right) dq_1 \wedge dq_2 + dq_1 \wedge dp_1 \\ &\quad - dq_2 \wedge dp_2 - f_{p_2} dq_1 \wedge du + f_{p_1} dq_2 \wedge du, \\ \partial U &= \left(p_2 f_{p_2} - p_1 f_{p_1}\right) dq_1 \wedge dq_2 + dq_1 \wedge dp_1 \\ &\quad + dq_2 \wedge dp_2 - f_{p_2} dq_1 \wedge du - f_{p_1} dq_2 \wedge du.\end{aligned}$$

Below the form $\Pi\left(\omega\right)$ is denoted by ω. We see that $\omega \wedge \partial U = 0$, $k = f_{p_1p_1} f_{p_2p_2}$ and $F = -1$. Suppose that the function k is non-vanishing. Then

$$F_0 = -\frac{1}{f_{p_1p_1} f_{p_2p_2}}.$$

In the free basis P_1, P_2, Q_1, Q_2 the operator A has a diagonal form:

$$A = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{Vmatrix}.$$

Moreover

$$\begin{aligned}
W &= f_{p_2}\frac{\partial}{\partial p_1} - f_{p_1}\frac{\partial}{\partial p_2}, \\
V &= \frac{1}{f_{p_1p_1}f_{p_2p_2}}\left(f_{p_2}\frac{\partial}{\partial p_1} + f_{p_1}\frac{\partial}{\partial p_2}\right), \\
X_{F_0} &= k^{-2}\left(f_{p_1p_1}f_{p_2p_2p_2} + f_{p_2p_2}f_{p_1p_1p_2}\right)\frac{d}{dq_2} \\
&\quad + k^{-2}\left(f_{p_1p_1}f_{p_1p_2p_2} + f_{p_2p_2}f_{p_1p_1p_1}\right)\frac{d}{dq_1} \\
&\quad - k^{-2}(p_2\left(f_{p_1p_1}f_{p_2p_2u} + f_{p_2p_2}f_{p_1p_1u}\right) + f_{p_1p_1}f_{q_2p_2p_2} + f_{p_2p_2}f_{q_2p_1p_1})\frac{\partial}{\partial p_2} \\
&\quad - k^{-2}(p_1\left(f_{p_1p_1}f_{p_2p_2u} + f_{p_2p_2}f_{p_1p_1u}\right) + f_{p_1p_1}f_{q_1p_2p_2} + f_{p_2p_2}f_{q_1p_1p_1})\frac{\partial}{\partial p_1}.
\end{aligned}$$

We obtain the following e-structure:

$$\begin{aligned}
\mathcal{Y}_{1,0} &= -k^{-1/2}(f_{p_1p_2p_2}f_{p_1p_1} + f_{p_2p_2}f_{p_1p_1p_1})\left(\frac{d}{dq_1} + f\frac{\partial}{\partial p_2}\right) \\
&\quad + k^{-1/2}(-2f_{p_2}f_{p_2p_2}f_{p_1p_1} + p_1f_{p_2p_2u}f_{p_1p_1} + ff_{p_2p_2p_2}f_{p_1p_1} \\
&\quad + p_1f_{p_2p_2}f_{p_1p_1u} + ff_{p_2p_2}f_{p_1p_1p_2} + f_{p_1p_1}f_{q_1p_2p_2} + f_{p_2p_2}f_{q_1p_1p_1})\frac{\partial}{\partial p_1},
\end{aligned}$$

$$\begin{aligned}
\mathcal{Y}_{0,1} &= -k^{-1/2}(f_{p_2p_2p_2}f_{p_1p_1} + f_{p_2p_2}f_{p_1p_1p_2})\left(\frac{d}{dq_2} + f\frac{\partial}{\partial p_1}\right) \\
&\quad k^{-1/2}(p_2f_{p_2p_2u}f_{p_1p_1} + f_{p_1p_1}(ff_{p_1p_2p_2} + f_{q_2p_2p_2}) + \\
&\quad f_{p_2p_2}(-2f_{p_1}f_{p_1p_1} + p_2f_{p_1p_1u} + ff_{p_1p_1p_1} + f_{q_2p_1p_1}))\frac{\partial}{\partial p_2},
\end{aligned}$$

$$\mathcal{Z} = k^{-1/2}\left(\frac{\partial}{\partial u} + f_{p_2}\frac{\partial}{\partial p_1} + f_{p_1}\frac{\partial}{\partial p_2}\right),$$

$$\begin{aligned}
\mathcal{X}_{1,0} = {} & \frac{1}{f_{p_1p_1}}(p_2 f_{p_2p_2u} f_{p_1p_1} + p_2 f_{p_1}(f_{p_2p_2p_2} f_{p_1p_1} + f_{p_2p_2} f_{p_1p_1p_2}) \\
& - \frac{1}{f_{p_2p_2}}(f_{p_2p_2p_2} f_{p_1p_1} + f_{p_2p_2} f_{p_1p_1p_2})(-f_u + p_2 f_{p_2u} - f_{p_2} f_{p_1} \\
& + p_2 f_{p_2p_2} f_{p_1} + f f_{p_1p_2} + f_{q_2p_2}) + f_{p_1p_1}(f f_{p_1p_2p_2} + f_{q_2p_2p_2}) \\
& + f_{p_2p_2}(-2 f_{p_1} f_{p_1p_1} + p_2 f_{p_1p_1u} + f f_{p_1p_1p_1} + f_{q_2p_1p_1}))\frac{\partial}{\partial p_1}
\end{aligned}$$

$$\begin{aligned}
\mathcal{X}_{0,1} = {} & \frac{1}{f_{p_2p_2}}\,(-2 f_{p_2} f_{p_2p_2} f_{p_1p_1} + p_1 f_{p_2p_2u} f_{p_1p_1} + f f_{p_2p_2p_2} f_{p_1p_1} \\
& + p_1 f_{p_2p_2} f_{p_1p_1u} + f f_{p_2p_2} f_{p_1p_1p_2} + p_1 f_{p_2}(f_{p_1p_2p_2} f_{p_1p_1} + f_{p_2p_2} f_{p_1p_1p_1}) \\
& + f_{p_1p_1} f_{q_1p_2p_2} - \frac{1}{f_{p_1p_1}}(f_{p_1p_2p_2} f_{p_1p_1} + f_{p_2p_2} f_{p_1p_1p_1}(-f_u + p_1 f_{p_1u} \\
& + f f_{p_1p_2} + f_{p_2}(-f_{p_1} + p_1 f_{p_1p_1}) + f_{q_1p_1}) + f_{p_2p_2} f_{q_1p_1p_1})\frac{\partial}{\partial p_2}\,.
\end{aligned}$$

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