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DIFFERENTIAL EQUATIONS WITH CONSTRAINTS IN JET BUNDLES: LAGRANGIAN AND HAMILTONIAN SYSTEMS

(submitted by V.V. Lychagin)

ABSTRACT. The paper is a survey of the theory of Lagrangian systems with non-holonomic constraints in jet bundles. The subject of the paper are systems of second-order ordinary and partial differential equations that arise as extremals of variational functionals in fibered manifolds. A geometric setting for Euler-Lagrange and Hamilton equations, based on the concept of Lepage class is presented. A constraint is modeled in the underlying fibered manifold as a fibered submanifold endowed with a distribution (the canonical distribution). A constrained system is defined by means of a Lepage class on the constraint submanifold. Constrained Euler-Lagrange equations and constrained Hamilton equations, and properties of the corresponding exterior differential systems, such as regularity, canonical form, or existence of a constraint Legendre transformation, are presented. The case of mechanics (ODEs) and field theory (PDEs) are investigated separately, however, stress is put on a unified exposition, so that a direct comparison of results and formulas is at hand.

1. INTRODUCTION

Since the 30's of the last century when the pioneer paper by Chetaev was published [4], the study of non-holonomic constrained systems has been of growing interest in mechanics, control theory and geometry.

Namely during the past 15 years much effort has been devoted to developments of geometric methods and studies of geometric structures of non-holonomic mechanics; among the many contributions to the subject, let us mention here at least [3, 6, 9, 15, 17, 20, 26, 27, 28, 30, 31, 33, 34, 38], and references therein. Recently, several authors have started to study a more general situation of partial differential equations (field theories) with constraints given by systems of first-order partial differential equations [2, 22, 25, 37]. Since the geometric origin of these constraints is the same as in mechanics, it is natural also in this generalized situation to call such constraints "non-holonomic".

The papers investigating non-holonomic systems differ in approaches, methods, geometric setting, kind of constraints studied, and many other aspects. Usually (and this is in no case specific for constrained systems), tools, structures and methods used in mechanics (i.e. ordinary differential equations) and field theory (partial differential equations) are essentially different. The aim of this paper is to present foundations of a general geometric theory of non-holonomic systems as a part of the calculus of variations on fibered manifolds. It is based on the theory of Lepage equivalents of Lagrangians (Krupka [10, 12]) and of dynamical forms (Krupková [14, 15, 16, 19]), and on study of exterior differential systems associated with variational equations (Krupková [14, 16, 19, 21]). It is also important to note that a constraint is modeled in the underlying fibered manifold as a *fibered submanifold endowed with a distribution* (called *canonical distribution*) [15, 22]. This structure plays a key role in studying the geometry of non-holonomic constrained systems, and represents a *correct mathematical realization of the physical d'Alembert's principle* (that is ambiguous in case of velocity dependent constraints in mechanics, and completely unclear in field theory). The setting of [15]

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and [22] brings a unified approach to mechanics and field theory, both unconstrained and with constraints, and can be directly transferred to higher-order situation [18].

There are basically two different approaches to systems with constraints:

• constrained system is modeled as a modified unconstrained system, defined *on the same manifold as the unconstrained system* (in mechanics this concerns so called "constraint forces" and dynamics governed by equations with Lagrange multipliers),

• constrained system is modeled as a system defined *on the constraint submanifold* (dynamics are modeled by the so called "reduced equations", without Lagrange multipliers).

As shown in [15] and [22], both these approaches are equivalent. In this paper, however, we prefer the latter one, since it is more geometrical, and enables us to study constrained systems by the same tools as unconstrained systems. We focus on *variational systems*, i.e. such that their dynamics are given by differential equations that arise as equations for extremals of Lagrangians (Euler–Lagrange equations). First, we recall basic facts on *unconstrained* Lagrangian systems and their associated Hamiltonian systems in jet bundles. Then we turn to the concept of *non-holonomic constraint structure*. Finally we study Lagrangian systems subjected to non-holonomic constraints, namely *constrained Euler–Lagrange equations and constrained Hamilton equations*, where we devote our attention to such problems as regularity of constrained systems, or existence of an appropriate "constraint Legendre transformation". The cases of mechanics (ordinary differential equations) and field theory (partial differential equations) are investigated separately, however, stress is put on a unified exposition, so that common features on one hand and differences on the other hand are transparent and their geometric origin becomes clear. We also tried to provide analogous results and formulas in such a way that the reader could compare them directly.

This work is basically a review paper, however, it contains also new, yet unpublished results (this concerns Sec. 4.5 and 4.6 on constrained Hamilton–De Donder equations and constraint Legendre transformation for general non-holonomic constraints in field theory).

2. Calculus in jet bundles

We start with a brief introduction of notations, basic structures and the corresponding calculus to be used. For more details we refer to the

books by Saunders [32] and Krupková [16], and the papers by Krupka [10, 12].

We consider a fibered manifold $\pi : Y \to X$ with $\dim X = n \geq 1$, $\dim Y = m + n$, and its jet prolongations $\pi_1 : J^1Y \to X$ and $\pi_2 : J^2Y \to X$. All manifolds and mappings are smooth, and the summation convention on repeated indices applies throughout.

A mapping $\gamma : U \to Y$, where $U \subset X$ is an open set, is called a *section* of π if $\pi \circ \gamma = \mathrm{id}_U$. We denote by $J^1\gamma$ and $J^2\gamma$ the first and the second jet prolongation of γ, respectively. Note that $J^1\gamma$ (resp. $J^2\gamma$) is a section of π_1 (resp. π_2). A section δ of π_1 is called *holonomic* if $\delta = J^1\gamma$ for a section γ of π.

A vector field ξ on Y is called π-*vertical* if $T\pi \cdot \xi = 0$, and π-*projectable* if $T\pi \cdot \xi = \xi_0 \circ \pi$ for a vector field ξ_0 on X. Considering the projections $\pi_1 : J^1Y \to X$, $\pi_2 : J^2Y \to X$, $\pi_{1,0} : J^1Y \to Y$, $\pi_{2,1} : J^2Y \to J^1Y$ and $\pi_{2,0} : J^2Y \to Y$ the concepts of the corresponding verticality and projectability are obtained quite similarly. For the module of vector fields (resp. π_r-vertical vector fields) on J^rY, $r = 1, 2$, we shall use the notation $\mathcal{X}(J^rY)$ (resp. $\mathcal{V}(J^rY)$).

Denote by $\Omega^q(J^1Y)$ the module of q-forms on J^1Y. A form $\eta \in \Omega^q(J^1Y)$ is called π_1-*horizontal* (resp. $\pi_{1,0}$-*horizontal*) if $i_\xi\eta = 0$ for every π_1-vertical (resp. $\pi_{1,0}$-vertical) vector field ξ on J^1Y; $\eta \in \Omega^q(J^1Y)$ is called *contact* if $J^1\gamma^*\eta = 0$ for every section γ of π [10]. A contact form $\eta \in \Omega^q(J^1Y)$ is called 1-*contact* if for every π_1-vertical vector field ξ the form $i_\xi\eta$ is π_1-horizontal; it is called k-*contact*, where $2 \leq k \leq q$, if for every π_1-vertical vector field ξ the form $i_\xi\eta$ is $(k-1)$-contact [12]. We denote
- $\Omega^q_X(J^1Y)$ the module of π_1-horizontal q-forms on J^1Y,
- $\Omega^{q-i,i}(J^1Y)$ the module of i-contact q-forms on J^1Y,
- $\Omega^{q-i,i}_Y(J^1Y)$ the submodule of $\Omega^{q-i,i}(J^1Y)$ consisting of $\pi_{1,0}$-horizontal forms.

It is important to mention that every form $\eta \in \Omega^q(J^1Y)$ has a *unique decomposition into contact components* as follows (Krupka [12]):

$$\pi^*_{2,1}\eta = h\eta + p_1\eta + \cdots + p_{q-1}\eta + p_q\eta, \qquad (2.1)$$

where h and p_k ($k \geq 1$) denotes the horizontalization and k-contactization operators, respectively, assigning to η its horizontal (resp. k-contact, $1 \leq k \leq q$) component.

Therefore, we shall also use the following notations:
- $\Omega^{q(\geq k)}(J^1Y) = \Omega^{q-k,k}(J^1Y) \oplus \Omega^{q-k-1,k+1}(J^1Y) \oplus \cdots \oplus \Omega^{0,q}(J^1Y)$, i.e. the module of q-forms on J^1Y that are *at least* k-contact,

• $\Omega_Y^{q(\geq k)}(J^1Y) = \Omega_Y^{q-k,k}(J^1Y) \oplus \Omega_Y^{q-k-1,k+1}(J^1Y) \oplus \cdots \oplus \Omega_Y^{0,q}(J^1Y)$, i.e. $\pi_{1,0}$-*horizontal* q-forms on J^1Y that are *at least* k-contact.

We denote by (x^i, y^σ), where $1 \leq i \leq n$, $1 \leq \sigma \leq m$, local fibered coordinates on Y, and by $(x^i, y^\sigma, y_j^\sigma)$ and $(x^i, y^\sigma, y_j^\sigma, y_{jk}^\sigma)$, where $1 \leq j \leq k \leq n$, associated coordinates on J^1Y and J^2Y, respectively. We put

$$\omega_0 = dx^1 \wedge \cdots \wedge dx^n, \quad \omega_j = i_{\partial/\partial x^j}\omega_0. \tag{2.2}$$

In case that $\dim X = 1$, we write (t, q^σ), where $1 \leq \sigma \leq m$, to denote local fibered coordinates on Y, and $(t, q^\sigma, \dot{q}^\sigma)$ (resp. $(t, q^\sigma, \dot{q}^\sigma, \ddot{q}^\sigma)$) for associated coordinates on J^1Y (resp. J^2Y).

In calculations we use either a canonical basis of one forms, i.e. $(dx^i, dy^\sigma, dy_j^\sigma)$ on J^1Y and $(dx^i, dy^\sigma, dy_j^\sigma, dy_{jl}^\sigma)$ on J^2Y (alternatively, if $\dim X = 1$, $(dt, dq^\sigma, d\dot{q}^\sigma)$ and $(dt, dq^\sigma, d\dot{q}^\sigma, d\ddot{q}^\sigma)$), or better a *basis adapted to the contact structure*, i.e. $(dx^i, \omega^\sigma, dy_j^\sigma)$ on J^1Y and $(dx^i, \omega^\sigma, \omega_j^\sigma, dy_{jl}^\sigma)$, on J^2Y, where

$$\omega^\sigma = dy^\sigma - y_i^\sigma \, dx^i, \quad \omega_j^\sigma = dy_j^\sigma - y_{ji}^\sigma \, dx^i \tag{2.3}$$

are local canonical contact 1-forms. Alternatively, if $\dim X = 1$, adapted bases take the form $(dt, \omega^\sigma, d\dot{q}^\sigma)$ and $(dt, \omega^\sigma, \dot{\omega}^\sigma, d\ddot{q}^\sigma)$, where

$$\omega^\sigma = dq^\sigma - \dot{q}^\sigma \, dt, \quad \dot{\omega}^\sigma = d\dot{q}^\sigma - \ddot{q}^\sigma \, dt. \tag{2.4}$$

In an adapted basis to the contact structure every k-contact component $p_k\eta$ of a q-form η (where $1 \leq k \leq q$) is expressed by means of a wedge product containing *exactly* k of the canonical contact 1-forms above.

If f is a function on J^1Y, we have by (2.1) the exterior derivative df canonically splitted into the horizontal and contact component,

$$\pi_{2,1}^* df = hdf + pdf, \tag{2.5}$$

with

$$hdf = \frac{df}{dx^j} \, dx^j, \tag{2.6}$$

where d/dx^j, $1 \leq j \leq n$, denotes the j-th total derivative operator (also called j-th formal derivative operator),

$$\frac{d}{dx^j} = \frac{\partial}{\partial x^j} + y_j^\sigma \frac{\partial}{\partial y^\sigma} + y_{ij}^\sigma \frac{\partial}{\partial y_i^\sigma}. \tag{2.7}$$

For convenience of notations we also use the 'cut' total derivative operators,

$$\frac{d'}{dx^j} = \frac{\partial}{\partial x^j} + y_j^\sigma \frac{\partial}{\partial y^\sigma} = \frac{d}{dx^j} - y_{ij}^\sigma \frac{\partial}{\partial y_i^\sigma}, \quad 1 \leq j \leq n. \tag{2.8}$$

If $\dim X = 1$, these formulas take the following form:

$$hdf = \frac{df}{dt}\, dt, \tag{2.9}$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \dot{q}^\sigma \frac{\partial}{\partial q^\sigma} + \ddot{q}^\sigma \frac{\partial}{\partial \dot{q}^\sigma}, \tag{2.10}$$

$$\frac{d'}{dt} = \frac{\partial}{\partial t} + \dot{q}^\sigma \frac{\partial}{\partial q^\sigma} = \frac{d}{dt} - \ddot{q}^\sigma \frac{\partial}{\partial \dot{q}^\sigma}. \tag{2.11}$$

Definition 2.1. 1-contact $(n+1)$-forms on $J^r Y$, horizontal with respect to the projection $\pi_{r,0}$, are called *dynamical forms* of order r [16].

Horizontal n-forms on $J^r Y$ are called *Lagrangians* of order r [10]. By a *local* Lagrangian (of order r) we shall mean a Lagrangian defined on an open subset of $J^r Y$.

Definition 2.2. [10] Let λ be a Lagrangian on $J^1 Y$. An n-form ρ is called *Lepage equivalent* of λ if $h\rho = \lambda$ and $p_1 d\rho$ is a dynamical form. The form $p_1 d\rho$ is then called the *Euler–Lagrange form of λ* and denoted by E_λ.

As proved in [12], every Lagrangian has a Lepage equivalent. For a Lagrangian of order r Lepage equivalents are of order $2r - 1$, and the Euler–Lagrange form is of order $2r$. It should be stressed that *while Lepage equivalent of a Lagrangian need not be unique, the Euler–Lagrange form always is unique.*

3. MECHANICAL SYSTEMS WITH CONSTRAINTS

Throughout this section we consider a fibered manifold $\pi : Y \to X$, $\dim X = 1$, and we assume $\dim Y = m + 1$, where $m \geq 1$. Main sources for our exposition are the following: [1, 8, 11, 12, 35, 36] for the inverse variational problem, [15, 16, 21] for a geometric approach to variational ordinary differential equations, [7, 10, 12, 14, 16] for (unconstrained) Lagrangian and Hamiltonian mechanics in jet bundles, [15, 28] for the model of the non-holonomic constraint structure, and [3, 4, 6, 9, 15, 17, 20, 27, 28, 29, 30, 31, 33, 38] for non-holonomic Lagrangian and Hamiltonian systems.

3.1. **Dynamical forms.** Let E be a dynamical form on $J^2 Y$. A section γ of π is called a *path* of E if

$$E \circ J^2 \gamma = 0. \tag{3.1}$$

In fibered coordinates E reads

$$E = E_\sigma \omega^\sigma \wedge dt, \tag{3.2}$$

where E_σ are functions of $(t, q^\nu, \dot{q}^\nu, \ddot{q}^\nu)$, and the equation for paths of E takes the form of *a system of m second-order ordinary differential equations* for the components (γ^ν) of γ as follows:

$$E_\sigma\left(t, \gamma^\nu(t), \frac{d\gamma^\nu}{dt}, \frac{d^2\gamma^\nu}{dt^2}\right) = 0, \quad 1 \leq \sigma \leq m. \tag{3.3}$$

We stress that these equations need not be "solvable with respect to the second derivatives", meaning that they *need not be expressible in a normal form,*

$$\frac{d^2\gamma^\sigma}{dt^2} = F^\sigma\left(t, \gamma^\nu(t), \frac{d\gamma^\nu}{dt}\right). \tag{3.4}$$

Equations for paths of dynamical forms can be represented by means of *exterior differential systems* locally generated by 1-forms [14].

Proposition 3.1. *Let E be a dynamical form on J^2Y. A section γ of π is a path of E if and only if*

$$J^2\gamma^* i_\xi \alpha = 0 \quad \forall \xi \in \mathcal{V}(J^2Y), \tag{3.5}$$

where α is any 2-form such that $p_1\alpha = E$.

Proof. By a direct computation we immediately obtain that (3.1) is equivalent with the condition $J^2\gamma^* i_\xi E = 0$ $\forall \xi \in \mathcal{V}(J^2Y)$. Now, since contraction by vertical vector fields is compatible with the decomposition of forms to contact components (2.1), and prolongations of sections annihilate contact forms, we can see that adding to E (which is 1-contact) *any* 2-contact form F gives us

$$J^2\gamma^* i_\xi(E + F) = J^2\gamma^* i_\xi E + J^2\gamma^* i_\xi F = J^2\gamma^* i_\xi E \quad \forall \xi \in \mathcal{V}(J^2Y). \tag{3.6}$$

$\square$

Definition 3.1. [15] Let $E \in \Omega^2_Y(J^2Y)$ be a dynamical form. The equivalence class of 2-forms (on an open subset $U \subset J^2Y$) defined by

$$\alpha_1 \sim \alpha_2 \quad \text{iff} \quad p_1\alpha_1 = p_1\alpha_2 = E|_U \tag{3.7}$$

is called *Lepage class of E on U*. The family of all local Lepage classes of E will be referred to as *Lepage class of E* and will be denoted by $[\alpha]_E$, or simply $[\alpha]$.

By the above proposition, the equation for paths of E (on U) coincides with equations for *holonomic* integral sections of the distribution

$$\Delta_\alpha = \text{annih}\,\{i_\xi\alpha \mid \forall \xi \in \mathcal{V}(J^2Y)\} = \text{span}\,\{\zeta \in \mathcal{X}(J^2Y) \mid i_\zeta\alpha = 0\}, \tag{3.8}$$

where α is any representative of the Lepage class of E (on U).

Definition 3.2. [15] Let $[\alpha]$ be a Lepage class of E. Every representative $\alpha \in [\alpha]$ is called a *Hamiltonian system* associated with E. The distribution Δ_α is called a *dynamical distribution* of E. Equations for (all) integral sections of Δ_α are called *Hamilton equations* associated with E.

In what follows we shall be interested in dynamical forms that can be represented by *first-order Lepage classes*. This means that the dynamics are described by dynamical distributions defined on (open subsets of) J^1Y.

Proposition 3.2. *Let E be a dynamical form on J^2Y. The following conditions are equivalent:*

(1) *Around each point in J^1Y there exists a Lepage class of E.*
(2) *In every fibered chart, E takes the form (3.2), where the functions E_σ are affine in the second derivatives, i.e.,*

$$E_\sigma = A_\sigma(t, q^\nu, \dot{q}^\nu) + B_{\sigma\rho}(t, q^\nu, \dot{q}^\nu)\ddot{q}^\rho. \tag{3.9}$$

Proof. We have

$$\alpha = E + F = E_\sigma \omega^\sigma \wedge dt + F_{\sigma\nu}\omega^\sigma \wedge \omega^\nu + F^{01}_{\sigma\nu}\omega^\sigma \wedge \dot{\omega}^\nu + F^{11}_{\sigma\nu}\dot{\omega}^\sigma \wedge \dot{\omega}^\nu$$

$$= (E_\sigma - 2F_{\sigma\nu}\dot{q}^\nu - F^{01}_{\sigma\nu}\ddot{q}^\nu)dq^\sigma \wedge dt + (F^{01}_{\nu\sigma}\dot{q}^\sigma - 2F^{11}_{\sigma\nu}\ddot{q}^\nu)d\dot{q}^\sigma \wedge dt \tag{3.10}$$

$$+ F_{\sigma\nu}dq^\sigma \wedge dq^\nu + F^{01}_{\sigma\nu}dq^\sigma \wedge d\dot{q}^\nu + F^{11}_{\sigma\nu}d\dot{q}^\sigma \wedge d\dot{q}^\nu.$$

Hence, α is projectable onto an open subset of J^1Y iff $F_{\sigma\nu}$ and $F^{01}_{\sigma\nu}$ do not depend on $\ddot{q}^\rho$, $F^{11}_{\sigma\nu} = 0$, and

$$\frac{\partial E_\sigma}{\partial \ddot{q}^\rho} = F^{01}_{\sigma\rho}; \tag{3.11}$$

consequently,

$$\frac{\partial^2 E_\sigma}{\partial \ddot{q}^\rho \partial \ddot{q}^\nu} = 0. \tag{3.12}$$

The first-order Lepage class is represented by 2-forms

$$\alpha = A_\sigma \omega^\sigma \wedge dt + B_{\sigma\nu}\omega^\sigma \wedge d\dot{q}^\nu + F_{\sigma\nu}\omega^\sigma \wedge \omega^\nu, \tag{3.13}$$

where $F_{\sigma\nu}$ are arbitrary functions of $(t, q^\rho, \dot{q}^\rho)$. $\qquad\qquad\square$

Definition 3.3. A dynamical form on J^2Y that has a Lepage class around each point of J^1Y is called J^1Y-*pertinent*, or, a *first-order mechanical system*.

Finally, we recall the concept of a regular dynamical form.

Definition 3.4. [14, 15] A first-order mechanical system (respectively, a J^1Y-pertinent dynamical form) E is called *regular* if around each point of J^1Y there exists a dynamical distribution Δ_α, $\alpha \in [\alpha]_E$ such that rank $\Delta_\alpha = 1$.

Proposition 3.3. [14, 15] *The following conditions are equivalent:*

(1) *A first-order mechanical system E is regular.*
(2) *The following condition holds:*

$$\det\left(\frac{\partial E_\sigma}{\partial \ddot{q}^\nu}\right) = \det(B_{\sigma\nu}) \neq 0. \tag{3.14}$$

(3) *Equations for paths of E have an equivalent normal form* (3.4) *where $F^\sigma = -B^{\sigma\nu}A_\nu$.*

3.2. **Variational ODE's and related Hamiltonian systems.** A dynamical form $E \in \Omega^2_Y(J^2Y)$ is called (*globally*) *variational* if there exists a Lagrangian λ such that (possibly up to a projection), $E = E_\lambda$. E is called *locally variational* if it is variational in a neighborhood of every point in J^2Y [11, 12]. In fibered coordinates this means that the components E_σ of E take the form of *Euler–Lagrange expressions* of $\lambda = L\, dt$, i.e.

$$E_\sigma = \frac{\partial L}{\partial q^\sigma} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\sigma} \tag{3.15}$$

if λ is a first order Lagrangian.

It is known that a locally variational form need not be globally variational [35]: obstructions come from the topology of the manifold Y. Every (globally) variational form on J^2Y possesses a global second-order Lagrangian. This Lagrangian is locally equivalent with first-order Lagrangians (we say that it can be *locally reduced to first-order Lagrangians*).

A dynamical form $E \in \Omega^2_Y(J^2Y)$ is locally variational if and only if its components E_σ, $1 \leq \sigma \leq m$, satisfy the *Helmholtz conditions* [8]

$$\begin{aligned}
\frac{\partial E_\sigma}{\partial \ddot{q}^\nu} - \frac{\partial E_\nu}{\partial \ddot{q}^\sigma} &= 0, \\
\frac{\partial E_\sigma}{\partial \dot{q}^\nu} + \frac{\partial E_\nu}{\partial \dot{q}^\sigma} - 2\frac{d}{dt}\frac{\partial E_\nu}{\partial \ddot{q}^\sigma} &= 0, \\
\frac{\partial E_\sigma}{\partial q^\nu} - \frac{\partial E_\nu}{\partial q^\sigma} + \frac{d}{dt}\frac{\partial E_\nu}{\partial \dot{q}^\sigma} - \frac{d^2}{dt^2}\frac{\partial E_\nu}{\partial \ddot{q}^\sigma} &= 0.
\end{aligned} \tag{3.16}$$

Local (second-order) Lagrangians then can be constructed using the following formula [36]

$$L = q^\sigma \int_0^1 E_\sigma(t, uq^\nu, u\dot{q}^\nu, u\ddot{q}^\nu)du. \tag{3.17}$$

Notice that from the Helmholtz conditions one easily gets that *every second-order locally variational form is J^1Y-pertinent*, i.e., defines a *first-order mechanical system*; it is called a first-order *Lagrangian system*.

Recall that every representative α of the Lepage class of E is called a *Hamiltonian system* associated with E.

As shown in [10], every first-order Lagrangian λ has a unique first-order Lepage equivalent, the *Cartan form*, denoted by Θ_λ. Consequently, local first-order Lepage classes of E are represented by 2-forms

$$\alpha = d\Theta_\lambda + F, \tag{3.18}$$

where F is an arbitrary 2-contact form on the domain of definition of $d\Theta_\lambda$. Hence, if λ is (any) Lagrangian for E on $U \subset J^1Y$, the Lepage class of E on U is given by $[\alpha]|_U = [d\Theta_\lambda]$. To simplify notations, we write with an obvious inaccuracy,

$$[\alpha] = [d\Theta_\lambda] = d\Theta_\lambda \bmod \Omega_Y^{0,2}(J^1Y). \tag{3.19}$$

Moreover, we have the following stronger result:

Theorem 3.1. (Krupková [14, 15]). *Every first-order Lepage class of E has a unique closed representative, defined on J^1Y.*

The unique closed 2-form mentioned above is denoted by α_E and called the *Lepage equivalent of E* [14]. If λ is a Lagrangian for E (possibly local, of order $r \geq 1$) then (up to a projection) $\alpha_E|_U = d\Theta_\lambda$; here U denotes the domain of definition of $d\Theta_\lambda$.

In fibered coordinates, where $E = E_\sigma \omega^\sigma \wedge dt$ and $\lambda = L\,dt$, we have

$$\alpha_E = E_\sigma \omega^\sigma \wedge dt + \frac{1}{2}\frac{\partial E_\sigma}{\partial \dot{q}^\nu}\omega^\sigma \wedge \omega^\nu + \frac{\partial E_\sigma}{\partial \ddot{q}^\nu}\omega^\sigma \wedge \dot{\omega}^\nu, \tag{3.20}$$

$$\Theta_\lambda = L\,dt + \frac{\partial L}{\partial \dot{q}^\sigma}\omega^\sigma, \tag{3.21}$$

and, on the domain of definition of λ,

$$E = E_\lambda = p_1 d\Theta_\lambda = E_\sigma dq^\sigma \wedge dt \quad \text{where} \quad E_\sigma = \frac{\partial L}{\partial q^\sigma} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\sigma}. \tag{3.22}$$

Since the functions E_σ are affine in the $\ddot{q}$'s, we write

$$E_\sigma = A_\sigma + B_{\sigma\nu}\ddot{q}^\nu, \tag{3.23}$$

where A_σ and $B_{\sigma\nu}$ are functions of $(t, q^\sigma, \dot{q}^\sigma)$,

$$B_{\sigma\nu} = -\frac{\partial^2 L}{\partial \dot{q}^\sigma \partial \dot{q}^\nu}, \quad A_\sigma = \frac{\partial L}{\partial q^\sigma} - \frac{\partial^2 L}{\partial t\, \partial \dot{q}^\sigma} - \frac{\partial^2 L}{\partial q^\nu \partial \dot{q}^\sigma}\dot{q}^\nu. \tag{3.24}$$

Remark 3.1. In what follows we shall always assume that E is defined on J^2Y and is everywhere nontrivially of order 2. This means that $(B_{\sigma\nu})$ in (3.23) is everywhere a non-zero matrix, or, equivalently, for every Lagrangian λ the Cartan 2-form $d\Theta_\lambda$ is everywhere nontrivially of order one.

Definition 3.5. Paths of a locally variational form are called *extremals*. Equations for paths of a locally variational form (respectively, equations for holonomic integral sections of associated dynamical distributions) are called *Euler–Lagrange equations*. Equations for integral sections of the dynamical distributions are called *Hamilton equations*, their integral sections are then called *Hamilton extremals*. The dynamical distribution Δ_{α_E} is called the *Euler–Lagrange distribution*.

Note that locally for every Lagrangian λ of E, $\Delta_{\alpha_E} = \Delta_{d\Theta_\lambda}$.

A principal question in the theory of Lagrangian systems on fibered manifolds is the relationship between Hamilton equations on one side and Euler–Lagrange equations on the other side [13]. It is clear that every extremal prolonged to J^1Y is a Hamilton extremal. The converse, however need not hold: a Hamilton extremal need not be a solution of the Euler–Lagrange equations.

The problem of equivalence between the set of extremals and Hamilton extremals is solved by the following theorem.

Theorem 3.2. (Krupková [15]). *If E is regular then the dynamical distributions Δ_α, where α belongs to the first-order Lepage class of E, coincide on the common domain of definition (and their rank equals to 1). Consequently, if E is regular then for every $\alpha \in [\alpha]_E$ the Hamilton equations are equivalent with the Euler–Lagrange equations.*

Notice that if E is regular then every dynamical distribution is locally spanned by the following semispray:

$$\zeta = \frac{\partial}{\partial t} + \dot{q}^\sigma \frac{\partial}{\partial q^\sigma} - B^{\sigma\rho} A_\rho \frac{\partial}{\partial \dot{q}^\sigma}, \qquad (3.25)$$

where $(B^{\sigma\rho})$ is the inverse matrix to $(B_{\rho\nu})$.

Definition 3.6. A Lagrangian λ is called *regular* if its Euler–Lagrange form E_λ is regular [14].

Proposition 3.3 and Theorem 3.1 easily imply that we have the following equivalent characterizations of a regular Lagrangian:

(1) rank $\Delta_{d\Theta_\lambda} = \operatorname{corank} d\Theta_\lambda = 1$.

(2) If λ is a first-order Lagrangian,

$$\det\left(\frac{\partial^2 L}{\partial\dot{q}^\sigma\partial\dot{q}^\nu}\right) \neq 0. \tag{3.26}$$

In view of Theorem 3.1 and a theorem on a canonical form of the Lepage equivalent of E ([14]) one obtains the following result:

Proposition 3.4. *Let E be regular. Then in a neighborhood of every point in J^1Y there is a local coordinate transformation $(t, q^\sigma, \dot{q}^\sigma) \to (t, q^\sigma, p_\sigma)$ such that every α belonging to the first-order Lepage class of E takes the canonical form*

$$\alpha = -dH \wedge dt + dp_\sigma \wedge dq^\sigma + F, \tag{3.27}$$

where F is a 2-contact $\pi_{1,0}$-horizontal 2-form.

This transformation is called *Legendre transformation.*

Functions H and p_σ above can be expressed in terms of a *first-order* Lagrangian λ for E; it holds

$$p_\sigma = \frac{\partial L}{\partial\dot{q}^\sigma}, \qquad H = -L + p_\sigma\dot{q}^\sigma. \tag{3.28}$$

In Legendre coordinates Hamilton equations take the "canonical form"

$$\frac{d(p_\sigma \circ \delta)}{dt} = -\frac{\partial H}{\partial q^\sigma}, \qquad \frac{d(q^\sigma \circ \delta)}{dt} = \frac{\partial H}{\partial p_\sigma}. \tag{3.29}$$

Summarizing, for a *regular* Lagrangian system (represented by a regular locally variational form E on J^2Y), all related Hamiltonian systems are (locally) equivalent and Hamilton equations are equivalent with Euler–Lagrange equations. Hamilton extremals coincide with prolongations of extremals, and are solutions of the canonical equations (3.29).

3.3. **Nonholonomic constraints.** Let us introduce the non-holonomic constraint structure in J^1Y, as defined in [15].

Definition 3.7. By a *constraint submanifold* or a *non-holonomic constraint* in J^1Y we shall understand a submanifold $Q \subset J^1Y$, fibered over Y, precisely speaking, a surjective submersion $\pi_{1,0}|_Q : Q \to Y$.

We denote by k the codimension of Q and assume that $1 \leq k \leq m-1$.

A nonholonomic constraint Q in J^1Y of codimension k can be locally expressed by equations

$$f^a(t, q^\sigma, \dot{q}^\sigma) = 0, \quad 1 \leq a \leq k, \tag{3.30}$$

where

$$\mathrm{rank}\left(\frac{\partial f^a}{\partial \dot{q}^\sigma}\right) = k, \tag{3.31}$$

or, equivalently, by equations in a *normal form*,

$$\dot{q}^{m-k+a} - g^a(t, q^\sigma, \dot{q}^1, \dots, \dot{q}^{m-k}) = 0, \quad 1 \le a \le k. \tag{3.32}$$

A section γ of π defined on an open set $W \subset X$ is called a *holonomic path in Q* if for every $x \in W$

$$J^1\gamma(x) \in Q. \tag{3.33}$$

Given a constraint submanifold Q in J^1Y there naturally arise the following local distributions, defined on the domain U of definition of the functions f^a:

(1) $\mathcal{D}_U = \mathrm{annih}\,\{df^a, 1 \le a \le k\}$; $\mathrm{rank}\,\mathcal{D}_U$ is constant on U due to (3.31) and equal to $2m + 1 - k$.

(2) $\tilde{\mathcal{C}}_U = \mathrm{annih}\,\{\phi^a, 1 \le a \le k\}$, where

$$\phi^a = f^a dt + \frac{\partial f^a}{\partial \dot{q}^\sigma}\,\omega^\sigma, \quad 1 \le a \le k. \tag{3.34}$$

(3) $\mathcal{C}_U = \mathrm{annih}\,\{\phi^a, df^a, 1 \le a \le k\}$.

$\tilde{\mathcal{C}}_U$ is called *extended constraint distribution*; it has a constant rank equal to $2m + 1 - k$. $\mathcal{C}_U$ is called *constraint distribution* related to the constraint submanifold Q on U, and its rank equals to $2m + 1 - 2k$.

The following assertions hold ([15]):

Proposition 3.5. *$Q \cap U$ is an integral submanifold of $\mathcal{D}_U$. For every point $x \in Q$, the forms $df^a(x)$, $1 \le a \le k$, annihilate the tangent space T_xQ to the manifold Q at x, i.e. along Q, $\mathcal{D} = \mathrm{annih}\,\{df^a, 1 \le a \le k\} = TQ$.*

Corollary 3.1. *Let Q be a non-holonomic constraint of codimension k in J^1Y, and let $f^a = 0$ and $\bar{f}^a = 0$, where $1 \le a \le k$, be two sets of equations of Q on an open set $U \subset J^1Y$. Then there are functions γ_j^a on U such that at each point of U, (γ_j^a) is a regular matrix, and $d\bar{f}^a = \gamma_j^a df^j$. In particular, at each point $x \in Q \cap U$,*

$$\frac{\partial \bar{f}^a}{\partial \dot{q}^\sigma} = \gamma_j^a \frac{\partial f^j}{\partial \dot{q}^\sigma}. \tag{3.35}$$

Proposition 3.6. *$\mathcal{C}_U$ is a subdistribution of both $\tilde{\mathcal{C}}_U$ and $\mathcal{D}_U$. At the points of $Q \cap U$, the distributions $\mathcal{C}_U$ and $\tilde{\mathcal{C}}_U \cap \mathcal{D}$ coincide, and define a distribution of corank k on $Q \cap U$.*

Theorem 3.3. *Let Q be a non-holonomic constraint in J^1Y, let $\iota : Q \to J^1Y$ be the canonical embedding of the submanifold Q into J^1Y. Put*

$$\bar{\varphi}^a = \iota^*\phi^a, \quad 1 \le a \le k. \tag{3.36}$$

Then

$$\mathcal{C} = \operatorname{annih}\{\bar{\varphi}^a, 1 \le a \le k\} \tag{3.37}$$

is a distribution of corank k on Q.

Note that in fibered coordinates

$$\bar{\varphi}^a = \left(\frac{\partial f^a}{\partial \dot{q}^\sigma} \circ \iota\right)\bar{\omega}^\sigma = \bar{\omega}^{m-k+a} - \frac{\partial g^a}{\partial \dot{q}^l}\bar{\omega}^l \tag{3.38}$$

where we have denoted

$$\bar{\omega}^\sigma = \iota^*\omega^\sigma = dq^\sigma - (\dot{q}^\sigma \circ \iota)dt, \tag{3.39}$$

i.e.,

$$\begin{aligned}
\bar{\omega}^l &= dq^l - \dot{q}^l dt, \quad 1 \le l \le m - k, \\
\bar{\omega}^{m-k+a} &= dq^{m-k+a} - g^a dt, \quad 1 \le a \le k.
\end{aligned} \tag{3.40}$$

Definition 3.8. [15] The distribution $\mathcal{C}$ (3.37) on Q is called *canonical distribution*. 1-forms belonging to the annihilator $\mathcal{C}^0$ of $\mathcal{C}$, are called *canonical constraint 1-forms*. The ideal in the exterior algebra of differential forms on Q generated by $\mathcal{C}^0$ is called *canonical constraint ideal*, and denoted by $\mathcal{I}(\mathcal{C}^0)$; its homogeneous component of degree p is denoted by $\mathcal{I}^p(\mathcal{C}^0)$. Elements of the ideal $\mathcal{I}(\mathcal{C}^0)$ are called *canonical constraint forms*.

One can show by a direct computation that the canonical distribution can be equivalently locally spanned by the following system of vector fields:

$$\begin{aligned}
\frac{\partial_c}{\partial t} &\equiv \frac{\partial}{\partial t} + \sum_{a=1}^{k}\left(g^a - \frac{\partial g^a}{\partial \dot{q}^l}\dot{q}^l\right)\frac{\partial}{\partial q^{m-k+a}}, \\
\frac{\partial_c}{\partial q^s} &\equiv \frac{\partial}{\partial q^s} + \sum_{a=1}^{k}\frac{\partial g^a}{\partial \dot{q}^s}\frac{\partial}{\partial q^{m-k+a}}, \quad 1 \le s \le m - k, \\
\frac{\partial}{\partial \dot{q}^s}, &\quad 1 \le s \le m - k.
\end{aligned} \tag{3.41}$$

For the sake of simplicity we shall also use the following notations:

$$\frac{d_c}{dt} = \frac{\partial}{\partial t} + \dot{q}^l \frac{\partial}{\partial q^l} + g^a \frac{\partial}{\partial q^{m-k+a}} + \ddot{q}^l \frac{\partial}{\partial \dot{q}^l} = \frac{\partial_c}{\partial t} + \dot{q}^l \frac{\partial_c}{\partial q^l} + \ddot{q}^l \frac{\partial}{\partial \dot{q}^l},$$

$$\frac{d'_c}{dt} = \frac{\partial}{\partial t} + \dot{q}^l \frac{\partial}{\partial q^l} + g^a \frac{\partial}{\partial q^{m-k+a}} = \frac{\partial_c}{\partial t} + \dot{q}^l \frac{\partial_c}{\partial q^l} \tag{3.42}$$

$$= \frac{d_c}{dt} - \ddot{q}^l \frac{\partial}{\partial \dot{q}^l} = \frac{d'}{dt} \circ \iota.$$

In general, the canonical distribution is not completely integrable. There are two interesting particular cases of non-holonomic constraints as follows:

Definition 3.9. [15] A non-holonomic constraint Q is called

(1) *simple* if the canonical distribution $\mathcal{C}$ is projectable onto a distribution on Y,

(2) *semiholonomic* if the canonical distribution $\mathcal{C}$ is completely integrable.

It can be proved [15] that *every semiholonomic constraint is simple.* Consequently it can be *equivalently* modeled as either

• a fibered submanifold $Q \subset J^1Y$ with the canonical distribution $\mathcal{C}$ completely integrable (i.e., the canonical constraint ideal $\mathcal{I}(\mathcal{C}^0)$ closed), or

• a completely integrable, nowhere vertical distribution on the total space Y.

Another result shows [15] that *a non-holonomic constraint is simple if and only if it is locally defined by equations affine in velocities.* Consequently, a simple non-holonomic constraint can be *equivalently* modeled as either

• a fibered submanifold $Q \subset J^1Y$ with the canonical distribution $\mathcal{C}$ projectable onto a distribution on Y, or

• a nowhere vertical distribution on Y, which need not be completely integrable.

3.4. **Constrained Lagrangian systems.** Let us consider a Lagrangian system on J^1Y. Recall that it is defined by a locally variational form E on J^2Y, as the first-order Lepage class $[\alpha]$ of E (see (3.19))

$$[\alpha] = [d\Theta_\lambda] = d\Theta_\lambda \bmod \Omega_Y^{0,2}(J^1Y). \tag{3.43}$$

If $\iota : Q \to J^1Y$ is a non-holonomic constraint and $\mathcal{I}(\mathcal{C}^0)$ the corresponding canonical constraint ideal, we have another equivalence, denoted by $\approx$, on 2-forms on Q (with the same domain of definition):

$$\eta_1 \approx \eta_2 \quad \text{iff} \quad \eta_1 - \eta_2 = \bar{F} + \varphi, \tag{3.44}$$

where $\bar{F}$ is a (local) 2-contact 2-form on Q, and φ is a constraint 2-form. We denote by $[[\eta]]$ the class of η. If $[\alpha]$ is a Lepage class on J^1Y associated with a locally variational form $E \in \Omega^2(J^2Y)$ then for any of its two elements defined on the same subset of J^1Y,

$$\alpha_1 \sim \alpha_2 \Rightarrow \iota^*\alpha_1 \approx \iota^*\alpha_2. \tag{3.45}$$

Definition 3.10. [15, 20] Let $[\alpha] = [d\Theta_\lambda]$ be a Lagrangian system on J^1Y. By the associated *constrained Lagrangian system* we mean the class $[[\iota^*\alpha]] = [[\iota^*d\Theta_\lambda]]$. Each form $\iota^*d\Theta_\lambda + \varphi$, where $\varphi \in \mathcal{I}^2(\mathcal{C}^0)$, is called *constrained Cartan* 2-*form* of λ.

Note that every element of $[[\iota^*d\Theta_\lambda]]$ is of the form

$$\bar{\alpha} = \iota^*d\Theta_\lambda + \bar{F} + \varphi, \tag{3.46}$$

where $\bar{F} \in \Omega_Y^{0,2}(Q)$ and $\varphi \in \mathcal{I}^2(\mathcal{C}^0)$.

In fibered coordinates, where Q is given by (3.32) and the Euler–Lagrange form of λ is represented by (3.23), (3.24), we have [15, 20]

$$\bar{\alpha} = \bar{A}_l\bar{\omega}^l \wedge dt + \bar{B}_{ls}\bar{\omega}^l \wedge d\dot{q}^s + \bar{F}_{ls}\,\bar{\omega}^l \wedge \bar{\omega}^s + \varphi, \tag{3.47}$$

where $\bar{\omega}^l = \iota^*\omega^l$, $\varphi \in \mathcal{I}^2(\mathcal{C}^0)$, $\bar{F}_{ls}$ are arbitrary, and

$$
\begin{aligned}
\bar{A}_l &= \left(A_l + A_{m-k+i}\frac{\partial g^i}{\partial \dot{q}^l} + \left(B_{l,m-k+i} + B_{m-k+j,m-k+i}\frac{\partial g^j}{\partial \dot{q}^l} \right)\frac{d'g^i}{dt} \right) \circ \iota, \\
\bar{B}_{ls} &= \left(B_{ls} + B_{l,m-k+i}\frac{\partial g^i}{\partial \dot{q}^s} + B_{s,m-k+i}\frac{\partial g^i}{\partial \dot{q}^l} + B_{m-k+i,m-k+j}\frac{\partial g^i}{\partial \dot{q}^l}\frac{\partial g^j}{\partial \dot{q}^s} \right) \circ \iota,
\end{aligned}
\tag{3.48}
$$

and summations run over $l, s = 1, 2, \ldots, m - k$ and $i, j = 1, 2, \ldots, k$. Since $(B_{\sigma\nu})$ is a symmetric matrix, the above formula gives us that the matrix $(\bar{B}_{ls})$ is *symmetric*.

Definition 3.11. [15] The *constraint dynamical distribution* related with a 2-form $\bar{\alpha}$, denoted by $\Delta_{\bar{\alpha}}$, is defined to be the subdistribution of the canonical distribution $\mathcal{C}$, annihilated by the 1-forms $i_\xi\bar{\alpha}$, where ξ runs over all π_1-vertical vector fields on Q *belonging to* $\mathcal{C}$. This means that

$$\Delta_{\bar{\alpha}} = \text{annih}\,\{\bar{\varphi}^a,\ 1 \leq a \leq k,\ i_\xi\bar{\alpha},\ \forall \xi \in \mathcal{V}(J^1Y) \cap \mathcal{C}\}. \tag{3.49}$$

In particular, the constraint dynamical distribution related with a constrained Cartan 2-form is called *constraint Euler–Lagrange distribution*.

Definition 3.12. [15, 20] Let $[[\iota^*d\Theta_\lambda]]$ be a constrained Lagrangian system. Then for any representative $\bar{\alpha}$ of the class $[[\iota^*d\Theta_\lambda]]$, equations for *holonomic* integral sections of the constraint dynamical distribution $\Delta_{\bar{\alpha}}$, i.e., the equations

$$J^1\gamma^*i_\xi\bar{\alpha} = 0 \quad \text{for every } \pi_1\text{-vertical vector field } \xi \in \mathcal{C}, \tag{3.50}$$

where $\gamma : W \to Y$, $J^1\gamma(W) \subset Q$, are called *constrained Euler–Lagrange equations*. Solutions of constrained Euler–Lagrange equations are called *constrained extremals*.

We note that (locally) constrained Euler–Lagrange equations do not depend upon the choice of a representative $\bar\alpha$ of the class $[[\iota^*d\Theta_\lambda]]$. This means that with help of a (local, possibly higher-order) Lagrangian λ for E we can write the constrained Euler–Lagrange equations in the form

$$J^1\gamma^*i_\xi(\iota^*d\Theta_\lambda) = 0 \quad \text{for every } \pi_1\text{-vertical vector field } \xi \in \mathcal{C}, \quad (3.51)$$

where $\gamma : W \to Y$, $J^1\gamma(W) \subset Q$.

For $\lambda = L\,dt$ denote

$$\bar{L} = L \circ \iota, \quad \bar{L}_a = \frac{\partial L}{\partial \dot{q}^{m-k+a}} \circ \iota, \quad 1 \leq a \leq k, \quad (3.52)$$

and

$$\Theta_{\iota^*\lambda} = \bar{L}\,dt + \sum_{s=1}^{m-k} \frac{\partial \bar{L}}{\partial \dot{q}^s}\,\bar\omega^s. \quad (3.53)$$

We get the following relation between the forms $\iota^*\Theta_\lambda$ and $\Theta_{\iota^*\lambda}$:

Proposition 3.7. [20]

$$\iota^*\Theta_\lambda = \Theta_{\iota^*\lambda} + \bar{L}_a\,\bar\varphi^a. \quad (3.54)$$

For convenience we shall use the notation

$$\mu_s = \frac{\partial_c}{\partial q^s} - \frac{d_c}{dt}\frac{\partial}{\partial \dot{q}^s}, \quad \mu'_s = \frac{\partial_c}{\partial q^s} - \frac{d'_c}{dt}\frac{\partial}{\partial \dot{q}^s} \quad (3.55)$$

for the so called $\mathcal{C}$-*modified Euler–Lagrange operator* and *cut $\mathcal{C}$-modified Euler–Lagrange operator*, respectively.

In fibered coordinates constrained Euler–Lagrange equations take the form a mixed system of $m - k$ second-order and k first-order ODE's for sections γ of π as follows:

Theorem 3.4. [15, 20] *Let $[d\Theta_\lambda]$ be a Lagrangian system on J^1Y, $Q \subset J^1Y$ a non-holonomic constraint. A section $\gamma : W \to Y$ of π is a constrained extremal if and only if it satisfies $J^1\gamma(W) \subset Q$, i.e.*

$$f^a \circ J^1\gamma = 0, \quad 1 \leq a \leq k, \quad (3.56)$$

and the constrained Euler–Lagrange equations (3.51) . The latter take one of the following equivalent coordinate forms:

(1) *By means of L,*

$$(\bar{A}_l + \bar{B}_{ls}\ddot{q}^s) \circ J^2\gamma = 0, \quad 1 \leq l \leq m - k, \quad (3.57)$$

where $\bar{A}_l$, $\bar{B}_{ls}$ are given by (3.24), (3.48).

(2) *By means of $\bar{L}$ and $\bar{L}_a$,*

$$\left(\mu_l(\bar{L}) - \bar{L}_a\,\mu_l(g^a)\right) \circ J^2\gamma = 0, \quad 1 \le l \le m - k. \qquad (3.58)$$

Consequently, the functions $\bar{A}_l$, $\bar{B}_{ls}$ are equivalently expressed as follows:

$$\bar{A}_l = \mu'_l(\bar{L}) - \bar{L}_a\,\mu'_l(g^a), \quad \bar{B}_{ls} = -\frac{\partial^2 \bar{L}}{\partial \dot{q}^l \partial \dot{q}^s} + \bar{L}_a\frac{\partial^2 g^a}{\partial \dot{q}^l \partial \dot{q}^s}. \qquad (3.59)$$

Definition 3.13. The operator

$$\mathcal{E}^{\mathcal{C}}_s(\bar{L}, \bar{L}_a) = \mu_s(\bar{L}) - \bar{L}_a\,\mu_s(g^a), \quad 1 \le s \le m - k \qquad (3.60)$$

is called the *constraint Euler–Lagrange operator*.

The definition of a regular constrained system is quite similar to the unconstrained case.

Definition 3.14. [15] A constrained Lagrangian system $[[\iota^*d\Theta_\lambda]]$ is called *regular* if around each point of Q there exists a constraint dynamical distribution $\Delta_{\bar{\alpha}}$ such that $\text{rank}\,\Delta_{\bar{\alpha}} = 1$.

Regular constrained systems are characterized as follows (see [15, 38]):

Theorem 3.5. *Let $[[\iota^*d\Theta_\lambda]]$ be a constrained Lagrangian system. The following conditions are equivalent:*

(1) *$[[\iota^*d\Theta_\lambda]]$ is regular.*

(2) *The $(m - k) \times (m - k)$-matrix $(\bar{B}_{sl})$ is regular, i.e.,*

$$\det(\bar{B}_{sl}) \ne 0. \qquad (3.61)$$

(3) *Every first-order Lagrangian $\lambda = L\,dt$ satisfies the regularity condition*

$$\det\left(\left(\frac{\partial^2 L}{\partial \dot{q}^l \partial \dot{q}^s} + \frac{\partial^2 L}{\partial \dot{q}^{m-k+a} \partial \dot{q}^s}\frac{\partial g^a}{\partial \dot{q}^l} + \frac{\partial^2 L}{\partial \dot{q}^{m-k+a} \partial \dot{q}^l}\frac{\partial g^a}{\partial \dot{q}^s}\right.\right.$$
$$\left.\left. + \frac{\partial^2 L}{\partial \dot{q}^{m-k+a} \partial \dot{q}^{m-k+b}}\frac{\partial g^a}{\partial \dot{q}^l}\frac{\partial g^b}{\partial \dot{q}^s}\right) \circ \iota\right) \ne 0. \qquad (3.62)$$

(4) *Every first-order Lagrangian $\lambda = L\,dt$ satisfies the regularity condition*

$$\det\left(\frac{\partial^2 \bar{L}}{\partial \dot{q}^l \partial \dot{q}^s} - \left(\frac{\partial L}{\partial \dot{q}^{m-k+a}} \circ \iota\right)\frac{\partial^2 g^a}{\partial \dot{q}^l \partial \dot{q}^s}\right) \ne 0, \qquad (3.63)$$

where $\bar{L} = L \circ \iota$.

(5) *Every constraint dynamical distribution is locally spanned by the following constraint semispray:*

$$\zeta = \frac{\partial}{\partial t} + \sum_{l=1}^{m-k} \dot{q}^l \frac{\partial}{\partial q^l} + \sum_{a=1}^{k} g^a \frac{\partial}{\partial q^{m-k+a}} - \sum_{l,s=1}^{m-k} \bar{B}^{ls}\bar{A}_s\frac{\partial}{\partial \dot{q}^l}, \qquad (3.64)$$

where $(\bar{B}^{ls})$ is the inverse matrix to $(\bar{B}_{ls})$.

(6) The constrained Euler–Lagrange equations have an equivalent form

$$\begin{aligned}
\dot{q}^{m-k+a} &= g^a(t, q^\sigma, \dot{q}^1, \ldots, \dot{q}^{m-k}), & 1 \leq a \leq k, \\
\ddot{q}^l &= -\bar{B}^{ls}\bar{A}_s, & 1 \leq l \leq m-k.
\end{aligned}$$
(3.65)

We stress that, as one can see from any of the above equivalent regularity conditions, a constrained system arising from a regular Lagrangian system need not be regular.

Corollary 3.2. *If $Q \subset J^1Y$ is a simple non-holonomic constraint then the regularity condition reads*

$$\det\left(\frac{\partial^2 \bar{L}}{\partial \dot{q}^l \partial \dot{q}^s}\right) \neq 0,$$
(3.66)

where $\bar{L} = L \circ \iota$.

3.5. **Constrained Hamilton equations.** Constrained Hamiltonian systems were studied in detail in [3, 9, 38].

Let $[\alpha] = [d\Theta_\lambda]$ be a Lagrangian system on J^1Y, $\iota : Q \to J^1Y$ a non-holonomic constraint, $[[\iota^*\alpha]] = [[\iota^*d\Theta_\lambda]]$ the corresponding constrained system. For every $\bar{\alpha} = \iota^*\alpha \in [[\iota^*d\Theta_\lambda]]$ we have the constraint dynamical distribution $\Delta_{\bar{\alpha}}$ defined on the domain of definition of $\bar{\alpha}$, say $U \subset Q$.

Directly from the definition of constraint dynamical distribution we can see that if $\bar{\alpha}_1$ and $\bar{\alpha}_2$ differ by a constraint form, their constraint dynamical distributions $\Delta_{\bar{\alpha}_1}$ and $\Delta_{\bar{\alpha}_2}$ *coincide*.

Definition 3.15. [38] Let $[d\Theta_\lambda]$ be a Lagrangian system on J^1Y. For every $\alpha \in [d\Theta_\lambda]$ the equivalence class

$$\bar{\alpha}_\mathcal{I} = \iota^*\alpha \mod \mathcal{I}^2(\mathcal{C}^0)$$
(3.67)

is called *constrained Hamiltonian system* related with α and the constraint Q.

Equations for integral sections of the corresponding constraint dynamical distribution $\Delta_{\bar{\alpha}}$, i.e.

$$\delta^*\bar{\varphi}^a = 0, \quad 1 \leq a \leq k, \quad \delta^* i_\xi \bar{\alpha} = 0 \quad \forall \xi \in \mathcal{V}(Q) \cap \mathcal{C},$$
(3.68)

where $\bar{\alpha} \in \bar{\alpha}_\mathcal{I}$ and δ is a section of $\pi_1|_Q : Q \to X$, are called *constrained Hamilton equations*.

Note that for every $\bar{\alpha} \in [[\iota^*d\Theta_\lambda]]$ on U, *holonomic* integral sections of $\Delta_{\bar{\alpha}}$ coincide with prolongations of *constrained extremals* in U.

Theorem 3.6. [38] *Let $[[\iota^* d\Theta_\lambda]]$ be a regular constrained Lagrangian system. Then for any two its Hamiltonian systems $\bar{\alpha}_{1\mathcal{I}}$, $\bar{\alpha}_{2\mathcal{I}}$ on an open subset U their constrained dynamical distributions coincide, i.e. $\Delta_{\bar{\alpha}_1} = \Delta_{\bar{\alpha}_2}$.*

Corollary 3.3. *If the regularity condition* (3.61) *is satisfied then constrained Euler–Lagrange equations are equivalent with (any) constrained Hamilton equations.*

For regular constrained systems we can introduce a *constraint Legendre transformation.*

Theorem 3.7. [38] *Let $\iota : Q \to J^1 Y$ be a non-holonomic constraint, $[[\iota^* d\Theta_\lambda]]$ a constrained Lagrangian system. Let $x \in Q$ be a point. Suppose that in a neighborhood of x,*

$$\frac{\partial \bar{B}_{ls}}{\partial \dot{q}^r} = \frac{\partial \bar{B}_{lr}}{\partial \dot{q}^s}, \quad 1 \le l, r, s \le m - k. \tag{3.69}$$

Then there exists a neighborhood $U \subset Q$ of x, and, on U, functions P_l, $1 \le l \le m - k$, and a 1-form η, such that the class $[[\iota^* d\Theta_\lambda]]$ has a representative of the form

$$\bar{\alpha}' = \eta \wedge dt + dP_l \wedge dq^l. \tag{3.70}$$

If, moreover, the constrained system $[[\iota^* d\Theta_\lambda]]$ is regular, then $(t, q^\sigma, \dot{q}^l) \to (t, q^\sigma, P_l)$ is a coordinate transformation on U.

Proof. In a neighborhood of x, let us consider the elements of the equivalence class $[[\iota^* d\Theta_\lambda]]$ in the form (3.47). By assumption, from the Poincaré Lemma we get a neighborhood $U \subset Q$ of x and functions P_l, $1 \le l \le m - k$, on U such that

$$\bar{B}_{ls} = -\frac{\partial P_l}{\partial \dot{q}^s}. \tag{3.71}$$

Hence, in the class $[[\iota^* d\Theta_\lambda]]$ there is a local representative of the form

$$\bar{\alpha} = \bar{A}_l\, \bar{\omega}^l \wedge dt + \frac{\partial P_l}{\partial \dot{q}^s}\, d\dot{q}^s \wedge \bar{\omega}^l$$

$$= \bar{A}_l\bar{\omega}^l \wedge dt + dP_l \wedge \bar{\omega}^l - \frac{\partial P_l}{\partial t}dt \wedge \bar{\omega}^l - \frac{\partial P_l}{\partial q^s}dq^s \wedge \bar{\omega}^l$$

$$- \frac{\partial P_l}{\partial q^{m-k+a}}dq^{m-k+a} \wedge \bar{\omega}^l$$

$$= \left(\bar{A}_l + \frac{\partial P_l}{\partial t} \right) dq^l \wedge dt + dP_l \wedge dq^l - \dot{q}^l\, dP_l \wedge dt \qquad (3.72)$$

$$- \frac{\partial P_l}{\partial q^s}(\bar{\omega}^s + \dot{q}^s dt) \wedge \bar{\omega}^l - \frac{\partial P_l}{\partial q^{m-k+a}}(\bar{\omega}^{m-k+a} + g^a dt) \wedge \bar{\omega}^l$$

$$= \left(\bar{A}_l + \frac{\partial P_l}{\partial t} + \frac{\partial P_l}{\partial q^s}\dot{q}^s + \frac{\partial P_l}{\partial q^{m-k+a}}g^a \right) dq^l \wedge dt - \dot{q}^l dP_l \wedge dt$$

$$+ dP_l \wedge dq^l + \frac{\partial P_l}{\partial q^s}\bar{\omega}^l \wedge \bar{\omega}^s - \frac{\partial P_l}{\partial q^{m-k+a}}\bar{\omega}^{m-k+a} \wedge \bar{\omega}^l.$$

This means that we also have a representative

$$\bar{\alpha}' = \left(\bar{A}_l + \frac{\partial P_l}{\partial t} + \frac{\partial P_l}{\partial q^s}\dot{q}^s + \frac{\partial P_l}{\partial q^{m-k+a}}g^a \right) dq^l \wedge dt - \dot{q}^s dP_s \wedge dt$$

$$+ dP_l \wedge dq^l$$

$$= \left(\bar{A}_l + \frac{\partial P_l}{\partial t} + \left(\frac{\partial P_l}{\partial q^s} - \frac{\partial P_s}{\partial q^l} \right)\dot{q}^s + \frac{\partial P_l}{\partial q^{m-k+a}}g^a \right) dq^l \wedge dt \qquad (3.73)$$

$$- \frac{\partial P_s}{\partial q^{m-k+a}}\dot{q}^s dq^{m-k+a} \wedge dt - \frac{\partial P_s}{\partial \dot{q}^l}\dot{q}^s d\dot{q}^l \wedge dt + dP_l \wedge dq^l.$$

We can write it in the form $\bar{\alpha}' = \eta \wedge dt + dP_l \wedge dq^l$ with

$$\eta = \bar{\eta}_0 dt + \bar{\eta}_l dq^l + \bar{\eta}_{m-k+a}dq^{m-k+a} + \tilde{\eta}_l d\dot{q}^l, \qquad (3.74)$$

where $\bar{\eta}_0$ is an arbitrary function on U, and

$$
\begin{aligned}
\bar{\eta}_l &= \bar{A}_l + \frac{\partial P_l}{\partial t} + \left(\frac{\partial P_l}{\partial q^s} - \frac{\partial P_s}{\partial q^l} \right)\dot{q}^s + \frac{\partial P_l}{\partial q^{m-k+a}}g^a, \\
\tilde{\eta}_l &= -\frac{\partial P_s}{\partial \dot{q}^l}\dot{q}^s, \qquad \bar{\eta}_{m-k+a} = -\frac{\partial P_l}{\partial q^{m-k+a}}\dot{q}^l.
\end{aligned}
\qquad (3.75)
$$

Finally, the regularity condition for the transformation $(t, q^\sigma, \dot{q}^l) \to (t, q^\sigma, P_l)$ coincides with (3.61). $\qquad\square$

Remark 3.2. Condition (3.69) rewritten in terms of a first-order Lagrangian reads

$$\left(\frac{\partial}{\partial \dot{q}^r} \left(\frac{\partial L}{\partial \dot{q}^{m-k+a}} \circ \iota \right) \right) \frac{\partial^2 g^a}{\partial \dot{q}^l \partial \dot{q}^s} = \left(\frac{\partial}{\partial \dot{q}^s} \left(\frac{\partial L}{\partial \dot{q}^{m-k+a}} \circ \iota \right) \right) \frac{\partial^2 g^a}{\partial \dot{q}^l \partial \dot{q}^r}. \quad (3.76)$$

The integrability condition for the $\bar{B}_{sl}$'s ((3.69), (3.76)) ensures that one can express functions P_l explicitly. To this purpose we consider a mapping $\chi : [0,1] \times W \to W$ defined by $(u, t, q^\sigma, \dot{q}^l) \to (t, q^\sigma, u\dot{q}^l)$, where $W \subset Q$ is an appropriate open set. Then Poincaré Lemma gives us a solution [38]

$$
\begin{aligned}
P_l &= -\dot{q}^s \int_0^1 (\bar{B}_{ls} \circ \chi)\, du \\
&= \frac{\partial \bar{L}}{\partial \dot{q}^l} - \dot{q}^s \int_0^1 \left(\left(\frac{\partial L}{\partial \dot{q}^{m-k+a}} \circ \iota \right) \frac{\partial^2 g^a}{\partial \dot{q}^l \partial \dot{q}^s} \right) \circ \chi \, du.
\end{aligned}
\tag{3.77}
$$

Definition 3.16. [38] We call the above functions P_l, $1 \le l \le k$, *constraint momenta*, and the corresponding coordinate transformation *constraint Legendre transformation*. The 1-form η in (3.70) is called a *constraint energy 1-form*.

The 1-form η is *determined up to a constraint 1-form*, and *need not be closed*. In constraint Legendre coordinates we can write

$$
\eta_{\mathcal{I}} = \eta_0 \, dt + \eta_l \, dq^l + \eta^l \, dP_l \bmod \mathcal{I}^1(\mathcal{C}^0).
\tag{3.78}
$$

Corollary 3.4. *If the Lagrangian system $[[\iota^* d\Theta_\lambda]]$ is regular, then the constraint Euler–Lagrange equations are equivalent with constraint Hamilton equations. In constraint Legendre coordinates Hamilton equations take the following canonical form*

$$
\frac{d}{dt}(P_l \circ \delta) = \eta_l, \quad \frac{d}{dt}(q^l \circ \delta) = -\eta^l, \quad \frac{d}{dt}(q^{m-k+a} \circ \delta) = g^a,
\tag{3.79}
$$

where $1 \le l \le m - k$, $1 \le a \le k$. Constraint Hamilton equations depend upon the choice of a representative $\alpha \in [d\Theta_\lambda]$, rather than on a particular Lagrangian λ.

For *simple non-holonomic constraints*, which, as we have seen in Sec. 3.3, can be modeled by a *distribution on Y*, and are given by (3.30) (resp. (3.32)) where the functions f^a (resp. g^a) are affine in the velocities) the situation essentially simplifies:

Theorem 3.8. *Assume that Q is a simple non-holonomic constraint. Then (3.69) is fulfilled identically and the constraint momenta are defined by*

$$
P_l = \frac{\partial \bar{L}}{\partial \dot{q}^l}, \quad 1 \le l \le m - k.
\tag{3.80}
$$

Regularity condition takes the form

$$
\det \left(\frac{\partial^2 \bar{L}}{\partial \dot{q}^l \partial \dot{q}^s} \right) \ne 0.
\tag{3.81}
$$

Moreover, if the constraint Q is semiholonomic then the family of energy 1-forms (3.78) contains a closed 1-form equal to $-d\bar{H}$, where

$$\bar{H} = -\bar{L} + P_l \dot{q}^l. \tag{3.82}$$

3.6. **Holonomic Lagrangian systems.** The case of Lagrangian systems subjected to holonomic constraints can be considered as a special case of the nonholonomic theory (see [15]).

Definition 3.17. Let $\pi : Y \to X$ be a fibered manifold, $\dim X = 1$, $\dim Y = m + 1$, $m > 1$. By a *holonomic constraint* in Y we mean a fibered submanifold $\pi_0 : Q_0 \to X$ of π.

If $X = R$ and $Y = R \times M$ where M is a manifold of dimension m, we also speak about a *rheonomic constraint*. If a rheonomic constraint is of the form $Q_0 = R \times N$ where N is a submanifold of M, it is called *skleronomic.*

We denote by $\iota_0 : Q_0 \to Y$ the canonical inclusion of Q_0 into Y, and assume codim $Q_0 = k$, where $1 \le k < m$. The constraint Q_0 is locally defined by a system of *algebraic* equations

$$u^a(t, q^\sigma) = 0, \quad 1 \le a \le k, \tag{3.83}$$

where the functions u^a satisfy the rank condition

$$\text{rank} \left(\frac{\partial u^a}{\partial q^\sigma} \right) = k. \tag{3.84}$$

Hence, around every point $x \in Q_0$ there is a fibered chart (U, χ) on Y, adapted to the submanifold Q_0, i.e. $\chi = (t, q^1, \ldots, q^{m-k}, u^1, \ldots, u^k)$.

The fibered submanifold Q_0 can be prolonged to $J^1 Q_0 \subset J^1 Y$, codim $J^1 Q_0 = 2k$, locally defined by the equations

$$u^a = 0, \quad \frac{du^a}{dt} = 0, \quad 1 \le a \le k. \tag{3.85}$$

$J^1 Q_0$ is a submanifold of the manifold $Q \subset J^1 Y$, defined by the equations

$$f^a \equiv \frac{du^a}{dt} = 0, \quad 1 \le a \le k. \tag{3.86}$$

We can see that Q *is a semiholonomic constraint in* $J^1 Y$, codim $Q = k$.

The only admissible holonomic paths in Q are sections γ of the fibered manifold $Q_0 \to X$, i.e. such that $J^1 \gamma \in J^1 Q_0$. This means that for a Lagrangian system $[d\Theta_\lambda]$ on $J^1 Y$, the corresponding constrained system $[[\iota^* d\Theta_\lambda]]$ can be restricted to $J^1 Q_0$.

Proposition 3.8. *Let $Q_0 \subset Y$ be a holonomic constraint, Q the semi-holonomic constraint related with Q_0. Let $\mathcal{C}$ be the canonical distribution on Q. Then for every $x \in J^1Q_0$*

$$\mathcal{C}(x) = T_x J^1 Q_0. \tag{3.87}$$

Equivalently, the annihilator $\mathcal{C}^0$ of the canonical distribution on J^1Q_0 is trivial, $\mathcal{C}^0(x) = \{0\}$.

Corollary 3.5. *Let Q_0 be a holonomic constraint in Y, $Q \subset J^1Y$ the associated semiholonomic constraint. Then the canonical distribution $\mathcal{C}$ on Q is completely integrable and projects onto a (completely integrable) distribution on Y. Along J^1Q_0 the canonical distribution coincides with the tangent bundle to J^1Q_0 and projects onto the tangent bundle $TQ_0 \to Q_0$.*

Proposition 3.9. *Let $\iota_0 : Q_0 \to Y$ be a holonomic constraint, $[d\Theta_\lambda]$ a Lagrangian system on J^1Y. Then for every $\alpha \in [d\Theta_\lambda]$*

$$(J^1\iota_0^* \alpha)_{\mathcal{I}} = J^1\iota_0^* \alpha. \tag{3.88}$$

Moreover, for every first-order Lagrangian λ

$$J^1\iota_0^* \Theta_\lambda = \Theta_{J^1\iota_0^* \lambda}. \tag{3.89}$$

This means that the corresponding constrained system on J^1Q_0 satisfies

$$\begin{aligned} [[J^1\iota_0^* d\Theta_\lambda]] &= [J^1\iota_0^* d\Theta_\lambda] = J^1\iota_0^* [d\Theta_\lambda] = [d\Theta_{J^1\iota_0^* \lambda}] \\ &\equiv d\Theta_{J^1\iota_0^* \lambda} \bmod \Omega_Y^{0;2}(J^1Q_0). \end{aligned} \tag{3.90}$$

For simplicity of notations we write

$$\bar{\lambda} = J^1\iota_0^* \lambda, \quad \bar{L} = L \circ J^1\iota_0 \tag{3.91}$$

for the restricted Lagrangian.

By the above propositions, contrary to the non-holonomic case, holonomic constraints represent no constraints in the tangent bundle to the constraint submanifold. Consequently, *holonomic constrained systems are treated in the same way as unconstrained systems on fibered manifolds.* Simply, instead of a Lagrangian λ and a constraint Q_0 in Y one can consider the restricted Lagrangian $\bar{\lambda}$ on J^1Q_0.

Let us summarize some of the main properties of holonomic systems:

Corollary 3.6. *The holonomic constraint Euler–Lagrange form satisfies*

$$E_\lambda^{\mathcal{C}} = E_{\bar{\lambda}}, \tag{3.92}$$

and constraint Euler–Lagrange equations become simply equations for sections γ of the fibered manifold $\pi_0 : Q_0 \to X$ as follows:

$$J^1\gamma^* i_\xi d\Theta_{\bar\lambda} = 0 \quad \forall \xi \in \mathcal{V}(J^1 Q_0), \tag{3.93}$$

or, in adapted fibered coordinates,

$$\frac{\partial \bar{L}}{\partial q^l} - \frac{d}{dt}\frac{\partial \bar{L}}{\partial \dot{q}^l} = 0, \quad 1 \le l \le m - k. \tag{3.94}$$

Hamilton equations are then equations for sections of the prolonged manifold $(\pi_0)_1 : J^1 Q_0 \to X$,

$$\delta^* i_\xi d\Theta_{\bar\lambda} = 0 \quad \forall \xi \in \mathcal{V}(J^1 Q_0). \tag{3.95}$$

The regularity condition reads

$$\det\left(\frac{\partial^2 \bar{L}}{\partial \dot{q}^l \partial \dot{q}^s}\right) \ne 0.$$

The class (3.90) has the canonical form

$$-d\bar{H} \wedge dt + dP_l \wedge dq^l \ \mathrm{mod}\ \Omega_Y^{0,2}(J^1 Q_0), \tag{3.96}$$
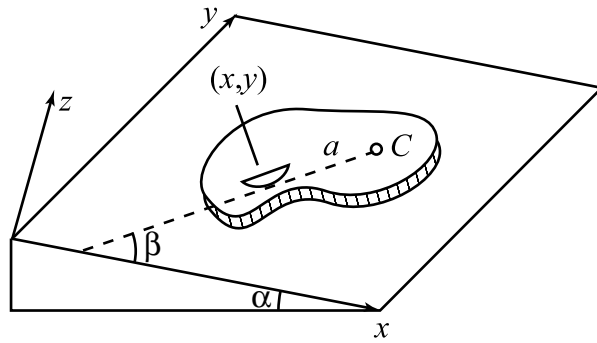
where the Hamiltonian and momenta take the form

$$\bar{H} = -\bar{L} + P_l \dot{q}^l, \quad P_l = \frac{\partial \bar{L}}{\partial \dot{q}^l}, \quad 1 \le l \le m - k. \tag{3.97}$$

The holonomic Hamilton equations then take the canonical form

$$\frac{d}{dt}(P_l \circ \delta) = -\frac{\partial \bar{H}}{\partial q^l}, \quad \frac{d}{dt}(q^l \circ \delta) = \frac{\partial \bar{H}}{\partial P_l}. \tag{3.98}$$

3.7. **Example: A sleigh on an inclined plane.** Let us consider an example of a nonholonomic motion. The situation is presented on the following picture:

There is an object on the inclined plane and a cutting knife. The center of mass of the object lies on the straight line along the knife edge at a distance a from the point (x, y) of contact of the knife and the plane

and β measures the angle between the straight line along the knife edge and x axis, see [29].

This mechanical system is modeled on the fibered manifold $\pi : \mathbb{R} \times S^1 \times \mathbb{R}^2 \to \mathbb{R}$, (t, β, x, y) are coordinates on $\mathbb{R} \times S^1 \times \mathbb{R}^2$.

We introduce "generalized coordinates"

$$q^1 = \beta, \; q^2 = x, \; q^3 = y, \tag{3.99}$$

$\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

The Lagrange function L of the system consists of three parts, the first one represents the energy of rotation, the second one characterizes the kinetic energy of translation of the mechanical system and the third one is the potential energy of the system.

Because the center of mass C does not coincide with the point of contact, the energy of rotation has to be modified. At first we write down the formulas for angular velocities associated with Euler angles which represent rotational motion of this system,

$$\begin{aligned} \omega'_x &= 0, \\ \omega'_y &= 0, \\ \omega'_z &= \dot{\beta}, \end{aligned} \tag{3.100}$$

for $\bar{\beta} = \beta$, $\bar{\Theta} = 0$, $\bar{\psi} = 0$. Then

$$T^{rot} = \frac{1}{2}J\dot{\beta}^2 + mv_{tr}(\omega' \times R'), \tag{3.101}$$

where ω' is the angular velocity vector, $R' = (x', y', z')$ is the position vector between point of contact and the center of mass and v_{tr} is the velocity vector of a translation. The following identities hold

$$\begin{aligned} \omega' \times R' &= (\omega'_x, \omega'_y, \omega'_z) \times (x', y', z') \\ &= (\omega'_y z' - \omega'_z y', -\omega'_x z' + \omega'_z x', \omega'_x y' - \omega'_y x') \\ &= (-\dot{\beta}y', \dot{\beta}x', 0) = (-\dot{\beta}a \sin\beta, \dot{\beta}a \cos\beta, 0). \end{aligned} \tag{3.102}$$

The second term of (3.101) now can be expressed as follows

$$mv_{tr}(\omega' \times R') = m(\dot{x}, \dot{y}, \dot{z})(\omega' \times R') = ma\dot{\beta}(\dot{y}\cos\beta - \dot{x}\sin\beta). \tag{3.103}$$

The kinetic energy of translation and the potential energy take the form

$$T^{tr} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2), \quad V = -mgx\sin\alpha, \tag{3.104}$$

where angle α represents an inclination of the plane.

Finally the Lagrange function L of this mechanical system is expressed by

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}J\dot{\beta}^2 + ma\dot{\beta}(\dot{y}\cos\beta - \dot{x}\sin\beta) - mgx\sin\alpha, \tag{3.105}$$

g is the gravitational acceleration. For the variations of dx and dy we obtain

$$dy = \tan\beta dx. \tag{3.106}$$

This is equivalent to

$$\dot{y} = \dot{x}\tan\beta. \tag{3.107}$$

Then constraint function f_1 is expressed by

$$f_1 = \dot{y} - g_1 = \dot{y} - \dot{x}\tan\beta. \tag{3.108}$$

With respect to (3.24) we have

$$\begin{aligned}
A_1 &= 0, \\
A_2 &= -mg\sin\alpha + ma\dot{\beta}^2\cos\beta, \\
A_3 &= ma\dot{\beta}^2\sin\beta,
\end{aligned} \tag{3.109}$$

and

$$B_{\sigma\nu} = \begin{pmatrix} -J & ma\sin\beta & -ma\cos\beta \\ ma\sin\beta & -m & 0 \\ -ma\cos\beta & 0 & -m \end{pmatrix}. \tag{3.110}$$

The matrix $B_{\sigma\nu}$ is regular, $\det(B_{\sigma\nu}) \neq 0$. The Euler–Lagrange equations are then

$$\begin{aligned}
\ddot{\beta} &= -\frac{amg\sin\alpha\sin\beta}{J - ma^2}, \\
\ddot{x} &= -\frac{Jg\sin\alpha - Ja\dot{\beta}^2\cos\beta - ma^2 g\sin\alpha\cos^2\beta + ma^3\dot{\beta}^2\cos\beta}{J - ma^2}, \\
\ddot{y} &= \frac{a\sin\beta(mag\sin\alpha\cos\beta + J\dot{\beta}^2 - ma^2\dot{\beta}^2)}{J - ma^2}.
\end{aligned} \tag{3.111}$$

Let us return to the constrained case. Now $a = 1$, $k = 1$. The rank condition (3.31) is satisfied. Indeed,

$$\mathrm{rank}\left(\frac{\partial f^1}{\partial\dot{q}^\sigma}\right) = \mathrm{rank}\left(\,0, \quad -\tan\beta, \quad 1\,\right) = 1. \tag{3.112}$$

So, constraint (3.108) generates a constraint submanifold $Q \subset J^1 Y$ by

$$Q = \left\{(t, \beta, x, y, \dot{\beta}, \dot{x}, \dot{y}); \dot{y} = \dot{x}\tan\beta\right\}. \tag{3.113}$$

The constraint distribution on Q is annihilated by one 1-form $\bar{\varphi}^1 = \iota^*\phi^1$, where

$$\phi^1 = dy - \tan\beta dx. \tag{3.114}$$

The constraint distribution in not completely integrable, i.e. there does not exist any 1-form μ such that

$$d\bar{\varphi}^1 = \mu \wedge \bar{\varphi}^1. \tag{3.115}$$

Indeed,

$$d\bar{\varphi}^1 = d(dy - \tan\beta dx) = -\frac{1}{\cos^2\beta}d\beta \wedge dx. \tag{3.116}$$

The constraint is affine in the velocities, i.e. the constraint is simple and it can be equivalently represented by a (non-integrable) distribution on Y, generated by ϕ^1.

From (3.48) we get

$$
\begin{aligned}
\bar{A}_1 &= -\frac{ma\dot{\beta}\dot{x}}{\cos\beta}, \\
\bar{A}_2 &= -\frac{m(g\sin\alpha\cos^3\beta - a\dot{\beta}^2\cos^2\beta + \dot{\beta}\dot{x}\sin\beta)}{\cos^3\beta},
\end{aligned} \tag{3.117}
$$

and

$$\bar{B} = \begin{pmatrix} -J & 0 \\ 0 & -m - m\tan^2\beta \end{pmatrix}. \tag{3.118}$$

The determinant of $\bar{B}$ is non-zero, so the constrained system is *regular*.

Constrained Euler–Lagrange equations (3.65) consist of two equations of the second order

$$
\begin{aligned}
\ddot{\beta} &= -\frac{ma\dot{\beta}\dot{x}}{J\cos\beta}, \\
\ddot{x} &= -g\sin\alpha\cos^2\beta + a\dot{\beta}^2\cos\beta - \dot{\beta}\dot{x}\tan\beta,
\end{aligned} \tag{3.119}
$$

and of one equation of the first order

$$\dot{y} = \dot{x}\tan\beta. \tag{3.120}$$

The "constraint Lagrangian" $\bar{L} = L \circ \iota$ is expressed by

$$\bar{L} = \frac{1}{2}\frac{m\dot{x}^2}{\cos^2\beta} + \frac{1}{2}J\dot{\beta}^2 - mgx\sin\alpha, \tag{3.121}$$

and we can check by a direct computation that the same equations are obtained from the functions $\bar{L}$ and

$$\bar{L}_1 = m\dot{x}\tan\beta + ma\dot{\beta}\cos\beta \tag{3.122}$$

using Euler–Lagrange equations in the form (3.58).

The constraint Legendre transformation is given by

$$(t, \beta, x, y, \dot{\beta}, \dot{x}) \rightarrow (t, \beta, x, y, P_\beta, P_x), \tag{3.123}$$

where *constraint momenta* (3.77) take the form

$$
\begin{aligned}
P_\beta &= J\dot{\beta}, \\
P_x &= \frac{m\dot{x}}{\cos^2\beta}.
\end{aligned} \tag{3.124}
$$

The class of *constraint energy* 1-*forms* is then expressed in constraint Legendre coordinates by

$$
\begin{aligned}
\eta_{\mathcal{I}} &= \bar{\eta}_0 dt \\
&- \frac{aP_\beta P_x \cos\beta}{J} d\beta + \left(-mg\sin\alpha + \frac{maP_\beta^2}{J^2\cos\beta} + \frac{P_\beta P_x \sin\beta}{J\cos\beta} \right) dx \quad (3.125) \\
&- \frac{P_\beta}{J} dP_\beta - \frac{P_x \cos^2\beta}{m} dP_x \quad \mathrm{mod}\ \mathcal{I}^1(\mathcal{C}^0).
\end{aligned}
$$

Hence, *constrained Hamilton equations* consist of four first-order equations which, in simplified notation, can be written as follows:

$$
\begin{aligned}
\dot{P}_\beta &= -\frac{aP_\beta P_x \cos\beta}{J}, \\
\dot{P}_x &= -mg\sin\alpha + \frac{maP_\beta^2}{J^2\cos\beta} + \frac{P_\beta P_x \sin\beta}{J\cos\beta}, \\
\dot{\beta} &= \frac{P_\beta}{J}, \\
\dot{x} &= \frac{P_x \cos^2\varphi}{m},
\end{aligned} \qquad (3.126)
$$

and of the equation of the constraint $\dot{y} = \dot{x}\tan\beta$.

4. Fields with differential constraints

In the sequel we consider a fibered manifold $\pi : Y \to X$ where $\dim X = n$, and its jet prolongations. As above, m denotes the fiber dimension (i.e. $\dim Y = m + n$), and we assume $m > 1$.

First, we summarize main concepts from the theory of unconstrained Lagrangian and Hamiltonian systems, then we turn to the constraint structure in J^1Y, and finally we are interested in Lagrangian and Hamiltonian constrained field equations. Main sources for this section are [7, 10, 12, 13, 19, 21] for the unconstrained theory, [22] for the nonholonomic constraint structure, and [2, 22, 37, 38], for the nonholonomic constrained systems. In this section we also present new results concerning constrained Hamilton–De Donder systems. We study constrained Hamilton–De Donder equations, regularity, and existence of constraint Legendre transformation.

4.1. **Dynamical forms and locally variational forms.** Let E be a dynamical form on J^2Y. In fibered coordinates,

$$
E = E_\sigma \omega^\sigma \wedge \omega_0, \qquad (4.1)
$$

where E_σ are functions of $(x^i, y^\nu, y^\nu_i, y^\nu_{ij})$, and ω_0 denotes the local volume element (2.2). The coordinate form of the equation for paths of E

(3.1) is *a system of m second-order partial differential equations* for the components (γ^ν) of sections γ of π as follows:

$$E_\sigma\left(x^i, \gamma^\nu(x^i), \frac{\partial \gamma^\nu}{\partial x^i}, \frac{\partial^2 \gamma^\nu}{\partial x^i \partial x^j}\right) = 0, \quad 1 \leq \sigma \leq m. \tag{4.2}$$

Again, equations for paths of dynamical forms can be represented by means of *exterior differential systems*; now, however, locally generated by n-forms.

Proposition 4.1. [19] *Let E be a dynamical form on J^2Y. A section γ of π is a path of E if and only if*

$$J^2\gamma^* i_\xi \alpha = 0 \quad \forall \xi \in \mathcal{V}(J^2Y), \tag{4.3}$$

where α is any $(n+1)$-form such that $p_1\alpha = E$.

Proof of this statement is the same as that of Proposition 3.1.

In the "PDE situation" we can proceed in full analogy with the case of mechanics, and consider Lepage classes and corresponding Hamiltonian systems:

Definition 4.1. [22] Let $E \in \Omega_Y^{n+1}(J^2Y)$ be a dynamical form. The equivalence class of $(n+1)$-forms (on an open subset $U \subset J^2Y$) defined by

$$\alpha_1 \sim \alpha_2 \quad \text{iff} \quad p_1\alpha_1 = p_1\alpha_2 = E|_U \tag{4.4}$$

is called *Lepage class of E on U*. The family of all local Lepage classes of E is referred to as *Lepage class of E* and is denoted by $[\alpha]_E$, or simply $[\alpha]$.

By the above proposition, the equation for paths of E (on U) coincides with equations for *holonomic* integral sections of the exterior differential system $\mathcal{H}_\alpha$, generated by the following system of n-forms

$$i_\xi \alpha \quad \forall \xi \in \mathcal{V}(J^2Y), \tag{4.5}$$

where α is any representative of the Lepage class of E (on U).

Definition 4.2. [19, 22] Let $[\alpha]$ be a Lepage class of E. Every representative $\alpha \in [\alpha]$ is called a *Hamiltonian system* associated with E. The exterior differential system $\mathcal{H}_\alpha$ is called a *Hamiltonian EDS* related to E. Equations for (all) integral sections of $\mathcal{H}_\alpha$ are called *Hamilton equations* associated with E.

Let us turn to *locally variational* dynamical forms. By definition this means that around each point $E = E_\lambda$, for a local Lagrangian λ (recall that a Lagrangian of order r is defined to be a horizontal n-form on J^rY).

In fibered coordinates, the components E_σ of E take the form of Euler–Lagrange expressions of $\lambda = L\omega_0$. *Necessary and sufficient conditions for a (second-order) dynamical form to be locally variational read* [1, 11]

$$
\begin{aligned}
\frac{\partial E_\sigma}{\partial y_{ij}^\nu} - \frac{\partial E_\nu}{\partial y_{ij}^\sigma} &= 0, \\
\frac{\partial E_\sigma}{\partial y_i^\nu} + \frac{\partial E_\nu}{\partial y_i^\sigma} - 2\frac{d}{dx^j}\frac{\partial E_\nu}{\partial y_{ij}^\sigma} &= 0, \\
\frac{\partial E_\sigma}{\partial y^\nu} - \frac{\partial E_\nu}{\partial y^\sigma} + \frac{d}{dx^i}\frac{\partial E_\nu}{\partial y_i^\sigma} - \frac{d}{dx^i}\frac{d}{dx^j}\frac{\partial E_\nu}{\partial y_{ij}^\sigma} &= 0,
\end{aligned}
\tag{4.6}
$$

and the corresponding Tonti Lagrangian for E is

$$
L = y^\sigma \int_0^1 E_\sigma(x^i, uy^\nu, uy_i^\nu, uy_{ij}^\nu)du.
\tag{4.7}
$$

Notice that, contrary to the case of ordinary differential equations, variationality conditions (4.6) do not imply that the locally variational form E should be affine in the second derivatives. Moreover, second-order locally variational forms need not come from Lagrangians of the first order. This means that Tonti Lagrangian need not be reducible to a first-order Lagrangian. (From the formula for Euler–Lagrange expressions we can see immediately that a necessary condition for reducibility is that E_σ should be affine in the second derivatives y_{ij}^ν).

Let us summarize basic definitions:

Definition 4.3. The Lepage class $[\alpha]$ of a locally variational form E is called *Lagrangian system*. Every element $\alpha \in [\alpha]$ is called a *Hamiltonian system* associated with E. Paths of a locally variational form E are called *extremals*. Equations for paths of a locally variational form (respectively, equations for holonomic integral sections of associated Hamiltonian EDS) are called *Euler–Lagrange equations*. Equations for integral sections of the Hamiltonian EDS are called *Hamilton equations*, their integral sections are then called *Hamilton extremals*.

4.2. **Euler–Lagrange and Hamilton–De Donder equations in first-order field theory.** In what follows we shall be concerned merely with locally variational forms that arise from local *first-order Lagrangians*. Such dynamical forms, among others, have the following properties:

 • Around each point in J^2Y it holds $E = E_\lambda$, where λ is a Lagrangian defined on an open subset of J^1Y. In fibered coordinates where $\lambda = L\omega_0$,

$$
E_\lambda = E_\sigma \omega^\sigma \wedge \omega_0, \quad E_\sigma = \frac{\partial L}{\partial y^\sigma} - \frac{d}{dx^j}\frac{\partial L}{\partial y_j^\sigma}.
\tag{4.8}
$$

This means that the *Euler–Lagrange equations* in fibered coordinates take the familiar form

$$\left(\frac{\partial L}{\partial y^\sigma} - \frac{d}{dx^j} \frac{\partial L}{\partial y^\sigma_j} \right) \circ J^2\gamma = 0, \quad 1 \le \sigma \le m. \tag{4.9}$$

• In every fibered chart, components E_σ of E are affine in the second derivatives, i.e.

$$E_\sigma = A_\sigma + B^{ij}_{\sigma\nu} \, y^\nu_{ij} = A_\sigma + B^{(ij)}_{\sigma\nu} \, y^\nu_{ij}, \tag{4.10}$$

where A_σ and $B^{ij}_{\sigma\nu}$ are functions of (x^i, y^ρ, y^ρ_l) that in terms of a first-order Lagrangian for E take the form

$$
\begin{aligned}
B^{ij}_{\sigma\nu} &= -\frac{\partial^2 L}{\partial y^\nu_i \partial y^\sigma_j}, \\
A_\sigma &= \frac{\partial L}{\partial y^\sigma} - \frac{d}{dx^j} \frac{\partial L}{\partial y^\sigma_j} = \frac{\partial L}{\partial y^\sigma} - \frac{\partial^2 L}{\partial x^j \, \partial y^\sigma_j} - \frac{\partial^2 L}{\partial y^\nu \partial y^\sigma_j} y^\nu_j.
\end{aligned}
\tag{4.11}
$$

Note that the $B^{ij}_{\sigma\nu}$ need not be symmetric in the upper indices. $B^{(ij)}_{\sigma\nu}$ denotes the symmetric part in the i, j.

• Every first-order Lagrangian for E has a first-order Lepage equivalent that is not unique. Lepage equivalents of λ take the form

$$\rho = \Theta_\lambda + d\nu + \mu, \tag{4.12}$$

where

$$\Theta_\lambda = L\,\omega_0 + \frac{\partial L}{\partial y^\sigma_j}\omega^\sigma \wedge \omega_j, \tag{4.13}$$

μ is an arbitrary at least 2-contact n-form, and ν is an arbitrary contact $(n-1)$-form. Θ_λ is called the *Poincaré–Cartan form* associated with λ. It is the *unique* at most 1-contact n-form such that $h\Theta_\lambda = \lambda$ and $p_1 d\Theta_\lambda$ is $\pi_{1,0}$-horizontal. This means that the Euler–Lagrange form E_λ of λ is also unique, since by (4.12) it does not depend upon the choice of a Lepage equivalent ρ of λ:

$$E_\lambda = p_1 d\rho = p_1 d\Theta_\lambda. \tag{4.14}$$

Note that contrary to mechanics, one generally has $d\Theta_{\lambda_1} \ne d\Theta_{\lambda_2}$ for equivalent Lagrangians λ_1, λ_2.

• E can be locally represented by a *first-order Lepage class* that is called *Lagrangian system* associated with the locally variational form E. With a similar inaccuracy as in Sec. 3, in order to simplify notations, we write

$$[\alpha] = [d\Theta_\lambda] = d\Theta_\lambda \bmod \Omega^{n+1,(\ge 2)}(J^1 Y). \tag{4.15}$$

Consequently, extremals and Hamilton extremals are described by Hamiltonian exterior differential systems defined on (open subsets of) $J^1 Y$.

Remark 4.1. In the sequel we shall again assume that E, defined on J^2Y, is everywhere nontrivially of order 2. This means that $(B_{\sigma\nu}^{ij})$ in (4.10) is everywhere a non-zero matrix, or, equivalently, for every first-order Lagrangian λ the Poincaré–Cartan $(n+1)$-form $d\Theta_\lambda$ is everywhere nontrivially of order one.

As we can see, the Euler–Lagrange and Hamilton equations in first-order field theory now have the following EDS formulation:

Proposition 4.2. *Let $[\alpha]$ be a Lagrangian system on J^1Y, E the corresponding locally variational form.*

A section γ of π is an extremal of E (on an open set $W \subset X$ in $\operatorname{dom}\gamma$) if and only if

$$J^1\gamma^* i_\xi \alpha = 0 \quad \forall \xi \in \mathcal{V}(J^1Y), \tag{4.16}$$

where α is any $(n+1)$-form belonging to the class $[\alpha]$ (defined on $\pi_1^{-1}(W)$).

A section δ of π_1 is a Hamilton extremal of E, related with the Hamiltonian system $\alpha \in [\alpha]$ (defined in $\pi_1^{-1}(\operatorname{dom}\delta)$) if and only if

$$\delta^* i_\xi \alpha = 0 \quad \forall \xi \in \mathcal{V}(J^1Y). \tag{4.17}$$

The concept of *regularity of a Lagrangian system* is, similarly as in mechanics, related with the properties of the associated Hamiltonian exterior differential systems. The situation in field theory is, however, much more rich and interesting than that in mechanics: the reason is the non-uniqueness of the Poincaré–Cartan form Θ_λ of a Lagrangian λ. For more details we refer to [19, 23, 24]. In this paper we shall study the most simple case related just to the properties of the Hamiltonian differential systems $\mathcal{H}_{d\Theta_\lambda}$ related with the forms $d\Theta_\lambda$ (*Hamilton–De Donder equations* [5, 7]).

To this end, let us recall the following definition [25].

Definition 4.4. Let $[\alpha]$ (on J^1Y) be a Lagrangian system related with a locally variational form E. An element α of the class $[\alpha]$ is called *Hamilton–De Donder system* related with E if

$$\alpha = d\Theta_\lambda + F, \tag{4.18}$$

where λ is a Lagrangian for E and $F \in \Omega_Y^{n+1,(\geq 2)}(J^1Y)$. The corresponding Hamilton equations, i.e.

$$\delta^* i_\xi (d\Theta_\lambda + F) = 0 \quad \forall \xi \in \mathcal{V}(J^1Y) \tag{4.19}$$

are called *Hamilton–De Donder equations*.

It is easy to see that Hamilton–De Donder systems can be locally expressed in the so-called *canonical form* as follows:

Proposition 4.3. *Let $\alpha = d\Theta_\lambda + F$ be a Hamilton–De Donder system on an open set $U \subset J^1Y$; we may assume that U is endowed with fibered coordinates $(x^i, y^\sigma, y^\sigma_j)$. Then there exist functions H and p^i_σ, $1 \leq \sigma \leq m$, $1 \leq i \leq n$, such that*

$$\alpha = -dH \wedge \omega_0 + dp^i_\sigma \wedge dy^\sigma \wedge \omega_i + F. \tag{4.20}$$

H and p^i_σ are defined by

$$p^i_\sigma = \frac{\partial L}{\partial y^\sigma_i}, \quad H = -L + p^j_\nu y^\nu_j, \tag{4.21}$$

where $L\omega_0$ is a first-order Lagrangian whose Poincaré–Cartan $(n+1)$-form coincides with $d\Theta_\lambda$.

Definition 4.5. Functions H and p^i_σ, $1 \leq \sigma \leq m$, $1 \leq i \leq n$, defined by formula (4.20) are called a *Hamiltonian* and *momenta* of the Hamilton–De Donder system α.

Note that the family of a Hamiltonian and momenta (4.21) of a Hamilton–De Donder system is *non-unique* and depends upon the choice of a Lagrangian for the form $d\Theta_\lambda$. On the other hand, in the following subclass of the equivalence class (4.15),

$$[\alpha]_Y = [d\Theta_\lambda]_Y \equiv d\Theta_\lambda \bmod \Omega_Y^{n+1,(\geq 2)}(U), \tag{4.22}$$

all elements posses the same families of momenta&Hamiltonian. This is due to the fact that if $\alpha_1, \alpha_2 \in [\alpha]|_U$ are such that $\alpha_1 = d\Theta_{\lambda_1} + F_1$ and $\alpha_2 = d\Theta_{\lambda_2} + F_2$ where $d\Theta_{\lambda_1} \neq d\Theta_{\lambda_2}$ and $F_1, F_2 \in \Omega_Y^{n+1,(\geq 2)}(U)$, then α_1 and α_2 are not equivalent in the sense of (4.22), and vice versa (see [25], Proposition 3.1 and its proof).

Let us turn to the concept of regularity of a Hamilton–De Donder system ([19]).

Definition 4.6. A Hamilton–De Donder system α is called *regular* if $\mathcal{H}_\alpha$ contains all the canonical contact n-forms

$$\omega^\sigma \wedge \omega_i, \quad 1 \leq \sigma \leq m, \quad 1 \leq i \leq n. \tag{4.23}$$

A Lagrangian system $[\alpha]$ is called *De Donder regular* if around each point in J^1Y there exists a related regular Hamilton–De Donder system.

Proposition 4.4. *Let α be a regular Hamilton–De Donder system. Then every integral section of $\mathcal{H}_\alpha$ is holonomic. Consequently, Hamilton–De Donder equations of α are equivalent with the Euler–Lagrange equations of $[\alpha]$.*

Proof. If a Hamilton–De Donder system α is regular then for every integral section δ of $\mathcal{H}_\alpha$,

$$
\delta^*(\omega^\sigma \wedge \omega_i) = \left(\frac{\partial(y^\sigma \circ \delta)}{\partial x^i} - (y_i^\sigma \circ \delta) \right) \omega_0 = 0,
$$
$$
1 \leq \sigma \leq m, \quad 1 \leq i \leq n,
$$
(4.24)

meaning that $\delta = J^1\gamma$ for a section γ of π. $\square$

Theorem 4.1. *Let $\alpha = d\Theta_\lambda + F$ be a Hamilton–De Donder system. The following conditions are equivalent:*

(1) *The Hamilton–De Donder system α is regular.*
(2) *A system of generators of $\mathcal{H}_\alpha$ has maximal rank (i.e. equal to $m + mn$).*
(3) *Every Lagrangian λ for $d\Theta_\lambda$ satisfies the regularity condition*

$$
\det \left(\frac{\partial^2 L}{\partial y_i^\sigma \partial y_j^\nu} \right) \neq 0.
$$
(4.25)

Proof. Computing explicitly generators of $\mathcal{H}_\alpha$ we obtain the following system of $m + mn$ differential n-forms:

$$
A_\sigma \omega_0 + \left(2F_{\sigma\nu}^j + \frac{\partial^2 L}{\partial y^\sigma \partial y_j^\nu} - \frac{\partial^2 L}{\partial y^\nu \partial y_j^\sigma} \right) \omega^\nu \wedge \omega_j + B_{\sigma\nu}^{ij} \, dy_j^\nu \wedge \omega_i + \mu_\sigma,
$$
$$
B_{\sigma\nu}^{ij} \, \omega^\nu \wedge \omega_j,
$$
(4.26)

where $1 \leq \sigma \leq m$, $1 \leq i \leq n$, A_σ and $B_{\sigma\nu}^{ij}$ are given by (4.11), and μ_σ are at least 2-contact (precisely, μ_σ is the at least 2-contact part of $i_{\partial/\partial y^\sigma} F$). This means that the matrix of generators of $\mathcal{H}_\alpha$ is the following matrix with $m + mn$ rows (and $1 + mn + mn^2 + \cdots$ columns):

$$
\begin{pmatrix}
A_\sigma & 2F_{\sigma\nu}^j + \dfrac{\partial^2 L}{\partial y^\sigma \partial y_j^\nu} - \dfrac{\partial^2 L}{\partial y^\nu \partial y_j^\sigma} & B_{\sigma\nu}^{ij} & \cdots \\
0 & B_{\sigma\nu}^{ij} & 0 & 0
\end{pmatrix}.
$$
(4.27)

First, we prove the equivalence of (1) and (2).

If α is regular then all the generators $B_{\sigma\nu}^{ij} \omega^\nu \wedge \omega_j$ are independent, meaning that the matrix $(B_{\sigma\nu}^{ij})$ is regular. Consequently, all rows of $(B_{\sigma\nu}^{ij})$ (labelled by (σ, i)) are linearly independent, for every fixed i. Then, however, the matrix $(B_{\sigma\nu}^{ij})$ with m rows labelled by σ, and mn columns labelled by (ν, j), has the maximal rank, m, and the rank of the matrix (4.27) is equal to $m + mn$, as desired.

Conversely, if the rank of the matrix (4.27) is maximal then its square submatrix $(B_{\sigma\nu}^{ij})$ (with rows labelled by (σ, i)) is regular. This means that all the forms $\omega^\sigma \wedge \omega_i$ are independent. Hence α is regular.

The equivalence of (3) and (2) is now clear: by the above, $\mathcal{H}_\alpha$ has maximal rank iff the matrix $(B^{ij}_{\sigma\nu})$ is regular, However, in terms of the Lagrangian λ for α,

$$B^{ij}_{\sigma\nu} = -\frac{\partial^2 L}{\partial y^\sigma_i \partial y^\nu_j}. \tag{4.28}$$

This completes the proof.   □

Theorem 4.2. *If $d\Theta_\lambda + F$ is a regular Hamilton–De Donder system then every form $\alpha \in [d\Theta_\lambda]_Y$ is regular. Consequently,*

(1) *For every $\alpha \in [d\Theta_\lambda]_Y$, all Hamilton extremals are holonomic.*
(2) *For every $\alpha \in [d\Theta_\lambda]_Y$, the Hamilton equations are equivalent with the Euler–Lagrange equations of λ.*
(3) *For every $\alpha \in [d\Theta_\lambda]_Y$, every Hamilton extremal of α is a prolongation of an extremal of λ.*
(4) *Hamilton equations of all elements in the class $[d\Theta_\lambda]_Y$ are equivalent.*

Proof. Looking at the generators (4.26) we can see immediately that regularity does not depend upon the choice of functions $F^j_{\sigma\nu}$, i.e., upon the choice of $\alpha \in [d\Theta_\lambda]_Y$.

The rest of the proof is elementary.   □

Corollary 4.1. *Let $\alpha = d\Theta_\lambda + F$ be a regular Hamilton–De Donder system. Then momenta p^j_σ, $1 \le \sigma \le m$, $1 \le j \le n$, of α are independent, and $(x^i, y^\sigma, p^j_\sigma)$ are local coordinates on J^1Y, called Legendre coordinates. The Hamiltonian differential system $\mathcal{H}_\alpha$ has generators that in Legendre coordinates take the form (since $\omega^\nu \wedge \omega_i \in \mathcal{H}_\alpha$, $\forall \nu, i$),*

$$\frac{\partial H}{\partial y^\sigma}\omega_0 + dp^j_\sigma \wedge \omega_j + \mu_\sigma, \quad \frac{\partial H}{\partial p^j_\sigma}\omega_0 - dy^\sigma \wedge \omega_j, \tag{4.29}$$

where $\mu_\sigma = i_{\partial/\partial y^\sigma} F \in \Omega^{n(\ge 2)}_Y(J^1Y)$. Hamilton equations of α in Legendre coordinates then read

$$\frac{\partial p^j_\sigma}{\partial x^j} = -\frac{\partial H}{\partial y^\sigma}, \qquad \frac{\partial y^\sigma}{\partial x^j} = \frac{\partial H}{\partial p^j_\sigma}, \tag{4.30}$$

where the appearing functions are considered along sections δ of π_1.

4.3. **Non-holonomic constraints in field theory.** The aim of the section is to present the concept of the non-holonomic constraint structure [22].

Non-holonomic constraints in the case $\dim X > 1$ are defined in the same way as for one independent variable:

Definition 4.7. By a *constraint submanifold* or a *non-holonomic constraint* in J^1Y we shall understand a submanifold $Q \subset J^1Y$, fibered over Y, precisely speaking, a surjective submersion $\pi_{1,0}|_Q : Q \to Y$.

Put $\operatorname{codim} Q = \kappa$ and assume $1 \leq \kappa \leq mn - 1$. Locally Q can be expressed by a system of first-order partial differential equations

$$f^\alpha(x^i, y^\sigma, y_j^\sigma) = 0, \quad 1 \leq \alpha \leq \kappa, \tag{4.31}$$

such that

$$\operatorname{rank}\left(\frac{\partial f^\alpha}{\partial y_j^\sigma}\right) = \kappa, \quad \text{where } \alpha \text{ labels rows and } \sigma, j \text{ columns.} \tag{4.32}$$

Definition 4.8. Let Q be a non-holonomic constraint in J^1Y, $\operatorname{codim} Q = \kappa$, $1 \leq \kappa \leq mn - 1$. If

$$\operatorname{rank}\left(\frac{\partial f^\alpha}{\partial y_j^\sigma}\right) = k, \quad \text{where } \alpha, j \text{ label rows and } \sigma \text{ columns,} \tag{4.33}$$

for some k, $1 \leq k \leq m - 1$, we say that Q is a *regular non-holonomic constraint of corank* (κ, k).

It can be shown that the above definition is correct (coordinate independent) [22].

Given a regular non-holonomic constraint Q in J^1Y there naturally arise the following local distributions, defined on the domain U of definition of the functions f^α:

(1) $\mathcal{D}_U = \operatorname{annih}\{df^\alpha, 1 \leq \alpha \leq \kappa\}$; $\operatorname{rank} \mathcal{D}_U$ is constant on U due to (4.32) and equal to $m + n + mn - \kappa$.

(2) $\tilde{\mathcal{C}}_U = \operatorname{annih}\{\phi^{\alpha j}\}$, where

$$\phi^{\alpha j} = f^\alpha dx^j + \frac{1}{n}\frac{\partial f^\alpha}{\partial y_j^\sigma}\omega^\sigma, \quad 1 \leq \alpha \leq \kappa, 1 \leq j \leq n. \tag{4.34}$$

The forms (4.34) are not linearly independent, however, due to rank condition (4.33), there exist functions $c_{\alpha j}^a$, $1 \leq a \leq k$, $1 \leq \alpha \leq \kappa$, $1 \leq j \leq n$, on U, such that the $(k \times m)$-matrix

$$M = (M_\sigma^a), \quad \text{where} \quad M_\sigma^a = \frac{1}{n}c_{\alpha j}^a \frac{\partial f^\alpha}{\partial y_j^\sigma}, \tag{4.35}$$

has maximal rank equal to k. Thus,

$$\phi^a = c_{\alpha j}^a \phi^{\alpha j} = c_{\alpha j}^a f^\alpha dx^j + \frac{1}{n}c_{\alpha j}^a \frac{\partial f^\alpha}{\partial y_j^\sigma}\omega^\sigma = c_{\alpha j}^a f^\alpha dx^j + M_\sigma^a \omega^\sigma, \tag{4.36}$$
$$1 \leq a \leq k,$$

are linearly independent at each point in U. Hence, the distribution $\tilde{\mathcal{C}}_U$ has constant corank equal to k, i.e. $\operatorname{rank} \tilde{\mathcal{C}}_U = m + n + mn - k$.

(3) $\mathcal{C}_U = \operatorname{annih}\{\phi^a, df^\alpha,\ 1 \le a \le k, 1 \le \alpha \le \kappa\}$.

The following results have been obtained in [22]:

Proposition 4.5. *$Q \cap U$ is an integral submanifold of $\mathcal{D}_U$. Hence, for every $x \in Q$, the forms $df^\alpha(x)$, $1 \le \alpha \le \kappa$, annihilate the tangent space T_xQ to the manifold Q at x, i.e., along Q, $\mathcal{D} = \operatorname{annih}\{df^\alpha, 1 \le \alpha \le \kappa\} = TQ$.*

Corollary 4.2. *Let Q be a constraint of codimension κ in J^1Y, and let $f^\alpha = 0$ and $f'^\alpha = 0$, where $1 \le \alpha \le \kappa$, be two sets of equations of Q on an open set $U \subset V_1 \subset J^1Y$. Then there are functions γ_β^α on U such that at each point of U, (γ_β^α) is a regular matrix, and $df'^\alpha = \gamma_\beta^\alpha df^\beta$. In particular, at each point $x \in Q \cap U$,*

$$\frac{\partial f'^\alpha}{\partial y_j^\sigma} = \gamma_\beta^\alpha \frac{\partial f^\beta}{\partial y_j^\sigma}. \tag{4.37}$$

Proposition 4.6. *$\mathcal{C}_U$ is a subdistribution of both $\tilde{\mathcal{C}}_U$ and $\mathcal{D}_U$. At the points of $Q \cap U$, the distributions $\mathcal{C}_U$ and $\tilde{\mathcal{C}}_U \cap \mathcal{D}$ coincide, and define a distribution of corank k on $Q \cap U$.*

The local distributions on Q mentioned above unite into a (global) distribution on Q:

Theorem 4.3. *Let Q be a regular non-holonomic constraint in J^1Y of corank (κ, k), let $\iota : Q \to J^1Y$ be the canonical embedding of the submanifold Q into J^1Y. If ϕ^a, $1 \le a \le k$, are independent 1-forms (4.36), put*

$$\varphi^a = \iota^*\phi^a = (M_\sigma^a \circ \iota)\, \iota^*\omega^\sigma, \quad 1 \le a \le k. \tag{4.38}$$

Then

$$\mathcal{C} = \operatorname{annih}\{\varphi^a, 1 \le a \le k\} \tag{4.39}$$

is a distribution of corank k on Q.

The proof of the theorem can be found in [22].

Definition 4.9. The distribution $\mathcal{C}$ (4.39) on Q is called *canonical distribution*. 1-forms belonging to the annihilator, $\mathcal{C}^0$, of $\mathcal{C}$, are called *canonical constraint 1-forms*. The ideal in the exterior algebra of differential forms on Q generated by $\mathcal{C}^0$ is called *canonical constraint ideal*, and denoted by $\mathcal{I}(\mathcal{C}^0)$; its homogeneous component of degree p is denoted by $\mathcal{I}^p(\mathcal{C}^0)$. Elements of the ideal $\mathcal{I}(\mathcal{C}^0)$ are called *canonical constraint forms*.

Theorem 4.4. *The canonical distribution $\mathcal{C}$ on Q is locally spanned by the following vector fields:*

$$\frac{\partial_c}{\partial x^i} \equiv \frac{\partial}{\partial x^i} + \sum_{a=1}^{k} (F_i^a \circ \iota) \frac{\partial}{\partial y^{m-k+a}}, \quad 1 \le i \le n,$$

$$\frac{\partial_c}{\partial y^s} \equiv \frac{\partial}{\partial y^s} + \sum_{a=1}^{k} (G_s^a \circ \iota) \frac{\partial}{\partial y^{m-k+a}}, \quad 1 \le s \le m - k, \tag{4.40}$$

$$\frac{\partial}{\partial z^J}, \quad 1 \le J \le nm - \kappa,$$

where $(x^i, y^\sigma, z^J, f^\alpha)$, $1 \le i \le n$, $1 \le \sigma \le m$, $1 \le J \le nm - \kappa$, $1 \le \alpha \le \kappa$, denote fibered coordinates adapted to the submanifold $\iota : Q \to J^1 Y$, the functions G_s^a represent (at each point) a fundamental system of solutions of the system of independent homogeneous algebraic equations for m unknowns Ξ^σ, $1 \le \sigma \le m$,

$$M_\sigma^a \Xi^\sigma = 0, \quad 1 \le a \le k, \tag{4.41}$$

and, for every $i = 1, 2, \ldots, n$, the F_i^a are solutions of the equations

$$M_\sigma^a F_i^\sigma = M_\sigma^a y_i^\sigma - f^\alpha c_{\alpha i}^a, \quad 1 \le a \le k, \tag{4.42}$$

(where y_j^σ are considered as functions of z^J, f^β) corresponding to the choice of all the parameters equal to zero.

A section γ of π defined on an open set $W \subset X$ is called a *holonomic path in Q* if for every $x \in W$

$$J^1 \gamma(x) \in Q. \tag{4.43}$$

Remark 4.2. We shall use the following notations and objects, adapted to the constraint structure, introduced in [22].
   (i) Conventions concerning *notation of indices*:

$$\begin{aligned} 1 \le i, j, l \le n, \quad & 1 \le \alpha, \beta, \gamma \le \kappa, \quad & 1 \le J \le nm - \kappa, \\ 1 \le \sigma, \nu, \rho \le m, \quad & 1 \le a, b, c \le k, \quad & 1 \le p, r, s \le m - k. \end{aligned} \tag{4.44}$$

   (ii) Taking into account that the matrix (4.35) in (4.36) has maximal rank, k, one can express k of the contact 1-forms ω^σ by means of the constraint forms ϕ^a, $1 \le a \le m$, and the remaining ω^ν's. Without loss of generality we may suppose that this concerns the forms ω^{m-k+a}, where $1 \le a \le k$. In an adapted basis $(x^i, y^\sigma, z^J, f^\alpha)$, and in the notations of the above theorem it holds

$$\begin{aligned} \omega^{m-k+a} &= \mu_b^a (\phi^b - M_s^b \omega^s - c_{\alpha j}^b f^\alpha dx^j) \\ &= \mu_b^a \phi^b + G_s^a \omega^s + (F_j^a + G_s^a y_j^s - y_j^{m-k+a}) dx^j, \end{aligned} \tag{4.45}$$

where (μ_b^a) is an appropriate regular matrix. Here and in what follows, y_j^σ are considered as functions of the coordinates $(x^i, y^\sigma, z^J, f^\alpha)$. Similarly, the rank condition (4.32) guarantees that one can express the forms dz^α by means of $(df^\beta, dx^i, dy^\sigma, dz^J)$. Thus, we have on J^1Y the following bases of 1-forms, adapted to the constraint structure:

$$(dx^i, dy^s, \phi^a, dz^J, df^\alpha), \quad \text{or} \quad (dx^i, \omega^s, \phi^a, dz^J, df^\alpha); \tag{4.46}$$

Consequently, with obvious notations we may write

$$\bar\omega^{m-k+a} \equiv \iota^* \omega^{m-k+a} = \bar\varphi^a + \bar G_s^a \bar\omega^s, \tag{4.47}$$

where $\bar\omega^s = \iota^* \omega^s$, and $\bar\varphi^a = \iota^*(\mu_b^a \phi^b) = (\mu_b^a \circ \iota)\varphi^b$. We can see that, on Q, instead of a canonical basis (dx^i, dy^σ, dz^J), or a basis $(dx^i, \bar\omega^\sigma, dz^J)$ adapted to the induced contact structure, it is worth to work with *bases adapted to the constraint structure*, where the canonical constraint 1-forms appear:

$$(dx^i, dy^s, \bar\varphi^a, dz^J), \qquad (dx^i, \bar\omega^s, \bar\varphi^a, dz^J). \tag{4.48}$$

(iii) Keeping the above notations we can express the functions G_s^a and F_j^a appearing in (4.40) as follows:

$$G_s^a = \mu_b^a M_s^b, \qquad F_j^a = y_j^{m-k+a} - G_s^a y_j^s - \mu_b^a c_{\alpha j}^b f^\alpha. \tag{4.49}$$

We also put

$$y_j^\sigma \circ \iota = g_j^\sigma. \tag{4.50}$$

With this notation,

$$\bar G_s^a = (\mu_b^a M_s^b) \circ \iota, \qquad \bar F_j^a = g_j^{m-k+a} - \bar G_s^a g_j^s, \tag{4.51}$$

i.e.

$$g_j^{m-k+a} = \bar F_j^a + \bar G_s^a g_j^s. \tag{4.52}$$

(iv) The vector fields $\partial_c/\partial x^i$ and $\partial_c/\partial y^s$ on Q defined by (4.40) are called *constraint partial derivative operators*. We put

$$\begin{aligned}
\frac{d_c'}{dx^i} &= \frac{\partial_c}{\partial x^i} + g_i^s \frac{\partial_c}{\partial y^s}, \\
\frac{d_c}{dx^i} &= \frac{\partial_c}{\partial x^i} + g_i^s \frac{\partial_c}{\partial y^s} + z_i^J \frac{\partial}{\partial z^J} = \frac{d_c'}{dx^i} + z_i^J \frac{\partial}{\partial z^J},
\end{aligned} \tag{4.53}$$

and call the above operators the *i-th cut constraint total derivative operator* and *i-th constraint total derivative operator*, respectively.

(v) The exterior derivative of a function f on Q is expressed as follows:

$$df = \frac{d_c' f}{dx^j} dx^j + \frac{\partial_c f}{\partial y^s} \bar\omega^s + \frac{\partial f}{\partial y^{m-k+a}} \bar\varphi^a + \frac{\partial f}{\partial z^J} dz^J. \tag{4.54}$$

(vi) Next, denote

$$\mathcal{C}^a_{Jj} = \bar{G}^a_s \frac{\partial g^s_j}{\partial z^J} - \frac{\partial g^{m-k+a}_j}{\partial z^J} = -\frac{\partial \bar{F}^a_j}{\partial z^J} - \frac{\partial \bar{G}^a_s}{\partial z^J} g^s_j, \qquad \mathcal{C}^{ai}_{js} = \mathcal{C}^a_{Jj} \frac{\partial z^J}{\partial y^s_i}. \quad (4.55)$$

(vii) For $d\bar{\varphi}^a$ we have

$$\begin{aligned}
d\bar{\varphi}^a = {} & \left(\bar{G}^a_s \frac{d'_c g^s_j}{dx^i} - \frac{d'_c g^{m-k+a}_j}{dx^i} \right) dx^i \wedge dx^j \\
& + \left(\bar{G}^a_s \frac{\partial_c g^s_j}{\partial y^r} + \frac{d'_c \bar{G}^a_r}{dx^j} - \frac{\partial_c g^{m-k+a}_j}{\partial y^r} \right) \bar{\omega}^r \wedge dx^j \\
& + \mathcal{C}^a_{Jj} dz^J \wedge dx^j - \frac{\partial_c \bar{G}^a_s}{\partial y^r} \bar{\omega}^r \wedge \bar{\omega}^s - \frac{\partial \bar{G}^a_s}{\partial z^J} dz^J \wedge \bar{\omega}^s \\
& + \left(\bar{G}^a_s \frac{\partial g^s_j}{\partial y^{m-k+b}} - \frac{\partial g^{m-k+a}_j}{\partial y^{m-k+b}} \right) \bar{\varphi}^b \wedge dx^j - \frac{\partial \bar{G}^a_s}{\partial y^{m-k+b}} \bar{\varphi}^b \wedge \bar{\omega}^s.
\end{aligned} \qquad (4.56)$$

There are several interesting *particular cases of regular non-holonomic constraints in field theory*. We wish to mention here very briefly the following ones (precise definitions and further properties can be found in [22]:

• Constraints whose canonical distribution is projectable onto a distribution on Y, i.e. *constraints* that can be *modeled by a distribution or codistribution on Y*.

• Constraints whose canonical distribution is completely integrable; these constraints are called *semiholonomic*, and can be equivalently modeled by a completely integrable, nowhere vertical distribution on Y.

• *Lagrangian constraints*: these are characterized by the property that the codistributions $\tilde{\mathcal{C}}^0_U$ can be generated by a system of (independent) Lepage 1-forms; for Lagrangian constraints it holds

$$\mathcal{C}^a_{Jj} = 0, \quad \mathcal{C}^{ai}_{js} = 0 \qquad (4.57)$$

for all values of indices. In this context it is interesting to note that for $\dim X = 1$ (mechanics) all non-holonomic constraints are Lagrangian.

• π-*adapted constraints*: can be locally represented by equations "in normal form",

$$y^{m-k+a}_j = g^a_j(x^i, y^\sigma, y^s_l), \quad 1 \le j \le n, \quad 1 \le a \le k. \qquad (4.58)$$

These constraints are Lagrangian.

Lagrangian and Hamiltonian systems subjected to π-adapted constraints are studied in detail in [25].

• *Holonomic constraints*, defined as fibered submanifolds of π, can again be easily treated in terms of the theory of regular non-holonomic

constraints as a (very) particular case. The situation is completely anal-
ogous to that in mechanics (for details see [22]).

4.4. **Constrained Lagrangian systems.** The aim of the section is to
introduce the concept of the constrained Lagrangian systems. For more
details and the proofs of the assertions we refer to [22].

Let us consider a Lagrangian system on J^1Y. Recall from Sec. 4.2
that it is defined to be a first-order Lepage class. We write it in the form

$$[\alpha] = [d\Theta_\lambda] = d\Theta_\lambda \bmod \Omega^{n+1,(\geq 2)}(J^1Y). \tag{4.59}$$

If $\iota : Q \to J^1Y$ is a regular non-holonomic constraint and $\mathcal{I}(\mathcal{C}^0)$ the
corresponding canonical constraint ideal, we have another equivalence,
denoted by $\approx$, on $(n+1)$-forms on Q (with the same domain of definition):

$$\eta_1 \approx \eta_2 \quad \text{iff} \quad \eta_1 - \eta_2 = \bar{F} + \varphi, \tag{4.60}$$

where $\bar{F}$ is a (local) at least 2-contact $(n + 1)$-form on Q, and φ is a
constraint $(n + 1)$-form. We denote by $[[\eta]]$ the class of η. If $[\alpha]$ is a
Lepage class on J^1Y then for any of its two elements defined on the same
subset of J^1Y,

$$\alpha_1 \sim \alpha_2 \Rightarrow \iota^*\alpha_1 \approx \iota^*\alpha_2. \tag{4.61}$$

Definition 4.10. Let $[\alpha] = [d\Theta_\lambda]$ be a Lagrangian system on J^1Y and
$\iota : Q \to J^1Y$ a regular non-holonomic constraint. By the associated
constrained Lagrangian system we mean the class $[[\iota^*\alpha]] = [[\iota^*d\Theta_\lambda]]$. Each
form $\iota^*d\Theta_\lambda + \varphi$, where $\varphi \in \mathcal{I}^{n+1}(\mathcal{C}^0)$, is called *constrained Poincaré-
Cartan $(n + 1)$-form* of λ.

Note that every element of $[[\iota^*d\Theta_\lambda]]$ is of the form

$$\bar{\alpha} = \iota^*d\Theta_\lambda + \bar{F} + \varphi, \tag{4.62}$$

where $\bar{F} \in \Omega^{n+1,(\geq 2)}(Q)$ and $\varphi \in \mathcal{I}^{n+1}(\mathcal{C}^0)$.

Definition 4.11. Consider the following system of forms on $\operatorname{dom} \bar{\alpha} \subset Q$:

$$i_\xi\bar{\alpha} \quad \forall \xi \in \mathcal{V}(Q) \cap \mathcal{C}, \quad \varphi \in \mathcal{C}^0. \tag{4.63}$$

The exterior differential system generated by (4.63) is called *constraint
Hamiltonian EDS* related with $\bar{\alpha}$, and is denoted by $\mathcal{H}_{\bar{\alpha}}$.

Definition 4.12. Let $[[\iota^*d\Theta_\lambda]]$ be a constrained Lagrangian system.
Then for any representative $\bar{\alpha}$ of the class $[[\iota^*d\Theta_\lambda]]$, equations for *holo-
nomic* integral sections of the constraint Hamiltonian exterior differential
system $\mathcal{H}_{\bar{\alpha}}$, i.e., the equations

$$J^1\gamma^*i_\xi\bar{\alpha} = 0 \quad \text{for every } \pi_1\text{-vertical vector field } \xi \in \mathcal{C}, \tag{4.64}$$

where $\gamma : W \to Y$, $J^1\gamma(W) \subset Q$, are called *constrained Euler–Lagrange equations*. Solutions of constrained Euler–Lagrange equations are called *constrained extremals*.

We note that (on an open subset of Q) constrained Euler–Lagrange equations do not depend upon the choice of a representative $\bar{\alpha}$ of the class $[[\iota^* d\Theta_\lambda]]$. This means that with help of a local Lagrangian λ for E we can write the constrained Euler–Lagrange equations in the form

$$J^1\gamma^* i_\xi(\iota^* d\Theta_\lambda) = 0 \quad \text{for every } \pi_1\text{-vertical vector field } \xi \in \mathcal{C}, \qquad (4.65)$$

where $\gamma : W \to Y$, $J^1\gamma(W) \subset Q$.

For $\lambda = L\omega_0$ denote

$$\bar{L} = L \circ \iota, \quad \bar{L}_a^j = \frac{\partial L}{\partial y_j^{m-k+a}} \circ \iota, \qquad (4.66)$$

considered as functions of adapted fibered coordinates (x^i, y^σ, z^J) on Q, and put

$$\Theta_{\iota^*\lambda} = \bar{L}\,\omega_0 + \frac{\partial \bar{L}}{\partial y_j^s}\,\bar{\omega}^s \wedge \omega_j = \bar{L}\,\omega_0 + \frac{\partial \bar{L}}{\partial z^J}\frac{\partial z^J}{\partial y_j^s}\,\bar{\omega}^s \wedge \omega_j. \qquad (4.67)$$

In keeping with notations in Remark 4.2 we can easily find the following relation:

Proposition 4.7.

$$\iota^*\Theta_\lambda = \Theta_{\iota^*\lambda} + \bar{L}_a^i \mathcal{C}_{is}^{aj}\,\bar{\omega}^s \wedge \omega_j + \bar{L}_a^j\,\bar{\varphi}^a \wedge \omega_j. \qquad (4.68)$$

For convenience of notations let us introduce the *$\mathcal{C}$-modified Euler–Lagrange operator* and *cut $\mathcal{C}$-modified Euler–Lagrange operator*, respectively:

$$
\begin{aligned}
\mu_s &= \frac{\partial_c}{\partial y^s} - \frac{d_c}{dx^i}\left(\frac{\partial}{\partial y_i^s}\right) - \frac{\partial_c g_j^r}{\partial y^s}\frac{\partial}{\partial y_j^r} \\
&= \frac{\partial_c}{\partial y^s} - \frac{d_c}{dx^i}\left(\frac{\partial z^J}{\partial y_i^s}\frac{\partial}{\partial z^J}\right) - \frac{\partial_c g_j^r}{\partial y^s}\frac{\partial z^J}{\partial y_j^r}\frac{\partial}{\partial z^J}, \\
\mu_s' &= \frac{\partial_c}{\partial y^s} - \frac{d_c'}{dx^i}\left(\frac{\partial}{\partial y_i^s}\right) - \frac{\partial_c g_j^r}{\partial y^s}\frac{\partial}{\partial y_j^r} \\
&= \frac{\partial_c}{\partial y^s} - \frac{d_c'}{dx^i}\left(\frac{\partial z^J}{\partial y_i^s}\frac{\partial}{\partial z^J}\right) - \frac{\partial_c g_j^r}{\partial y^s}\frac{\partial z^J}{\partial y_j^r}\frac{\partial}{\partial z^J}.
\end{aligned}
\qquad (4.69)
$$

Theorem 4.5. *Let λ be a Lagrangian in J^1Y, $Q \subset J^1Y$ a regular nonholonomic constraint. Denote by γ local sections of the fibered manifold $\pi : Y \to X$ such that $J^1\gamma(W) \subset Q$. In adapted fibered coordinates, the constrained Euler–Lagrange equations take one of the following equivalent*

forms:

(1) *By means of L,*

$$(\mathcal{A}_s + \mathcal{B}_{sJ}^{\ i} z_i^J) \circ J^2\gamma = 0, \tag{4.70}$$

where $\mathcal{A}_s$, $\mathcal{B}_{sJ}^{\ i}$ *are given by*

$$\mathcal{A}_s = \bar{A}_s + \bar{A}_{m-k+a}\bar{G}_s^a + (\bar{B}_{s\nu}^{ji} + \bar{B}_{m-k+a\,\nu}^{ji}\bar{G}_s^a)\frac{d_c'g_j^\nu}{dx^i},$$
$$\mathcal{B}_{sJ}^{\ i} = (\bar{B}_{s\nu}^{ji} + \bar{B}_{m-k+a\,\nu}^{ji}\bar{G}_s^a)\frac{\partial g_j^\nu}{\partial z^J}, \tag{4.71}$$

where (*cf.* (4.11))

$$\bar{A}_\sigma = \varepsilon_\sigma'(L) \circ \iota, \quad \bar{B}_{\sigma\nu}^{ij} = -\left(\frac{\partial^2 L}{\partial y_i^\sigma \partial y_j^\nu}\right) \circ \iota \tag{4.72}$$

(2) *By means of* $\bar{L}$ *and* $\bar{L}_a^j$,

$$\left(\mu_s(\bar{L}) - \bar{L}_a^j\,\mu_s(g_j^{m-k+a}) - \mathcal{C}_{js}^{ai}\frac{d_c\bar{L}_a^j}{dx^i}\right) \circ J^2\gamma = 0, \tag{4.73}$$

meaning that the functions $\mathcal{A}_s$, $\mathcal{B}_{sJ}^{\ i}$ *are equivalently expressed as follows:*

$$\mathcal{A}_s = \mu_s'(\bar{L}) - \bar{L}_a^j\,\mu_s'(g_j^{m-k+a}) - \mathcal{C}_{js}^{ai}\frac{d_c'\bar{L}_a^j}{dx^i},$$
$$\mathcal{B}_{sJ}^{\ i} = -\frac{\partial}{\partial z^J}\left(\frac{\partial\bar{L}}{\partial z^K}\frac{\partial z^K}{\partial y_i^s}\right) + \bar{L}_a^i\frac{\partial}{\partial z^J}(\bar{G}_s^a\delta_i^j - \mathcal{C}_{is}^{aj}) - \mathcal{C}_{js}^{ai}\frac{\partial\bar{L}_a^j}{\partial z^J} \tag{4.74}$$
$$= -\frac{\partial}{\partial z^J}\left(\frac{\partial\bar{L}}{\partial z^K}\frac{\partial z^K}{\partial y_i^s}\right) + \bar{L}_a^j\frac{\partial}{\partial z^J}\left(\frac{\partial g_j^{m-k+a}}{\partial z^K}\frac{\partial z^K}{\partial y_i^s}\right) - \mathcal{C}_{js}^{ai}\frac{\partial\bar{L}_a^j}{\partial z^J}.$$

Definition 4.13. The operator defined by (4.73), i.e.

$$\mathcal{E}_s^{\mathcal{C}}(\bar{L}, \bar{L}_a^j) = \mu_s(\bar{L}) - \bar{L}_a^j\,\mu_s(g_j^{m-k+a}) - \mathcal{C}_{js}^{ai}\frac{d_c\bar{L}_a^j}{dx^i} \tag{4.75}$$

is called the *constraint Euler–Lagrange operator.*

Remark 4.3. Lagrangian and semiholonomic constraints. Recall that if the constraint Q in J^1Y is *Lagrangian,* then

$$\mathcal{C}_{js}^{ai} = 0, \quad \mathcal{C}_{Jj}^a = 0 \tag{4.76}$$

for all values of indices. Consequently, formulas become much simpler. In particular,

$$\iota^*\Theta_\lambda = \Theta_{\iota^*\lambda} + \bar{L}_a^j\,\bar{\varphi}^a \wedge \omega_j, \tag{4.77}$$

and the constraint Euler–Lagrange operator reads

$$\mathcal{E}_s^{\mathcal{C}}(\bar{L}, \bar{L}_a^j) = \mu_s(\bar{L}) - \bar{L}_a^j\,\mu_s(g_j^{m-k+a}). \tag{4.78}$$

If, moreover, Q is a *semiholonomic constraint*, i.e., if $d\bar{\varphi}^a \in \mathcal{I}(\mathcal{C}^0)$ for all a, we get

$$\iota^* d\Theta_\lambda = d\Theta_{\iota^*\lambda} + \text{a constraint form.} \tag{4.79}$$

This means that $d\Theta_{\iota^*\lambda} \approx \iota^* d\Theta_\lambda$, and even that $d\Theta_{\iota^*\lambda}$ *is a constrained Poincaré–Cartan* $(n + 1)$-*form of* λ. Then, of course, the constrained Euler–Lagrange equations (4.65) have the equivalent form

$$J^1\gamma^* i_\xi d\Theta_{\iota^*\lambda} = 0 \quad \text{for every } \pi_1\text{-vertical vector field } \xi \in \mathcal{C}. \tag{4.80}$$

Since in this case $\kappa = kn$, we have on Q local coordinates (x^i, y^σ, y_j^s), and all formulas take a much simpler form (cf. e.g. [22, 25]).

4.5. **Constrained Hamilton–De Donder equations.** Let $[\alpha] = [d\Theta_\lambda]$ be a Lagrangian system on J^1Y, $\iota : Q \to J^1Y$ a regular non-holonomic constraint of corank (κ, k), $[[\iota^*\alpha]] = [[\iota^* d\Theta_\lambda]]$ the corresponding constrained system on Q. For every $\bar{\alpha} = \iota^*\alpha \in [[\iota^* d\Theta_\lambda]]$ we have the constraint Hamiltonian exterior differential system $\mathcal{H}_{\bar{\alpha}}$ defined on the domain of definition of $\bar{\alpha}$, say $U \subset Q$, and generated by the system of n-forms and 1-forms (4.63).

Directly from the definition of constraint Hamiltonian EDS we can see that if $\bar{\alpha}_1$ and $\bar{\alpha}_2$ differ by a constraint form, then $\mathcal{H}_{\bar{\alpha}_1} = \mathcal{H}_{\bar{\alpha}_2}$.

Definition 4.14. Let $[d\Theta_\lambda]$ be a Lagrangian system on J^1Y. For every $\alpha \in [d\Theta_\lambda]$ the equivalence class

$$\bar{\alpha}_\mathcal{I} = \iota^*\alpha \bmod \mathcal{I}^{n+1}(\mathcal{C}^0) \tag{4.81}$$

is called *constrained Hamiltonian system* related with α and the constraint Q.

Equations for integral sections of $\mathcal{H}_{\bar{\alpha}}$, i.e.

$$\delta^* i_\xi \bar{\alpha} = 0 \quad \forall \xi \in \mathcal{V}(Q) \cap \mathcal{C}, \quad \delta^* \bar{\varphi}^a = 0, \ 1 \le a \le k, \tag{4.82}$$

where $\bar{\alpha} \in \bar{\alpha}_\mathcal{I}$ and δ is a section of $\pi_1|_Q : Q \to X$, are called *constrained Hamilton equations*.

Integral sections of $\mathcal{H}_{\bar{\alpha}}$ are called *constrained Hamilton extremals* of $[[\iota^* d\Theta_\lambda]]$.

Similarly as in the unconstrained case, in what follows, we will be interested in *constrained Hamiltonian systems* that can be *completely characterized by constrained Poincaré–Cartan* $(n + 1)$-*forms*:

Definition 4.15. Let $[\alpha]$ be a Lagrangian system on J^1Y, $\iota : Q \to J^1Y$ a regular non-holonomic constraint of corank (κ, k). A constrained

Hamiltonian system $\bar{\alpha}_{\mathcal{I}}$ defined on $U \subset Q$ is called *constrained Hamilton–De Donder system* of $[\alpha]$ if there exists a Lagrangian λ for $[\alpha]$ such that for every $\bar{\alpha} \in \bar{\alpha}_{\mathcal{I}}$,

$$\bar{\alpha} = \iota^* d\Theta_\lambda + \bar{F}, \quad \text{where} \quad \bar{F} \in \Omega_Y^{n+1,(\geq 2)}(U). \tag{4.83}$$

The corresponding constrained Hamilton equations, i.e.

$$\delta^* i_\xi(\iota^* d\Theta_\lambda + \bar{F}) = 0 \quad \forall \xi \in \mathcal{V}(Q) \cap \mathcal{C}, \quad \delta^* \bar{\varphi}^a = 0, \quad 1 \leq a \leq k, \tag{4.84}$$

are called *constrained Hamilton–De Donder equations*.

Definition 4.16. A constrained Hamilton–De Donder system $\bar{\alpha}_{\mathcal{I}}$ is called *regular* if $\mathcal{H}_{\bar{\alpha}}$ contains all the canonical contact n-forms

$$\iota^* \omega^\sigma \wedge \omega_i, \quad 1 \leq \sigma \leq m, \quad 1 \leq i \leq n. \tag{4.85}$$

A constrained Lagrangian system is called *De Donder regular* if around each point in Q there exists an associated regular constrained Hamilton–De Donder system.

Proposition 4.8. *Let $[\alpha]$ be a Lagrangian system on J^1Y, $\iota : Q \to J^1Y$ a regular non-holonomic constraint of corank (κ, k), $\bar{\alpha}_{\mathcal{I}}$ an associated regular constrained Hamilton–De Donder system. Then (for all $\bar{\alpha} \in \bar{\alpha}_{\mathcal{I}}$) every integral section of $\mathcal{H}_{\bar{\alpha}}$ is holonomic. Consequently, constrained Hamilton–De Donder equations of $\bar{\alpha}_{\mathcal{I}}$ are equivalent with the constrained Euler–Lagrange equations.*

Proof. Let δ be an integral section of $\mathcal{H}_{\bar{\alpha}}$. If $\bar{\alpha}$ is regular then, by definition,

$$\delta^*(\bar{\omega}^\sigma \wedge \omega_i) = 0, \quad 1 \leq \sigma \leq m, \quad 1 \leq i \leq n. \tag{4.86}$$

This implies, however, that for all σ, $\delta^* \bar{\omega}^\sigma = 0$, meaning that δ is a holonomic section in Q. $\qquad\qquad\square$

Theorem 4.6. *Let $\bar{\alpha}_{\mathcal{I}} = \iota^* d\Theta_\lambda + \bar{F} \bmod \mathcal{I}^{n+1}(\mathcal{C}^0)$, where $\bar{F} \in \Omega_Y^{n+1,(\geq 2)}(Q)$, be a constrained Hamilton–De Donder system. The following conditions are equivalent:*

(1) *For every $\bar{\alpha} \in \bar{\alpha}_{\mathcal{I}}$, $\bar{\alpha}$ is regular.*
(2) *For every $\bar{\alpha} \in \bar{\alpha}_{\mathcal{I}}$, a system of generators of $\mathcal{H}_{\bar{\alpha}}$ has maximal rank (equal to $(n+1)(m-k)+k$).*
(3) *Every first-order Lagrangian λ for $d\Theta_\lambda$ satisfies the constraint regularity condition*

$$\operatorname{rank}(\mathcal{B}^i_{sJ}) = n(m-k), \tag{4.87}$$

where $\mathcal{B}^i_{sJ}$ are given in terms of $\lambda = L\omega_0$ by (4.71) or (4.74).

Proof. We can write

$$\bar{\alpha} = \mathcal{A}_s \bar{\omega}^s \wedge \omega_0 + \mathcal{B}^i_{sJ} \bar{\omega}^s \wedge dz^J \wedge \omega_i + \mathcal{F} + \varphi, \tag{4.88}$$

where $\mathcal{A}_s$ and $\mathcal{B}^i_{sJ}$ are given in terms of a first-order Lagrangian λ for $d\Theta_\lambda$ by (4.71) or (4.74), $\mathcal{F}$ is the sum of $\bar{F}$ and the 2-contact part of $\iota^* d\Theta_\lambda$, and $\varphi \in \mathcal{I}^{n+1}(\mathcal{C}^0)$. Denote

$$\mathcal{F} = \mathcal{F}^i_{ls} \bar{\omega}^l \wedge \bar{\omega}^s \wedge \omega_i + \nu, \quad \text{where} \quad \mathcal{F}^i_{ls} = -\mathcal{F}^i_{sl}. \tag{4.89}$$

Computing generators of $\mathcal{H}_{\bar{\alpha}}$, we obtain a mixed system of k (linearly independent) 1-forms

$$\bar{\varphi}^a = \bar{\omega}^{m-k+a} - \sum_{s=1}^{m-k} \bar{G}^a_s \bar{\omega}^s, \quad 1 \le a \le k, \tag{4.90}$$

and $m - k + mn - \kappa$ n-forms (that, in general, need not be independent) as follows:

$$\begin{aligned} & \mathcal{A}_s \omega_0 + 2\mathcal{F}^i_{sl} \bar{\omega}^l \wedge \omega_i + \mathcal{B}^i_{sJ} dz^J \wedge \omega_i + \mu_s, \\ & \mathcal{B}^i_{sJ} \bar{\omega}^s \wedge \omega_i, \end{aligned} \tag{4.91}$$

$1 \le s \le m - k$, $1 \le J \le mn - \kappa$, where $\mu_s = i_{\partial/\partial y^s} \nu$.

Suppose that $\bar{\alpha}_\mathcal{I}$ is regular. Then $\mathcal{H}_{\bar{\alpha}}$ is generated by the forms $\bar{\varphi}^a$ and

$$\mathcal{A}_s \omega_0 + \mathcal{B}^i_{sJ} dz^J \wedge \omega_i + \mu_s, \quad \bar{\omega}^s \wedge \omega_i, \tag{4.92}$$

which means that the matrix $\mathcal{B}_1 = (\mathcal{B}^i_{sJ})$ with $mn - \kappa$ rows labelled by J and $n(m - k)$ columns labelled by i, s, has rank $n(m - k)$. Now, the rank of the system of generators of $\mathcal{H}_{\bar{\alpha}}$ is $> n(m - k)$. Let us compute the rank of the matrix $\mathcal{B}_2 = (\mathcal{B}^i_{sJ})$ with $m - k$ rows labelled by s and $n(nm - \kappa)$ columns labelled by i, J. By the above we can see that for every fixed i, the matrix $\mathcal{B}_1$ has $m - k$ linearly independent columns (labelled by s). Consequently, $\mathcal{B}_2$ has for every fixed i the submatrix $(\mathcal{B}^\bullet_{sJ})$ with $m - k$ independent rows labelled by s (equal to transposed submatrix of $\mathcal{B}_2$ with the corresponding values of indices), hence rank $\mathcal{B}_2 = m - k$. Summarizing, we have obtained that the forms (4.91) (resp. (4.92)) are independent, meaning that the rank of the system of generators of $\mathcal{H}_{\bar{\alpha}}$ (for all $\bar{\alpha} \in \bar{\alpha}_\mathcal{I}$) is maximal and equal to $n(m - k) + m - k + k = (n + 1)(m - k) + k$, as desired.

Next, suppose that the rank of $\mathcal{H}_{\bar{\alpha}}$ is maximal. Then the forms (4.91) are independent, meaning that the constrained regularity condition (4.87) holds.

Finally, if (4.87) holds, then the forms $\mathcal{B}^i_{sJ} \bar{\omega}^s \wedge \omega_i$ in (4.91) are independent, i.e. the forms $\bar{\omega}^s \wedge \omega_i$, $1 \le i \le n$, $1 \le s \le m - k$, belong to $\mathcal{H}_{\bar{\alpha}}$.

By definition of $\mathcal{H}_{\bar{\alpha}}$ also all the n-forms $\bar{\varphi}^a \wedge \omega_i \in \mathcal{H}_{\bar{\alpha}}$. Hence, for all a, i, we have $\omega^{m-k+a} \wedge \omega_i = \bar{\varphi}^a \wedge \omega_i + \bar{G}^a_s \bar{\omega}^s \wedge \omega_i \in \mathcal{H}_{\bar{\alpha}}$, and we are done. $\square$

Corollary 4.3. *Let*

$$\bar{\alpha}_{\mathcal{I}} = \iota^* d\Theta_\lambda + \bar{F} \mod \mathcal{I}^{n+1}(\mathcal{C}^0), \ \text{where } \bar{F} \in \Omega_Y^{n+1,(\geq 2)}(Q),$$

be a regular constrained Hamilton–De Donder system. Then Hamilton extremals of $\bar{\alpha}_{\mathcal{I}}$ (i.e. integral sections of (4.90) and (4.92)) do not depend upon the choice of $\bar{F}$. This means that Hamilton–De Donder equations of all elements in the class

$$\iota^* d\Theta_\lambda \mod \Omega_Y^{n+1,(\geq 2)}(Q) + \mathcal{I}^{n+1}(\mathcal{C}^0) \tag{4.93}$$

are equivalent.

From the proof of Theorem 4.6 we can conclude that if $\bar{\alpha}_{\mathcal{I}}$ is regular then the matrix $\mathcal{B}_1$ has $nm - \kappa$ rows and $n(m-k)$ columns where $nm - \kappa \geq n(m-k)$. This means that $\kappa \leq nk$, and we get the following result:

Corollary 4.4. *Let $[\alpha]$ be a Lagrangian system on J^1Y, $\iota : Q \to J^1Y$ a regular non-holonomic constraint of corank (κ, k). A necessary condition for the constrained Lagrangian system $[[\iota^*\alpha]]$ be De Donder regular is*

$$\kappa \leq nk. \tag{4.94}$$

4.6. **Constraint Legendre transformation.**

Theorem 4.7. *Let $\iota : Q \to J^1Y$ be a regular non-holonomic constraint of corank (κ, k), $[[\bar{\alpha}]]$ a constrained Lagrangian system. Let $x \in Q$ be a point. Suppose that in a neighborhood of x,*

$$\frac{\partial \mathcal{B}^i_{sJ}}{\partial z^K} = \frac{\partial \mathcal{B}^i_{sK}}{\partial z^J}, \quad 1 \leq i \leq n, \ 1 \leq s \leq m-k, \ 1 \leq J, K \leq mn - \kappa. \tag{4.95}$$

Then there exists a neighborhood $U \subset Q$ of x, and, on U, functions P^i_s, $1 \leq i \leq n$, $1 \leq s \leq m - k$, and a n-form η, such that the class $[[\bar{\alpha}]]$ is represented by the $(n+1)$-form

$$\bar{\alpha} = \eta \wedge \omega_0 + dP^i_s \wedge dy^s \wedge \omega_i. \tag{4.96}$$

Proof. As we have seen in Sec. 4.4, around each point in Q, the constrained Lagrangian system $[[\bar{\alpha}]]$ has a representative

$$\mathcal{A}_s \bar{\omega}^s \wedge \omega_0 + \mathcal{B}^i_{sJ} \bar{\omega}^s \wedge dz^J \wedge \omega_i, \tag{4.97}$$

where $\mathcal{A}_s$ and $\mathcal{B}^i_{sJ}$ are defined by (4.71) or (4.74). Assume that the given Lagrangian system $[\bar{\alpha}]$ has a Lagrangian $\lambda = L\omega_0$ defined around x such that for the corresponding functions $\mathcal{B}^i_{sJ}$ integrability conditions (4.95) are satisfied. Applying the Poincaré Lemma we get a neighborhood

$U \subset Q$ of x and functions P_s^i, $1 \le i \le n$, $1 \le s \le m - k$, on U that are given by

$$\mathcal{B}_{sJ}^i = -\frac{\partial P_s^i}{\partial z^J}. \qquad (4.98)$$

Hence, in the class $[[\bar{\alpha}]]$ on U we can find the following representatives, equivalent with $\iota^* d\Theta_\lambda$, where λ is the above mentioned Lagrangian:

$$
\begin{aligned}
&\mathcal{A}_s \bar{\omega}^s \wedge \omega_0 + \frac{\partial P_s^i}{\partial z^J} dz^J \wedge \bar{\omega}^s \wedge \omega_i \\
&\approx \mathcal{A}_s \bar{\omega}^s \wedge \omega_0 + dP_s^i \wedge \bar{\omega}^s \wedge \omega_i - \frac{d_c' P_s^i}{dx^j} dx^j \wedge \bar{\omega}^s \wedge \omega_i \\
&= \left(\mathcal{A}_s + \frac{d_c' P_s^i}{dx^i} \right) dy^s \wedge \omega_0 - g_i^s dP_s^i \wedge \omega_0 + dP_s^i \wedge dy^s \wedge \omega_i \qquad (4.99) \\
&\approx \left(\mathcal{A}_s + \frac{d_c' P_s^i}{dx^i} - g_i^l \frac{\partial_c P_l^i}{\partial y^s} \right) dy^s \wedge \omega_0 \\
&\quad - g_i^s \frac{\partial P_s^i}{\partial z^J} dz^J \wedge \omega_0 + dP_s^i \wedge dy^s \wedge \omega_i.
\end{aligned}
$$

In this way we have obtained a representative

$$
\begin{aligned}
\bar{\alpha} = &\left(\mathcal{A}_s + \frac{d_c' P_s^i}{dx^i} - g_i^r \frac{\partial_c P_r^i}{\partial y^s} \right) dy^s \wedge \omega_0 \\
&- g_i^s \frac{\partial P_s^i}{\partial z^J} dz^J \wedge \omega_0 + dP_s^i \wedge dy^s \wedge \omega_i.
\end{aligned}
\qquad (4.100)
$$

Let us denote

$$\bar{\alpha} = \eta \wedge \omega_0 + dP_s^i \wedge dy^s \wedge \omega_i, \qquad (4.101)$$

with

$$\eta = \tilde{\eta}_j dx^j + \bar{\eta}_s dy^s + \bar{\eta}_J dz^J, \qquad (4.102)$$

where $\tilde{\eta}_j$, $1 \le j \le n$, are arbitrary functions on U, and

$$
\begin{aligned}
\bar{\eta}_s &= \mathcal{A}_s + \frac{d_c' P_s^i}{dx^i} - g_i^r \frac{\partial_c P_r^i}{\partial y^s}, \quad 1 \le r, s \le m - k, \\
\bar{\eta}_J &= -g_i^s \frac{\partial P_s^i}{\partial z^J}, \quad 1 \le s \le m - k, \; 1 \le J \le mn - \kappa.
\end{aligned}
\qquad (4.103)
$$

This completes the proof. $\square$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Corollary 4.5. *The class*

$$\bar{\alpha}_{\mathcal{I}} = \eta \wedge \omega_0 + dP_s^i \wedge dy^s \wedge \omega_i \mod \mathcal{I}^{n+1}(\mathcal{C}^0) \qquad (4.104)$$

constructed in the above Theorem is a constrained Hamilton–De Donder system, corresponding to the Lagrangian $\lambda = L\omega_0$.

Proof. It is sufficient to check the computations to see that that $\bar{\alpha} - \iota^* d\Theta_\lambda$ is (up to a constraint form) a 2-contact form, horizontal with respect to the projection onto Y. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Note that η in (4.104) is determined up to a constraint form. In this way, for a constrained Hamilton–De Donder system we have to consider the class

$$\eta_{\mathcal{I}} = \eta \bmod \mathcal{I}^n(\mathcal{C}^0). \tag{4.105}$$

It should be stressed that *in the class $\eta_{\mathcal{I}}$ there need not exist a closed representative.*

Remark 4.4. Integrability condition (4.95) can be rewritten in terms of a Lagrangian $\lambda = L\omega_0$ giving rise to the functions $\mathcal{B}^i_{sJ}$ using (4.71) or (4.74). For example, with help of (4.74) it takes the form

$$
\frac{\partial \bar{L}^j_a}{\partial z^K} \frac{\partial}{\partial z^J} \left(\frac{\partial g^{m-k+a}_j}{\partial z^I} \frac{\partial z^I}{\partial y^s_i} \right) - \frac{\partial \mathcal{C}^{ai}_{js}}{\partial z^K} \frac{\partial \bar{L}^j_a}{\partial z^J} \\
= \frac{\partial \bar{L}^j_a}{\partial z^J} \frac{\partial}{\partial z^K} \left(\frac{\partial g^{m-k+a}_j}{\partial z^I} \frac{\partial z^I}{\partial y^s_i} \right) - \frac{\partial \mathcal{C}^{ai}_{js}}{\partial z^J} \frac{\partial \bar{L}^j_a}{\partial z^K}.
\tag{4.106}
$$

Let us find an explicit formula for the functions P^i_s.

Proposition 4.9. *Let $x \in U$, and consider a mapping $\chi : [0,1] \times W \to W$, defined by*

$$(u, x^i, y^\sigma, z^J) \to (x^i, y^\sigma, uz^J), \tag{4.107}$$

where $W \subset U \subset Q$ is an appropriate neighborhood of x. Then for arbitrary functions $\psi^i_s(x^j, y^\nu)$ (resp. $\tilde{\psi}^i_s(x^j, y^\nu)$), $1 \le s \le m-k$, $1 \le i \le n$, the functions

$$
P^i_s = -z^J \int_0^1 \left(\mathcal{B}^i_{sJ} \circ \chi \right) du + \psi^i_s(x^j, y^\nu) \\
= \frac{\partial \bar{L}}{\partial z^K} \frac{\partial z^K}{\partial y^s_i} + z^J \int_0^1 \left(\mathcal{C}^{ai}_{js} \frac{\partial \bar{L}^j_a}{\partial z^J} - \bar{L}^j_a \frac{\partial}{\partial z^J} \left(\frac{\partial g^{m-k+a}_j}{\partial z^K} \frac{\partial z^K}{\partial y^s_i} \right) \right) \circ \chi \, du \\
+ \tilde{\psi}^i_s(x^j, y^\nu),
\tag{4.108}
$$

are solutions of (4.98).

Proof. Integrability condition (4.95) guarantees that in a neighborhood of any point of U one can find solutions of (4.98) using Poincaré Lemma. Put

$$P^i_s = -z^J \int_0^1 \left(\mathcal{B}^i_{sJ} \circ \chi \right) du + \psi^i_s(x^j, y^\nu). \tag{4.109}$$

Then with help of (4.95)

$$
\frac{\partial P^i_s}{\partial z^J} = -\int_0^1 \left(\mathcal{B}^i_{sJ} \circ \chi \right) du - z^K \int_0^1 \left(\frac{\partial \mathcal{B}^i_{sK}}{\partial z^J} \circ \chi \right) u \, du \\
= -\int_0^1 d(u(\mathcal{B}^i_{sJ} \circ \chi)) = -\mathcal{B}^i_{sJ},
\tag{4.110}
$$

as desired.

Using formula (4.74) and (4.109) we get

$$
\begin{aligned}
P_s^i &= z^J \int_0^1 \left(\frac{\partial}{\partial z^J} \left(\frac{\partial \bar{L}}{\partial z^K} \frac{\partial z^K}{\partial y_i^s} \right) + \mathcal{C}_{js}^{ai} \frac{\partial \bar{L}_a^j}{\partial z^J} \right. \\
&\quad \left. - \bar{L}_a^j \frac{\partial}{\partial z^J} \left(\frac{\partial g_j^{m-k+a}}{\partial z^K} \frac{\partial z^K}{\partial y_i^s} \right) \right) \circ \chi \, du + \psi_s^i(x^j, y^\nu) \\
&= \frac{\partial \bar{L}}{\partial z^K} \frac{\partial z^K}{\partial y_i^s} + z^J \int_0^1 \left(\mathcal{C}_{js}^{ai} \frac{\partial \bar{L}_a^j}{\partial z^J} - \bar{L}_a^j \frac{\partial}{\partial z^J} \left(\frac{\partial g_j^{m-k+a}}{\partial z^K} \frac{\partial z^K}{\partial y_i^s} \right) \right) \circ \chi \, du \\
&\quad + \tilde{\psi}_s^i(x^j, y^\nu),
\end{aligned}
\tag{4.111}
$$

since

$$
\begin{aligned}
\int_0^1 d \left(\frac{\partial \bar{L}}{\partial z^K} \frac{\partial z^K}{\partial y_i^s} \circ \chi \right) &= \left[\frac{\partial \bar{L}}{\partial z^K} \frac{\partial z^K}{\partial y_i^s} \circ \chi \right]_{u=0}^{u=1} = \frac{\partial \bar{L}}{\partial z^K} \frac{\partial z^K}{\partial y_i^s} - h_s^i(x^j, y^\nu) \\
&= \int_0^1 \frac{d}{du} \left(\frac{\partial \bar{L}}{\partial z^K} \frac{\partial z^K}{\partial y_i^s} \circ \chi \right) du = z^J \int_0^1 \left(\frac{\partial}{\partial z^J} \left(\frac{\partial \bar{L}}{\partial z^K} \frac{\partial z^K}{\partial y_i^s} \right) \right) \circ \chi \, du.
\end{aligned}
$$

This completes the proof. $\square$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

Keeping the above notations we can introduce the following concepts:

Definition 4.17. The local representative $\bar{\alpha}$ (4.96) of the constrained Lagrangian system is called a representative in *canonical form*. Functions P_s^i are called *constraint momenta*, and (any) 1-form η in (4.104) is called *energy 1-form* associated with the corresponding Hamilton–De Donder system $\bar{\alpha}_\mathcal{I}$.

By the next theorem, *constraint Legendre transformation*, associated with a regular constrained Hamilton–De Donder system is defined. It is a local coordinate transformation on the constraint submanifold Q.

Theorem 4.8. *Let $[[\bar{\alpha}]]$ be a De Donder regular constrained Lagrangian system on Q, let $\bar{\alpha}_\mathcal{I}$ be a corresponding regular Hamilton–De Donder system on a coordinate neighborhood $U \subset Q$, P_s^i, $1 \le i \le n$, $1 \le s \le m - k$, its associated constraint momenta. Then the set of functions (x^i, y^σ, P_s^i) can be completed to coordinates on U. In particular, the set of $nm - \kappa$ indices labelled by J has a subset labelled by B, $n(m-k)+1 \le B \le nm - \kappa$, such that*

$$
(x^i, y^\sigma, z^J) \to (x^i, y^\sigma, P_s^i, z^B)
\tag{4.112}
$$

is a coordinate transformation on U.

Proof. By assumption, the matrix $(\mathcal{B}_{sJ}^i) = (\partial P_s^i / \partial z^J)$ has maximal rank (equal to $n(m - k)$). This means that it has a regular submatrix with

$n(m − k)$ linearly independent rows: we can label them by A, where $1 \le A \le n(m − k)$. The remaining rows will be labelled by B, i.e. $n(m−k)+1 \le B \le nm−\kappa$. Now, the map $(x^i, y^\sigma, z^J) \to (x^i, y^\sigma, P^i_s, z^B)$ is regular, i.e. is a coordinate transformation on U. $\square$ $\qquad\qquad$ $\square$

Finally, let us find the expression of *constrained Hamilton–De Donder equations in constraint Legendre coordinates.* First, we have up to a constraint form,

$$\eta = \eta_j dx^j + \eta_s dy^s + \eta^s_i\, dP^i_s + \eta_B dz^B, \qquad (4.113)$$

hence,

$$\eta = \eta_i dx^i + \eta_s dy^s + \eta^s_i \left(\frac{d'_c P^i_s}{dx^j} dx^j + \frac{\partial_c P^i_s}{\partial y^r}\bar\omega^r + \frac{\partial P^i_s}{\partial z^J} dz^J \right) + \eta_B dz^B$$
$$= \left(\eta_j + \eta^s_i \frac{d'_c P^i_s}{dx^j} - \eta^s_i \frac{\partial_c P^i_s}{\partial y^r} g^r_j \right) dx^j + \left(\eta_r + \eta^s_i \frac{\partial_c P^i_s}{\partial y^r} \right) dy^r \qquad (4.114)$$
$$+ \eta^s_i \frac{\partial P^i_s}{\partial z^A} dz^A + \left(\eta^s_i \frac{\partial P^i_s}{\partial z^B} + \eta_B \right) dz^B.$$

Comparing with (4.102), (4.103) we obtain

$$\bar\eta_r = \eta_r + \eta^s_i \frac{\partial_c P^i_s}{\partial y^r} = \mathcal{A}_r + \frac{d'_c P^i_r}{dx^i} - g^s_i \frac{\partial_c P^i_s}{\partial y^r},$$
$$\bar\eta_A = \eta^s_i \frac{\partial P^i_s}{\partial z^A} = -g^s_i \frac{\partial P^i_s}{\partial z^A}, \qquad (4.115)$$
$$\bar\eta_B = \eta^s_i \frac{\partial P^i_s}{\partial z^B} + \eta_B = -g^s_i \frac{\partial P^i_s}{\partial z^B}.$$

The second relation gives us

$$(\eta^s_i + g^s_i) \frac{\partial P^i_s}{\partial z^A} = 0, \quad \text{i.e.} \quad \eta^s_i = -g^s_i(x^j, y^\nu, P^j_l, z^B), \qquad (4.116)$$

since the matrix $\left(\dfrac{\partial P^i_s}{\partial z^A} \right)$ is regular. The first and third relation above then read

$$\eta_r = \bar\eta_r + g^s_i \frac{\partial_c P^i_s}{\partial y^r} = \mathcal{A}_r + \frac{d'_c P^i_r}{dx^i}, \quad \eta_B = 0. \qquad (4.117)$$

Next, we can see that the vertical subbundle of the canonical distribution is in Legendre coordinates generated by the vector fields $\partial_c/\partial y^s$, $\partial/\partial P^i_s$, and $\partial/\partial z^B$. Computing contractions of a representative $\bar\alpha$ of (4.104) by these vector fields, we get the constrained Hamilton–De Donder equations $\delta^* i_\xi \bar\alpha = 0$, $\delta^* \bar\varphi^a = 0$ in the following "canonical form":

Theorem 4.9. *Constrained Hamilton–De Donder equations* (4.84) *in constraint Legendre coordinates take the form*

$$\frac{\partial(P_s^i \circ \delta)}{\partial x^i} = \eta_s \circ \delta, \quad \frac{\partial(y^s \circ \delta)}{\partial x^i} = -\eta_i^s \circ \delta,$$
$$1 \leq s \leq m - k, \ 1 \leq i \leq n, \tag{4.118}$$

$$\frac{\partial(y^{m-k+a} \circ \delta)}{\partial x^i} = g_i^{m-k+a} \circ \delta, \quad 1 \leq a \leq k, \ 1 \leq i \leq n. \tag{4.119}$$

Remark 4.5. In view of Theorem 4.8 we can see that for a regular constrained Lagrangian system one has on Q local adapted coordinates $(x^i, y^\sigma, y_j^s, z^B)$, where $1 \leq i \leq n$, $1 \leq \sigma \leq m$, $1 \leq s \leq k$, $n(m - k) + 1 \leq B \leq nm - \kappa$, and corresponding adapted bases of 1-forms $(dx^i, dy^s, \bar\varphi^a, dy_j^s, dz^B)$, respectively, $(dx^i, \bar\omega^s, \bar\varphi^a, dy_j^s, dz^B)$. In these coordinates many formulas simplify, since $z^A = y_j^s$ (hence $\partial z^B / \partial y_j^s = 0$).

The results on constrained Hamilton–De Donder systems can be easily reformulated for special cases of constraints. Let us recall the properties of semiholonomic constraints, that are quite similar to the unconstrained case [25].

Theorem 4.10. *Let Q be a semiholonomic constraint in J^1Y (i.e. $\kappa = kn$ and $d\mathcal{I}(\mathcal{C}^0) \subset \mathcal{I}(\mathcal{C}^0)$), $\bar\alpha_\mathcal{I}$ a constrained Hamilton–de Donder system. Then the integrability condition* (4.95) *is satisfied identically and the regularity condition reads*

$$\det \left(\frac{\partial^2 \bar L}{\partial y_i^l \partial y_j^s} \right) \neq 0. \tag{4.120}$$

Constrained momenta are given by the formula

$$P_s^i = \frac{\partial \bar L}{\partial y_i^s}, \tag{4.121}$$

the class of energy 1-forms $\eta_\mathcal{I}$ contains a closed form $d\bar H$, where

$$\bar H = -\bar L + P_s^i y_i^s, \tag{4.122}$$

and constraint Legendre transformation is a local diffeomorphism

$$(x^i, y^\sigma, y_j^s) \rightarrow (x^i, y^\sigma, P_s^j) \tag{4.123}$$

on the submanifold $Q \subset J^1Y$.

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