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Mikhail A. Malakhaltsev

DIFFERENTIAL COMPLEX ASSOCIATED TO CLOSED DIFFERENTIAL FORMS OF NONCONSTANT RANK

(submitted by V.V. Lychagin)

Abstract. In the present paper we construct a complex of sheaves associated to a closed differential form ω. We study this complex in case ω is a) a closed 1-form vanishing at an embedded submanifold, b) a symplectic structure with Martinet singularities. In particular, we prove that, under additional conditions on ω, this complex gives a fine resolution for the sheaf of infinitesimal automorphisms of ω.

1. Introduction

For a tensor field t defining an integrable G-structure on a smooth manifold M, the Spencer P-complex of the Lie derivative $DV = L_V t$ gives the fine resolution for the sheaf of infinitesimal automorphisms of this structure. In this way one can obtain, for example, the Dolbeaux cohomology of complex manifold and the Vaisman cohomology of foliated manifold [11] (see also [12]). Now suppose that a tensor field t on M defines an integrable structure on an everywhere dense open $U \subset M$, and on $\Sigma = M \setminus U$ the regularity condition for t fails. In this case we say that t is an integrable structure with singularities. Then we have the problem to construct a fine resolution for the sheaf of infinitesimal automorphisms of t.

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In the present paper we construct a complex of sheaves associated to a closed differential form ω. We study this complex in case ω is a) a closed 1-form vanishing at an embedded submanifold, b) a symplectic structure with Martinet singularities [1]. In particular, we prove that, under additional conditions on ω, this complex gives a fine resolution for the sheaf of infinitesimal automorphisms of ω. Note that the complex constructed in the present paper can be obtained using a version of Spencer P-complex for the Lie derivative (we cannot apply the initial construction of Spencer P-complex [11] because, in general situation, the Lie derivative fails to be formally integrable for the integrable structure with singularities), however in the present paper we choose a more direct way based on sheaves of differential ideals in the sheaf of differential forms.

For the detailed exposition on 1-forms with singularities, we refer the reader to [2]. The Martinet singularities first appeared in [1], then various properties of symplectic manifolds with Martinet singularities were investigated (see the recent papers [3],[4], [5]). In part, we studied infinitesimal deformations of symplectic structures with Martinet singularities and sheaves naturally associated to these structures [9], [10].

In the present paper we deal with the category of smooth manifolds. Thus all manifolds, maps, bundles, etc. are assumed to be smooth. For a smooth manifold M, we denote by $\mathfrak{X}_M$ the sheaf of vector fields on M, and by Ω^q_M the sheaf of q-forms on M. For a bundle ξ over M, we denote by Γ_ξ the sheaf of sections of ξ. For a manifold M, a closed subset $A \subset X$, and a sheaf $\mathcal{G}$ on A, by $\mathcal{G}^M$ we denote the sheaf on M generated by the presheaf: $U \to 0$ if $U \cap A = \emptyset$, otherwise $U \to \mathcal{G}(U \cap A)$ (see [8]).

2. Preliminary: differential complex associated to a morphism from a vector bundle to the bundle of exterior forms

We start with the following standard algebraic construction. Let K be a ring and $d : K \to K$ be a differentiation such that $d^2 = 0$. Let I be an ideal in K, then

$$I' = \{a + k_i \, db_i \mid a, b_i \in I\}$$

also is an ideal in K such that $d : I' \to I'$. Then we have the following exact sequence of differential rings:

$$0 \to I' \to K \to K/I' \to 0$$

The same construction can be done for sheaves of rings over a smooth manifold M. Then, for a sheaf of rings $\mathcal{K}$ over M endowed with a differential d such that $d^2 = 0$, and a subsheaf $\iota : \mathcal{I} \hookrightarrow \mathcal{K}$ of ideals, we get the sheaf $\mathcal{I}'$ of ideals and the exact sequence of sheaves

$$0 \to \mathcal{I}' \to \mathcal{K} \to \mathcal{K}/\mathcal{I}' \to 0$$

and the corresponding cohomology exact sequence

$$\cdots \to H^k(M;\mathcal{I}') \to H^k(M;\mathcal{K}) \to H^k(M;\mathcal{K}/\mathcal{I}') \to H^{k+1}(M;\mathcal{I}') \to \dots$$

For any vector bundles $\xi : E_\xi \to M$ and $\eta : E_\eta \to M$, each vector bundle morphism $Q : \xi \to \eta$ determines the morphism $\mathcal{Q} : \Gamma_\xi \to \Gamma_\eta$ of sheaves of vector spaces: for each $s \in \Gamma_\xi(U)$, the section $\mathcal{Q}(s)(p) = Q_p(s(p))$, $p \in U$, lies in $\Gamma_\eta(U)$. The kernel of $\mathcal{Q}$ is the sheaf $\mathcal{K}(U) = \{s \in \Gamma_\xi(U) \mid \mathcal{Q}(s) = 0\}$ of modules over the fine sheaf C^∞, therefore $\mathcal{K}$ is also fine. Now let us consider the presheaf $\mathcal{F}(U) = \{t \in \Gamma_\eta(U) \mid t = \mathcal{Q}(s)\}$.

Lemma 1. *The presheaf $\mathcal{F}$ is a sheaf.*

Proof. Let U be an open subset in M, and U_α be the open covering of U. Assume we are given $t_\alpha = Q(s_\alpha)$ and, for any α, β such that $U_\alpha \cap U_\beta \neq \emptyset$, at each point of $p \in U_\alpha \cap U_\beta$ we have $t_\alpha(p) = t_\beta(p)$. Then, t_α is the restriction of a section $t \in \Gamma_\eta(U)$ and $s_{\alpha\beta} = s_\alpha - s_\beta$ is a 1-cocycle on the covering $\{U_\alpha\}$ with coefficients in $\mathcal{K}$. Since $\mathcal{K}$ is fine, we can find $\widetilde{s}_\alpha \in \mathcal{K}(U_\alpha)$ such that $s_{\alpha\beta} = \widetilde{s}_\alpha - \widetilde{s}_\beta$. Then the sections $\bar{s}_\alpha = s_\alpha - \widetilde{s}_\alpha$ glue to a section $s \in \Gamma_\xi(U)$ such that $t = \mathcal{Q}(s)$, hence t lies in $\mathcal{F}(U)$. □

Let $\xi : E_\xi \to M$ be a vector bundle, and $A : \xi \to \Lambda^q M$ be a vector bundle morphism. Denote by $\mathcal{A} : \Gamma_\xi \to \Omega_M$ the sheaf morphism corresponding to A. Then the subsheaf $\mathcal{A}(\Gamma_\xi)$ generates the subsheaf $\mathcal{I}_M \subset \Omega_M$ of ideals. Let us denote by $\mathcal{F}_M = \oplus \mathcal{F}^q_M \subset \Omega_M$ the corresponding graded sheaf $\mathcal{I}'_M$ of differential ideals.

We take an open $U \subset M$ such that ξ and ΛM are trivial over U. Then, on U we get q-forms ω_a, $a = \overline{1, \operatorname{rank} \xi}$, which span $\mathcal{A}(U)$ over the ring $C^\infty(U)$ of functions on U. One can easily see that

$$\mathcal{F}^k(U) = \{\phi^a \wedge \omega_a + \psi^a \wedge d\omega_a \mid \phi^a \in \Omega^k(U), \psi^a \in \Omega^{k-1}(U)\} \qquad (1)$$

Thus, to any morphism $A : \xi \to \Lambda^q M$ we associate the complex $(\mathcal{F}^*, d)$ of sheaves, which is a subcomplex of the de Rham complex (Ω_M, d) considered also as a complex of sheaves. Also, we have the exact sequence of sheaves

$$0 \to \mathcal{F} \to \Omega_M \to \mathcal{G} = \Omega_M/\mathcal{F} \to 0.$$

Remark 2.1. Let $q = 1$, and $A : \xi \to \Omega^1$ be a morphism. Then the sheaf $\mathcal{A}(\Gamma_\xi)$ is the sheaf of sections of a subbundle (with singularities) in T^*M. If $\operatorname{rank} A$ is constant, then A determines a distribution on M, and if, in addition, this distribution is integrable, $(\mathcal{F}^*(M), d)$ is the complex generated by the basic forms of the corresponding foliation, which is widely used in the foliation theory [6], [7].

Remark 2.2. If $A : \xi \to \Omega^q$ is surjective, then the associated complex $(\mathcal{F}^*(M), d)$ is the de Rham complex of M.

Hereafter we assume that A *is surjective on an open everywhere dense set* $M \setminus \Sigma$, where $\iota : \Sigma \hookrightarrow M$ is an embedded submanifold. Then, for each open U such that $U \cap \Sigma = \emptyset$ we have $\mathcal{F}^k(U) = \Omega^k(U)$, hence $\mathcal{G}(U) = 0$. From this follows that the sheaf $\mathcal{G}$ is supported on Σ.

For a closed $\omega \in \Omega^{q+1}(M)$, the Lie derivative $D(V) = L_V\omega = di_V\omega$ is a first order differential operator $TM \to \Lambda^{q+1}M$. The operator D can be included to the complex of sheaves associated to the vector bundle morphism $I_\omega : TM \to \Lambda^q$, $I(V) = i_V\omega$:

$$0 \to \mathfrak{X}_\Gamma \to \mathfrak{X}_M \xrightarrow{D} \mathcal{F}^1 \xrightarrow{d} \mathcal{F}^2 \xrightarrow{d} \dots, \tag{2}$$

where $\mathfrak{X}_\Gamma$ is the sheaf of infinitesimal automorphisms of ω. Evidently, all the sheaves in (2), except for $\mathfrak{X}_\Gamma$ are fine. Therefore, if (2) is locally exact, it gives a fine resolution for $\mathfrak{X}_\Gamma$. However, in general, (2) fails to be locally exact.

3. Differential complex associated with closed 1-form with singularities

Let η be a closed 1-form on an n-dimensional differential manifold M, and $\Sigma = \{p \in M \mid \eta(p) = 0\}$ be an embedded submanifold of codimension k. Assume that for each $p \in \Sigma$ there exists a coordinate system (u^a, u^α), $a = \overline{1,k}$, $\alpha = \overline{k+1,n}$, such that Σ is given by the equations $u^a = 0$ and

$$\det ||\partial_a \eta_b(p)|| \neq 0, \forall p \in \Sigma.$$

Remark 3.1. Under these assumptions we can write coordinate expression for η. For any $p \in M \setminus \Sigma$, one can take a coordinate system $\{u^a\}$ in a neighborhood U of p such that $\eta|_U = du^1$.

Let $p \in \Sigma$. Since η is closed, we have $\eta = df$, where f is a function on a neighborhood U of p. From our assumptions it follows that

$$f = A_{ab}u^a u^b + B_{ab\alpha\beta}(u^c, u^\gamma)u^a u^b u^\alpha u^\beta,$$

where $||A_{ab}||$ is a constant symmetric matrix, and $B_{ab\alpha\beta}(u^c, u^\gamma)$ are smooth functions (see [13], Ch. 6). Hence

$$\eta = (A_{ab} + B_{ab\alpha\beta}(u^c, u^\gamma)u^\alpha u^\beta)u^a du^b + u^a u^b d(B_{ab\alpha\beta}(u^i)u^\alpha u^\beta).$$

Let us denote by $\Omega^q_{\Sigma,0}$ the sheaf over M consisting of q-forms θ such that $\iota^*\theta = 0$. This means that, for each open $U \subset M$ such that $U \cap \Sigma \neq \emptyset$, a form $\theta \in \Omega^q(U)$ lies in $\Omega^q_{\Sigma,0}(U)$ if and only if $\iota^*\theta = 0$. If $U \cap \Sigma = \emptyset$, then $\Omega^q_{\Sigma,0}(U) = \Omega^q(U)$. It is clear that the exterior differential d maps $\Omega^q_{\Sigma,0}$ to $\Omega^{q+1}_{\Sigma,0}$.

The 1-form η is a vector bundle morphism from the tangent bundle TM to the trivial vector bundle $\Lambda^0 M = M \times \mathbb{R}$. Denote by $\widetilde{\eta}$ the corresponding sheaf morphism $\mathfrak{X}_M \to \Omega_M$.

Statement 3.1. *The complex of sheaves $(\mathcal{F}^*, d)$ associated to the 1-form η is $\Omega^*_{\Sigma,0}$.*

Proof. Take a chart (U, u^i), and let $\eta = \eta_i du^i$. Then,

$$\widetilde{\eta}(\mathfrak{X}_M)(U) = \{\eta_i f^i | f^i \in \Omega^0(U)\} \tag{3}$$

and (1) gives us

$$\mathcal{F}^k(U) = \{\phi^i \wedge \eta_i + \psi^i \wedge d\eta_i \mid \phi^i \in \Omega^k(U), \psi^i \in \Omega^{k-1}(U)\}$$

If $p \in U$ does not lie in Σ, then at least one η_i does not vanish at p. Hence in a neighborhood V of p we have $\mathcal{F}^k(V) = \Omega^k(V)$. If $p \in U \cap \Sigma$, then $\eta_i(p) = 0$ for all $i = \overline{1, n}$. Moreover, for any $X \in T_p\Sigma$, we have $d\eta_i(X) = 0$. Hence $\mathcal{F}^k(U) \subset \Omega^k_{\Sigma,0}(U)$.

Now let $p \in \Sigma$. By assumption, $\det ||\partial_a \eta_b|| \neq 0$, hence we can take a coordinate system such that $u^a = \eta_a$. From this follows that (3) is just the ideal of functions vanishing on $\Sigma \cap U$. Then, for each $\theta \in \Sigma^k_{\Sigma,0}(U)$, we have $\theta = du^a \wedge \psi_a + \tau$, where the coordinate expression of τ does not involve any du^a. Since $i^*\theta = 0$, all the coordinates of τ vanish at Σ, hence $\tau = u^a \phi_a$. Therefore, we have $\theta = du^a \wedge \psi_a + u^a \phi_a$, hence $\Omega^k_{\Sigma,0} = \mathcal{F}^k(U)$. □

Let Ω^q_Σ be the sheaf of q-forms on Σ, and $(\Omega^q_\Sigma)^M$ be the corresponding sheaf over M. Let $\iota : \Sigma \to M$ be the embedding. For each open $U \subset M$, we have the following exact sequence of complexes of differential forms:

$$0 \to \Omega^*_{\Sigma,0}(U) \xrightarrow{i} \Omega^*_M(U) \xrightarrow{\iota^*} (\Omega^*_\Sigma)^M(U) \cong \Omega^*_\Sigma(U \cap \Sigma) \to 0. \tag{4}$$

Note that if $U \cap \Sigma = \emptyset$, then $\Omega^*_M(U) = 0$ and $\Omega^*_{\Sigma,0}(U) \xrightarrow{i} \Omega^*_M(U)$ is the identity mapping. Thus we obtain the exact sequence of complexes of sheaves:

$$0 \to \Omega^*_{\Sigma,0} \xrightarrow{i} \Omega_M \xrightarrow{\iota^*} (\Omega^*_\Sigma)^M \to 0,$$

where the sheaf morphism $\iota^* : \Omega^q \to \Omega^q_\Sigma$ is induced by the embedding ι.

Now let $\mathfrak{X}_\Gamma$ be the sheaf of infinitesimal automorphisms of η, this means that $\mathfrak{X}_\Gamma(U) = \{V \in \mathfrak{X}(U) \mid L_V\eta = 0\}$. By assumption, $\eta = 0$ on Σ, hence the function $i_V\eta$ vanishes on Σ. Since $L_V\eta = di_V\eta$, we have $L_V\eta(W) = 0$ for each $p \in \Sigma$ and $W \in T_p\Sigma$. Thus, we get a sheaf morphism $\mathcal{D} : \mathfrak{X} \to \Omega^1_{\Sigma,0}$, $\mathcal{D}(V) = L_V\eta$, and the sheaf $\mathfrak{X}_\Gamma$ is the kernel of $\mathcal{D}$.

Statement 3.2. *Let η be a closed 1-form on a differential manifold M, and $\Sigma = \{p \in M \mid \eta(p) = 0\}$. Assume that for each $p \in \Sigma$ there exists a coordinate system (u^a, u^α), $a = \overline{1,k}$, $\alpha = \overline{k+1,n}$ such that Σ is given by the equations $u^a = 0$ and*

$$\eta = A_{ab}u^a du^b, \tag{5}$$

where A_{ab} is a constant symmetric matrix of rank k. Then the sequence of sheaves and their morphisms

$$0 \to \mathfrak{X}_\Gamma \xrightarrow{i} \mathfrak{X} \xrightarrow{\mathcal{D}} \Omega^1_{\Sigma,0} \xrightarrow{d} \Omega^2_{\Sigma,0} \xrightarrow{d} \dots \Omega^n_{\Sigma,0} \tag{6}$$

is a fine resolution of the sheaf $\mathfrak{X}_\Gamma$.

Proof. It is clear that the sheaves $\mathfrak{X}$ and $\Omega^q_{\Sigma,0}$ are fine. Let us prove the exactness of (6).

If $p \notin \Sigma$, we take a contractible neighborhood $U \ni p$ such that $U \cap \Sigma = \emptyset$. Then, $\Omega^q_{\Sigma,0}(U) = \Omega^q(U)$, and, by the Poincare lemma, we obtain that (6) is exact at $\Omega^q_{\Sigma,0}(U)$ for each $q > 1$. If θ lies in $\Omega^1_{\Sigma,0}(U) = \Omega^1(U)$ and $d\theta = 0$, then $\theta = df$, where $f \in \Omega^0(U)$, and since η does not vanish on U, we can find $V \in \mathfrak{X}(U)$ such that $f = \eta(V)$. Then $\mathcal{D}(V) = L_V\eta = di_V\eta = df = \theta$.

Now take $p \in \Sigma$, and a contractible $U \ni p$ such that $U \cap \Sigma$ is also contractible. Hence $H^q(\Omega(U), d) \cong H^q(\Omega_\Sigma(U \cap \Sigma), d) \cong 0$ for $q > 0$. Then, from the exact cohomology sequence associated with the exact sequence (4) of complexes we get that $H^q(\Omega_{\Sigma,0}(U), d) \cong 0$ for $q > 1$. From this follows that (6) is exact at $\Omega^q_{\Sigma,0}(U)$ for each $q > 1$.

Since $\iota^* : H^0(\Omega^0(U)) \to H^0(\Omega^0_\Sigma(U \cap \Sigma)$ is an isomorphism, we have that $H^1(\Omega_{\Sigma,0}(U), d) \cong 0$. Hence, if $\theta \in \Omega^1_\Sigma(U)$ and $d\theta = 0$, then we can find $f \in \Omega_\Sigma(U)$ such that $df = \theta$. As above, to prove the exactness at Ω^1_Σ, we need to find V such that $f = \eta(V)$.

Let (u^i, u^α) be the coordinate system with respect to which η has canonical form (5). Then, $f(0, u^\alpha) = 0$, hence $f(u^a, u^\alpha) = u^a g_a(u^i, u^\alpha)$.

Therefore, we set $V = A^{ab}g_a\partial_b$, where A^{ab} is the matrix inverse to A_{ab}, and V is the required vector field. $\square$

4. Differential complex associated to symplectic form with Martinet singularities

Let ω be a closed 2-form on a smooth manifold M such that $\det\omega = 0$ on a closed submanifold $i : \Sigma \hookrightarrow M$ and $\operatorname{rank}\omega = 2m$ is constant on Σ. Then ω determines the vector bundle morphism $I_\omega : TM \to \Lambda^1 M$, $I_\omega(V) = i_V\omega$, which is a vector bundle isomorphism over $M \setminus \Sigma$. The kernel of I_ω is a vector bundle over Σ, call it $\epsilon : E \to \Sigma$. Denote by $\mathcal{I}_\omega$ the corresponding sheaf morphism $\mathfrak{X}_M \to \Omega^1_M$. From Lemma proved in [10] it follows that $\tau \in \mathcal{I}_\omega(\mathfrak{X}_M)$ if and only if $\tau|_E = 0$. We will consider the symplectic structures with Martinet singularities [1].

Let ω be a closed 2-form on a $2n$-dimensional manifold. Assume that for each point $p \in M$ one can take a chart (U, u^i) such that

$$\omega = u^1 du^1 \wedge du^2 + du^3 \wedge du^4 \cdots + du^{2n-1} \wedge du^{2n}. \tag{7}$$

Statement 4.1. *Let $(\mathcal{F}^*, d)$ be the complex of sheaves associated to the symplectic form ω with Martinet singularities locally given by* (7)*. Then $\mathcal{F}^1$ is the subsheaf of Ω^1_M consisting of forms which vanish on the subbundle $E \subset TM|_\Sigma$, and $\mathcal{F}^k = \Omega^k_M$ for $k \geq 2$.*

Proof. With respect to the canonical coordinates (7), the submanifold Σ is given by the equation $u^1 = 0$, and the subbundle E is spanned by the vector fields ∂_1, ∂_2 along Σ. Therefore, $\eta = \eta_a du^a$ lies in $\mathcal{I}(\mathfrak{X})(U)$ if and only if η_1, η_2 vanish on $U \cap \Sigma$. Note that

$$d(u^1 du^k) = du^1 \wedge du^k \quad \forall k = \overline{2, 2n}$$
$$d(u^i du^k) = du^i \wedge du^j \quad \forall i < j,\ i, j = \overline{2, 2n}$$

and $u^1 du^k, k = \overline{2, 2n}$, and $u^i du^j$, $i = \overline{2, 2n}$, $j = \overline{3, 2n}$, vanish at ∂_1, ∂_2 along Σ. This implies that for all $i, j = \overline{2, 2n}$ we have $du^i \wedge du^j = d\tau^{ij}$, where $\tau^{ij} \in \mathcal{I}(\mathfrak{X})(U)$. This proves that for each $\phi \in \Omega^k(U)$, $k \geq 2$, we can write $\phi = \psi^a d\omega_a$, where $\omega_a \in \mathcal{I}(\mathfrak{X})(U)$, hence, by (1), $\mathcal{F}^k(U) = \Omega^k(U)$.

Finally, the fact that $\mathcal{F}^1(U)$ consists of 1-forms vanishing on E follows from Lemma in [10]. $\square$

Statement 4.2. *For $\mathcal{D} : \mathfrak{X}_M \to \Omega^2_M$, $D(V) = L_V\omega$, the sequence of sheaves* (*see* (2))

$$0 \to \mathfrak{X}_\Gamma \xrightarrow{i} \mathfrak{X}_M \xrightarrow{D} \Omega^2 \xrightarrow{d} \Omega^3 \xrightarrow{d} \ldots \tag{8}$$

is a fine resolution for the sheaf $\mathfrak{X}_\Gamma$.

Proof. In ([9], Lemma 3) we have proved that, for each germ of 1-form $\langle\xi\rangle_p \in (\Omega^1_M)_p$, there exist germs $\langle W\rangle_p \in \mathfrak{X}_p$, $\langle f\rangle_p \in (\Omega_M)^0_p$ such that $\langle\xi\rangle_p = \langle i_W\omega + df\rangle_p$.

Let us take $\eta \in \Omega^2(U)$ such that $d\eta = 0$. Then $\eta = d\xi$, $\xi \in \Omega^1(U)$. From the above statement it follows that $\eta = di_W\omega$ for a vector field $W \in \mathfrak{X}(U)$. Thus (8) is exact in the term Ω^2_M. The Poincare lemma implies that (8) is exact in the other terms. □

Corollary 1. *For ω with Martinet singularities locally given by* (7), $H^q(M;\mathfrak{X}_\Gamma) \cong H^{q+1}_{DR}(M)$, $q \geq 1$, *where $H_{DR}(M)$ is the de Rham cohomology.*

Remark 4.1. The corollary assertion was proved in [10] by another method.

Remark 4.2. The complex associated to $I_\omega : TM \to \Lambda^M$ can be extended in the following way. In [10] we have considered the sheaf of local Hamiltonians for ω:

$$\mathcal{T}(U) = \{f \in \Omega^0(U) \mid df \in \mathcal{I}(\mathfrak{X})(U) \iff df|_E = 0\}$$

Evidently we have complex of sheaves:

$$0 \to \mathbb{R}_M \xrightarrow{d} \mathcal{T} \xrightarrow{d} \mathcal{F}^1 \xrightarrow{d} \mathcal{F}^2 = \Omega^2_M \xrightarrow{d} \dots$$

Let us consider another type of Martinet singularities. Let ω be a closed 2-form on a four-dimensional manifold, and assume that for each point $p \in M$ one can take a chart (U, u^i) such that

$$\begin{aligned}\omega_0 =& du^1 \wedge du^2 + u^3 du^1 \wedge du^4 \\ &+ u^3 du^2 \wedge du^3 + u^4 du^2 \wedge du^4 + (u^1 - (u^3)^2) du^3 \wedge du^4,\end{aligned} \tag{9}$$

Let us take the local frame

$$W_1 = \partial_1,\ W_2 = \partial_2,\ W_3 = u^3\partial_1 + \partial_3,\ W_4 = u^4\partial_1 - u^3\partial_2 + \partial_4.$$

and the dual frame

$$\theta^1 = du^1 - u^3 du^3 - u^4 du^4\ \theta^2 = du^2 + u^3 du^4,\ \theta^3 = du^3\ \theta^4 = du^4.$$

One can easily see that

$$\omega = \theta^1 \wedge \theta^2 + u^1 \theta^3 \wedge \theta^4,$$

and the vector bundle $E \to \Sigma$ is spanned by the vector fields W_3, W_4. Hence a form ξ lies in $\mathcal{I}_\omega(\mathfrak{X})(U)$ if and only if

$$\xi = \xi_1\theta^1 + \xi_2\theta^2 + u^1\alpha.$$

Statement 4.3. *Let $(\mathcal{F}^*, d)$ be the complex of sheaves associated to the symplectic form ω with Martinet singularities locally given by (9). Then $\mathcal{F}^1$ is the subsheaf of Ω^1_M consisting of forms vanishing on the subbundle $E \subset TM|_\Sigma$, and $\mathcal{F}^k = \Omega^k_M$ for $k \geq 2$.*

Proof. Note that θ^1, θ^2 lies in $I_\omega(\mathfrak{X}(U))$, and $\theta^3 \wedge \theta^4 = d\theta^2$. Then the proof is similar to that of Statement 4.1. □

Thus, in this case we also get the complex of sheaves (8). However, in this case the complex (8) is not locally exact. Let us take, e.g., $\eta = d\xi$, where $\xi = (u^3)^2\theta^4 \in \mathcal{F}^2(U)$. Let us prove that η cannot be represented as $\eta = di_V\omega$ with $V \in \mathfrak{X}(U)$. Suppose not, then $\xi - i_V\omega = df$, where f is a function on U. Then, the restriction of the last equality to Σ gives us the equation system

$$\partial_3 f = 0, \quad \partial_4 f - u^3\partial_f = (u^3)^2.$$

We differentiate the second equality with respect to u^3 and get $\partial_2 f = -2u^3$. Hence, $\partial_{23} = -2$, this contradicts $\partial_{23} f = 0$.

Remark 4.3. This example of 1-form ξ, which cannot be represented as $i_V\omega + df$, has been given in [9]. Note that in [9] the expression for ξ was misprinted (u^3du^4 instead of $(u^3)^2du^4$).

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Kazan State University, Kremlevskaya, 18, Kazan:420008, Russia

E-mail address: mikhail.malakhaltsev@ksu.ru

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