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THE TOTAL DIVERGENCE EQUATION

(submitted by V.V. Lychagin)

ABSTRACT. In this paper, the total divergence equation is investigated by means of the methods used in the theory of finite order variational sequences. Integrability conditions for this equation are found, and all solutions are described. The correspondence of the solutions with some differential forms on jet spaces is established.

1. INTRODUCTION

Let $U \subset \mathbb{R}^n$ be an open set, let $W \subset \mathbb{R}^m$ be an open ball with center at the origin, and denote $V = U \times W$. We consider V as a fibered manifold over U with the first Cartesian projection $\pi : V \to U$. V^s denotes the *s-jet prolongation* of V; explicitly, $V^s = U \times W \times L(\mathbb{R}^n, \mathbb{R}^m) \times L^2_{\mathrm{sym}}(\mathbb{R}^n, \mathbb{R}^m) \times ... \times L^s_{\mathrm{sym}}(\mathbb{R}^n, \mathbb{R}^m)$, where $L^k_{\mathrm{sym}}(\mathbb{R}^n, \mathbb{R}^m)$ is the vector space

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of k-linear, symmetric mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$. The Cartesian coordinates on V, and the associated jet coordinates on V^s, are denoted by x^i, y^σ, and $x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, ..., y^\sigma_{j_1 j_2...j_s}$, respectively; in these expressions $1 \le i, j_1, j_2, ..., j_s \le n$ and $1 \le \sigma \le m$.

Let $f : V^r \to \mathbb{R}$ be a differentiable function (order of differentiability in different variables can be easily deduced from the context). Our aim in this paper is to find solutions $g = (g^1, g^2, ..., g^n)$ of the *total divergence equation*

$$d_i g^i = f, \tag{1}$$

whose components g^i are differentiable real-valued functions on V^s, where s is a positive integer. Since the *total divergence* $d_i g^i$ is defined by

$$d_i g^i = \frac{\partial g^i}{\partial x^i} + \frac{\partial g^i}{\partial y^\sigma} y^\sigma_i + \frac{\partial g^i}{\partial y^\sigma_{i_1}} y^\sigma_{i_1 i} + \frac{\partial g^i}{\partial y^\sigma_{i_1 i_2}} y^\sigma_{i_1 i_2 i} + ... + \frac{\partial g^i}{\partial y^\sigma_{i_1 i_2...i_r}} y^\sigma_{i_1 i_2...i_r i}, \tag{2}$$

equation (1) is a first order partial differential equation. From this expression we immediately see that every solution $g = g^i$, defined on V^s, such that $s \le r + 1$, satisfies

$$\frac{\partial g^{i_1}}{\partial y^\sigma_{i_2 i_3...i_{s+1}}} + \frac{\partial g^{i_2}}{\partial y^\sigma_{i_1 i_3 i_4...i_{s+1}}} + ... + \frac{\partial g^{i_s}}{\partial y^\sigma_{i_1 i_2...i_{s-1} i_{s+1}}} + \frac{\partial g^{i_{s+1}}}{\partial y^\sigma_{i_1 i_2...i_s}} = 0. \tag{3}$$

In this paper, we solve the total divergence equation by means of the methods, developed in the theory of finite order variational sequences (Krupka [12]). For additional information on this theory, see also e.g. Francaviglia, Palese, Vitolo [3], Grassi [4], Grigore [5], Krbek and Musilova [9], Pommaret [14], and Vitolo [19]. We use standard concepts of the calculus of differential forms on jet prolongations of fibered manifolds, applied to open sets in Euclidean topological spaces $\mathbb{R}^n$ (see e.g. Krupka [10], [11]). A systematic exposition of variational analysis and differential equations in this context can be found in Hakova and Krupkova [6], and Krupkova [13]. We prove two theorems on the structure of solutions of the total divergence equation. We consider this equation as an overdetermined equation, and find the corresponding integrability condition, which guarantees existence of solutions. It turns out that integrability of the total divergence equation is equivalent with the vanishing of the Euler-Lagrange expressions of the right-hand side of equation (1), well known from the multi-dimensional, higher order variational theory. If the integrability condition is satisfied, we find all solutions. Moreover we show that the solutions can naturally be interpreted as some differential forms. The tools we use include among others the theory of Lepage

forms, the Young decompositions and the trace decompositions of tensor spaces, and the fibered homotopy operator.

It seems that the total divergence equations, differential operators, and algebraic constructions related with them, appeared for the first time in the geometric theory of partial differential equations on infinite jet spaces and, in particular, in the variational bicomplex theory (see e.g. Anderson [1], Dedecker and Tulczyjew [2], Krasilschik [7], Krasilschik, Lychagin, and Vinogradov [8], Saunders [15], Takens [16], Tulczyjew [17], Vinogradov [18], Tsujishita [20], and references therein). These equations play a crucial role in the proofs of exactness of the bicomplex. However, no explicit discussion has appeared in this context yet; the structure of solutions of the total divergence equation, and the meaning of condition (3), in these sources remain unclear.

2. Notation

In what follows we use basic notions from the theory of contact forms on jet prolongations of fibered manifolds (see e.g. [11], [13]). We denote

$$\omega_0 = dx^1 \wedge dx^2 \wedge ... \wedge dx^n, \quad \omega_i = i_{\partial/\partial x^i}\omega_0, \quad \omega_{ij} = i_{\partial/\partial x^i}i_{\partial/\partial x^j}\omega_0.$$

The *contact 1-forms* $\omega^\sigma_{j_1 j_2 ... j_k}$ are defined by

$$\omega^\sigma_{j_1 j_2 ... j_k} = dy^\sigma_{j_1 j_2 ... j_k} - y^\sigma_{j_1 j_2 ... j_k l}dx^l, \quad k = 0, 1, 2, ..., s-1.$$

h and p_k denote the *horizontalization*, and the *k-contact* mappings.

For any smooth function $f : V^r \to \mathbb{R}$ we define an n-form λ_f on V^r and a system of functions $E_\sigma(f) : V^{2r} \to \mathbb{R}$ by

$$\lambda_f = f\omega_0,$$

and

$$E_\sigma(f) = \frac{\partial f}{\partial y^\sigma} - d_{i_1}\frac{\partial f}{\partial y^\sigma_{i_1}} + d_{i_1}d_{i_2}\frac{\partial f}{\partial y^\sigma_{i_1 i_2}} - d_{i_1}d_{i_2}d_{i_3}\frac{\partial f}{\partial y^\sigma_{i_1 i_2 i_3}}$$
$$+ ... + (-1)^{r-1}d_{i_1}d_{i_2}...d_{i_{r-1}}\frac{\partial f}{\partial y^\sigma_{i_1 i_2 ... i_{r-1}}} + (-1)^r d_{i_1}d_{i_2}...d_{i_r}\frac{\partial f}{\partial y^\sigma_{i_1 i_2 ... i_r}}.$$

λ_f is the *lagrangian* associated with f, and the $(n+1)$-form

$$E_f = E_\sigma(f)\omega^\sigma \wedge \omega_0$$

is the *Euler-Lagrange form*, associated with f.

3. Projectable extensions of horizontal $(n-1)$-forms

Let us consider a π^r-horizontal $(n-1)$-form η on V^r, expressed as

$$\eta = g^i\omega_i = \frac{1}{(n-1)!}h_{j_2j_3...j_n}dx^{j_2} \wedge dx^{j_3} \wedge ... \wedge dx^{j_n}. \qquad (4)$$

Since

$$\omega_i = \frac{1}{(n-1)!}\varepsilon_{ij_2j_3...j_n}dx^{j_2} \wedge dx^{j_3} \wedge ... \wedge dx^{j_n},$$

we have the transformation formulas

$$h_{j_2j_3...j_n} = \varepsilon_{ij_2j_3...j_n}g^i, \quad g^k = \frac{1}{(n-1)!}\varepsilon^{kj_2j_3...j_n}h_{j_2j_3...j_n}.$$

We prove the following assertion, in which alt and sym denote *alternation* and *symmetrization* in the corresponding indices.

Lemma 1. *The functions g^i and $h_{j_1j_2...j_{n-1}}$ satisfy*

$$\frac{1}{(r+1)}\varepsilon_{il_2l_3...l_n}\left(\frac{\partial g^i}{\partial y^\sigma_{k_1k_2...k_r}} + \frac{\partial g^{k_1}}{\partial y^\sigma_{ik_2k_3...k_r}} + \frac{\partial g^{k_2}}{\partial y^\sigma_{k_1ik_3k_4...k_r}} + ... + \frac{\partial g^{k_r}}{\partial y^\sigma_{k_1k_2...k_{r-1}i}}\right)$$

$$= \frac{\partial h_{l_2l_3...l_n}}{\partial y^\sigma_{k_1k_2...k_r}} - \frac{r(n-1)}{(r+1)}\frac{\partial h_{il_3l_4...l_n}}{\partial y^\sigma_{ik_2k_3...k_r}}\delta^{k_1}_{l_2} \quad \text{alt}(l_2l_3...l_n) \quad \text{sym}(k_1k_2...k_r).$$

Proof. We have

$$\frac{\partial g^i}{\partial y^\sigma_{k_1k_2...k_r}}\frac{1}{(n-1)!}\varepsilon^{ij_2j_3...j_n}\frac{\partial h_{j_2j_3...j_n}}{\partial y^\sigma_{k_1k_2...k_r}}.$$

Multiplying this expression by $\varepsilon_{il_2l_3...l_n}$ we obtain

$$\varepsilon_{il_2l_3...l_n}\frac{\partial g^i}{\partial y^\sigma_{k_1k_2...k_r}} = \frac{1}{(n-1)!}\varepsilon_{il_2l_3...l_n}\varepsilon^{ij_2j_3...jn}\frac{\partial h_{j_2j_3...j_n}}{\partial y^\sigma_{k_1k_2...k_r}} = \frac{\partial h_{l_2l_3...l_n}}{\partial y^\sigma_{k_1k_2...k_r}}.$$

The rest of the proof is routine. $\qquad\square$

We say that a π^r-horizontal form η, defined on V^r, has a *projectable extension*, if there exists a form μ on $V^{r,r-1}$ such that

$$\eta = h\mu.$$

Let us consider a form η, expressed in two bases of $(n-1)$-forms by (4).

Lemma 2. *The following three conditions are equivalent:*
(a) *η has a $\pi^{r,r-1}$-projectable extension.*
(b) *The components $h_{i_1i_2...i_{n-1}}$ satisfy*

$$\frac{\partial h_{i_1i_2...i_{n-1}}}{\partial y^\sigma_{j_1j_2...j_r}} - \frac{r(n-1)}{r+1}\frac{\partial h_{si_2i_3...i_{n-1}}}{\partial y^\sigma_{sj_2j_3...j_r}}\delta^{j_1}_{i_1} = 0 \quad \text{sym}(j_1j_2...j_r) \quad \text{alt}(i_1i_2...i_{n-1}).$$

(c) *The components g^i satisfy*

$$\frac{\partial g^i}{\partial y^\sigma_{k_1 k_2 \dots k_r}} + \frac{\partial g^{k_1}}{\partial y^\sigma_{i k_2 k_3 \dots k_r}} + \frac{\partial g^{k_2}}{\partial y^\sigma_{k_1 i k_3 k_4 \dots k_r}} + \dots + \frac{\partial g^{k_r}}{\partial y^\sigma_{k_1 k_2 \dots k_{r-1} i}} = 0.$$

Proof. 1. To show that (a) implies (b), suppose that we have an $(n-1)$-form μ on V^{r-1} such that $\eta = h\mu$. To express μ in a fibered chart, we use multi-indices $J = (j_1 j_2 \dots j_{r-1})$ of length $|J| \leq r - 1$. Then $(\pi^{r,r-1})^*\mu$ is expressible as

$$(\pi^{r,r-1})^*\mu = \frac{1}{(n-1)!}h_{i_1 i_2 \dots i_{n-1}}dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_{n-1}}$$

$$+\sum_{k=1}^{n-2}\frac{1}{k!(n-1-k)!}h^{J_1 J_2 \dots J_k}_{\sigma_1 \sigma_2 \dots \sigma_k i_{k+1} i_{k+2} \dots i_{n-1}}\omega^{\sigma_1}_{J_1} \wedge \omega^{\sigma_2}_{J_2} \wedge \dots \wedge \omega^{\sigma_k}_{J_k}$$

$$\wedge dx^{i_{k+1}} \wedge dx^{i_{k+2}} \wedge \dots \wedge dx^{i_{n-1}}$$

$$+\frac{1}{(n-1)!}h^{J_1 J_2 \dots J_{n-1}}_{\sigma_1 \sigma_2 \dots \sigma_{n-1}}\omega^{\sigma_1}_{J_1} \wedge \omega^{\sigma_2}_{J_2} \wedge \dots \wedge \omega^{\sigma_{n-1}}_{J_{n-1}}.$$

Since μ is defined on V^{r-1}, so is $d\mu$. In particular, all terms in $(\pi^{r,r-1})^*d\mu$ $= d(\pi^{r,r-1})^*\mu$, containing $dy^\nu_{j_1 j_2 \dots j_r}$, should vanish identically. We find the terms in $d(\pi^{r,r-1})^*\mu$, which do not contain any of the forms $\omega^\sigma, \omega^\sigma_{j_1}, \omega^\sigma_{j_1 j_2}$, $\dots, \omega^\sigma_{j_1 j_2 \dots j_{r-1}}$; these terms should vanish separately. Obviously, they can arise only from the summands

$$\frac{1}{(n-1)!}h_{i_1 i_2 \dots i_{n-1}}dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_{n-1}}$$

$$+\frac{1}{(n-2)!}h^{J_1}_{\sigma_1 i_2 i_3 \dots i_{n-1}}\omega^{\sigma_1}_{J_1} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \dots \wedge dx^{i_{n-1}} \tag{5}$$

in η, in which $|J_1| = r - 1$. Writing $J_1 = (j_1 j_2 \dots j_{r-1})$, differentiating (5) and omitting the terms containing $\omega^{\sigma_1}_{J_1}$, we get

$$\frac{1}{(n-1)!}d'_{i_1}h_{i_2 i_3 \dots i_n}dx^{i_1} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \dots \wedge dx^{i_n}$$

$$+\frac{1}{(n-1)!}\frac{\partial h_{i_2 i_3 \dots i_n}}{\partial y^\sigma_{j_1 j_2 \dots j_r}}dy^\sigma_{j_1 j_2 \dots j_r} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \dots \wedge dx^{i_n}$$

$$-\frac{1}{(n-2)!}h^{j_1 j_2 \dots j_{r-1}}_{\sigma} {}_{i_3 i_4 \dots i_n}dy^\sigma_{j_1 j_2 \dots j_{r-1} i_2} \wedge dx^{i_2} \wedge dx^{i_3} \wedge dx^{i_4} \wedge \dots \wedge dx^{i_n}$$

$$=\frac{1}{(n-1)!}d'_{i_1}h_{i_2 i_3 \dots i_n}dx^{i_1} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \dots \wedge dx^{i_n}$$

$$+\frac{1}{(n-2)!}\left(\frac{1}{n-1}\frac{\partial h_{i_2 i_3 \dots i_n}}{\partial y^\sigma_{j_1 j_2 \dots j_r}} - h^{j_1 j_2 \dots j_{r-1}}_{\sigma} {}_{i_3 i_4 \dots i_n}\delta^{j_r}_{i_2}\right)$$

$$\cdot dy^\sigma_{j_1 j_2 \dots j_r} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \dots \wedge dx^{i_n},$$

where d_i' denotes the *cut formal derivative*,

$$d_{i_1}' h_{i_2 i_3 \ldots i_n} = d_{i_1} h_{i_2 i_3 \ldots i_n} - \frac{\partial h_{i_2 i_3 \ldots i_n}}{\partial y^{\sigma}_{j_1 j_2 \ldots j_r}} y^{\sigma}_{j_1 j_2 \ldots j_r i_1}.$$

Consequently, $d(\pi^{r,r-1})^* \mu$ can be written as

$$(\pi^{r,r-1})^* d\mu = \frac{1}{(n-1)!} d_{i_1}' h_{i_2 i_3 \ldots i_n} dx^{i_1} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \ldots \wedge dx^{i_n}$$

$$+ \frac{1}{(n-2)!} \left(\frac{1}{n-1} \frac{\partial h_{i_2 i_3 \ldots i_n}}{\partial y^{\sigma}_{j_1 j_2 \ldots j_r}} - h^{j_1 j_2 \ldots j_{r-1}}_{\sigma}{}_{i_3 i_4 \ldots i_n} \delta^{j_r}_{i_2} \right)$$

$$\cdot dy^{\sigma}_{j_1 j_2 \ldots j_r} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \ldots \wedge dx^{i_n} + \tau,$$

where τ is generated by the forms ω^{σ}, $\omega^{\sigma}_{j_1}$, $\omega^{\sigma}_{j_1 j_2}$, ..., $\omega^{\sigma}_{j_1 j_2 \ldots j_{r-1}}$. But $d(\pi^{r,r-1})^* \mu$ is $\pi^{r,r-1}$-projectable, so we see that μ satisfies

$$\frac{1}{n-1} \frac{\partial h_{i_1 i_2 \ldots i_{n-1}}}{\partial y^{\sigma}_{j_1 j_2 \ldots j_r}} - h^{j_1 j_2 \ldots j_{r-1}}_{\sigma}{}_{i_2 i_3 \ldots i_{n-1}} \delta^{j_r}_{i_1} = 0 \tag{6}$$

$$\mathrm{alt}(i_1 i_2 \ldots i_{n-1}) \quad \mathrm{sym}(j_1 j_2 \ldots j_r).$$

The structure of this system of algebraic equations for $h^{j_1 j_2 \ldots j_{r-1}}_{\sigma}{}_{i_3 i_4 \ldots i_n}$ is described by the trace decomposition theory. In what follows we use the trace operation tr, and a complementary operation, denoted by q, which satisfy $U = \mathrm{tr}\,\mathrm{q}\,U + \mathrm{q}\,\mathrm{tr}\,U$ for any tensor U, symmetric in the superscripts, and antisymmetric in the subscripts (see [11]). In terms of these operations, equation (6) can be written in the form $\mathrm{q}\,X = A$, where

$$X = h^{j_1 j_2 \ldots j_{r-1}}_{\sigma}{}_{i_1 i_2 \ldots i_{n-2}}, \quad A = \frac{r}{r+1} \frac{\partial h_{i_1 i_2 \ldots i_{n-1}}}{\partial y^{\sigma}_{j_1 j_2 \ldots j_r}}.$$

Recall that

$$\mathrm{q}\,X = \frac{r(n-1)}{r+1} h^{j_2 j_3 \ldots j_r}_{\sigma}{}_{i_2 i_3 \ldots i_{n-1}} \delta^{j_1}_{i_1} \quad \mathrm{alt}(i_1 i_2 \ldots i_{n-1}) \quad \mathrm{sym}(j_1 j_2 \ldots j_r).$$

But we have the identities $A = \mathrm{tr}\,\mathrm{q}\,A + \mathrm{q}\,\mathrm{tr}\,A$ and $\mathrm{q}\,\mathrm{q}\,X = 0$, and we know that a necessary and sufficient condition for existence of a solution X is $\mathrm{q}\,A = 0$ or, equivalently, $A - \mathrm{q}\,\mathrm{tr}\,A = 0$. Since

$$\mathrm{q}\,\mathrm{tr}\,A = \frac{r^2(n-1)}{(r+1)^2} \frac{\partial h_{s i_2 i_3 \ldots i_{n-1}}}{\partial y^{\sigma}_{s j_2 j_3 \ldots j_r}} \delta^{j_1}_{i_1} \quad \mathrm{sym}(j_1 j_2 \ldots j_r) \quad \mathrm{alt}(i_1 i_2 \ldots i_{n-1}),$$

and

$$A - \mathrm{q}\,\mathrm{tr}\,A = \frac{r}{r+1} \frac{\partial h_{i_1 i_2 \ldots i_{n-1}}}{\partial y^{\sigma}_{j_1 j_2 \ldots j_r}} - \frac{r^2(n-1)}{(r+1)^2} \frac{\partial h_{s i_2 i_3 \ldots i_{n-1}}}{\partial y^{\sigma}_{s j_2 j_3 \ldots j_r}} \delta^{j_1}_{i_1}$$

$$\mathrm{sym}(j_1 j_2 \ldots j_r) \quad \mathrm{alt}(i_1 i_2 \ldots i_{n-1}),$$

we see that (a) implies (b). Note that condition (b) means that the expression

$$\frac{\partial h_{i_1 i_2 \ldots i_{n-1}}}{\partial y^{\sigma}_{j_1 j_2 \ldots j_r}}$$

has vanishing its traceless component.

2. Property (c) of the form η follows from (b) by Lemma 1.

3. Suppose that the functions g^i satisfy condition (c),

$$\frac{\partial g^{i_1}}{\partial y^{\sigma}_{i_2 i_3 \ldots i_{r+1}}} + \frac{\partial g^{i_2}}{\partial y^{\sigma}_{i_1 i_3 \ldots i_{r+1}}} + \frac{\partial g^{i_3}}{\partial y^{\sigma}_{i_2 i_1 i_4 \ldots i_{r+1}}} + \ldots + \frac{\partial g^{i_{r+1}}}{\partial y^{\sigma}_{i_2 i_3 \ldots i_r i_1}} = 0. \quad (7)$$

We wish to show that these functions are necessarily polynomials of degree $\leq n-1$ in the variables $y^{\nu}_{j_1 j_2 \ldots j_r}$. In fact we prove that every solution g^i of equation (7) is necessarily a polynomial in the variables y^{ν}_J of degree $\leq n-1$. It will be convenient to work with multi-indices of length r, $J = (j_1 j_2 \ldots j_r)$. We assert that

$$\frac{\partial^n g^i}{\partial y^{\sigma_1}_{J_1} \partial y^{\sigma_2}_{J_2} \ldots \partial y^{\sigma_n}_{J_n}} = 0. \quad (8)$$

We prove (8) by showing that all Young diagrams, defining the Young decomposition of the expression on the left of (8), vanish. Since this expression is symmetric in the indices entering every of the multi-indices $J_1, J_2, \ldots, J_n$, only the diagrams, which contain any of the blocks $J_1, J_2, \ldots, J_n$ in a row, can make a nonzero contribution. Thus, we can restrict our attention to typical diagrams, which include the blocks $J_1, J_2, \ldots, J_n$ as follows:

J_1	J_2	J_3	$\ldots$		$\ldots$	J_{k_1}
J_{k_1+1}	J_{k_1+2}	$\ldots$		$J_{k_1+k_2}$		
$J_{k_1+k_2+1}$	$J_{k_1+k_2+2}$	$\ldots$				
$\ldots$						

(diagrams with different position of indices in each row give analogous Young projectors). On the other hand, the Young diagrams for the decomposition of (8) should also include the index i. If this index stands in a row, which contains at least one of the blocks $J_1, J_2, \ldots, J_n$, we get necessarily the zero Young projector, by (7). Thus, a non-zero projector may possibly arise only from the diagram, in which i is placed on the

bottom:

J_1	J_2	J_3	...		...	J_{k_1}
J_{k_1+1}	J_{k_1+2}	...		$J_{k_1+k_2}$		
$J_{k_1+k_2+1}$	$J_{k_1+k_2+2}$	...				
...						
i						

However, then it follows again from (7) and from antisymmetry of the corresponding Young projector in the first column, that we get the zero contribution whenever $k_1 \geq 2$; otherwise we simply transform the index i to the first column by a permutation in the first column. Thus, a nonzero contribution may arise only for the diagram

J_1
J_2
...
J_n
i

But the Young projector, corresponding with this diagram, is the zero projector because this diagram contains $n+1$ rows. This proves that g^i satisfies condition (8). In particular, g^i must be a polynomial in $y^\sigma_{i_1 i_2 \dots i_s}$ of degree $\leq n-1$.

Let g^i_p be the homogeneous component of g^i of degree p. Then

$$g^i = g^i_0 + g^i_1 + g^i_2 + \dots + g^i_{n-1}, \tag{9}$$

and we have for each $p = 0, 1, 2, \dots, n-1$

$$\frac{\partial g^{i_1}_p}{\partial y^\sigma_{i_2 i_3 \dots i_{r+1}}} + \frac{\partial g^{i_2}_p}{\partial y^\sigma_{i_1 i_3 \dots i_{r+1}}} + \frac{\partial g^{i_3}_p}{\partial y^\sigma_{i_2 i_1 i_4 \dots i_{r+1}}} + \dots + \frac{\partial g^{i_{r+1}}_p}{\partial y^\sigma_{i_2 i_3 \dots i_r i_1}} = 0.$$

Moreover, since for every $p = 1, 2, \dots, n-1$

$$g^i_p = \frac{1}{p} \frac{\partial g^i_p}{\partial y^{\sigma_1}_{I_1}} y^{\sigma_1}_{I_1},$$

we have

$$g^i_p = \frac{1}{p^2} \frac{\partial^2 g^i_p}{\partial y^{\sigma_1}_{I_1} \partial y^{\sigma_2}_{I_2}} y^{\sigma_1}_{I_1} y^{\sigma_2}_{I_2} + \frac{1}{p} g^i_p,$$

i.e.,

$$g^i_p = \frac{1}{p(p-1)} \frac{\partial^2 g^i_p}{\partial y^{\sigma_1}_{I_1} \partial y^{\sigma_2}_{I_2}} y^{\sigma_1}_{I_1} y^{\sigma_2}_{I_2}.$$

Continuing we get

$$g_p^i = \frac{1}{p!} \frac{\partial^p g_p^i}{\partial y_{I_1}^{\sigma_1} \partial y_{I_2}^{\sigma_2} ... \partial y_{I_p}^{\sigma_p}} y_{I_1}^{\sigma_1} y_{I_2}^{\sigma_2} ... y_{I_p}^{\sigma_p}, \tag{10}$$

where the coefficients are independent of y_I^σ, $|I| = r$. Writing

$$\frac{\partial^p g_p^i}{\partial y_{I_1}^{\sigma_1} \partial y_{I_2}^{\sigma_2} ... \partial y_{I_p}^{\sigma_p}} = \frac{\partial^p g_p^i}{\partial y_{J_1 j_1}^{\sigma_1} \partial y_{J_2 j_2}^{\sigma_2} ... \partial y_{J_p j_p}^{\sigma_p}} \tag{11}$$

and analyzing this expression in the same way as above, we see that we have a unique Young diagram giving a possibly nontrivial Young symmetrizer, namely

j_1	J_1
j_2	J_2
...	
j_{p-1}	J_{p-1}
j_p	J_p
i	

Thus, the coefficients (11) are antisymmetric in $i, j_1, j_2, ..., j_p$. This completes our description of the functions g^i (9), satisfying condition (7).

Write for the coefficients in (10)

$$\frac{\partial^p g_p^i}{\partial y_{I_1}^{\sigma_1} \partial y_{I_2}^{\sigma_2} ... \partial y_{I_p}^{\sigma_p}} = \frac{\partial^p g_p^j}{\partial y_{J_1 i_1}^{\sigma_1} \partial y_{J_2 i_2}^{\sigma_2} ... \partial y_{J_p i_p}^{\sigma_p}} \delta_j^i \delta_{i_1}^{j_1} \delta_{i_2}^{j_2} ... \delta_{i_p}^{j_p} \quad \text{alt}(i j_1 j_2 ... j_p)$$

$$= \frac{1}{(p+1)!(n-p-1)!} \frac{\partial^p g_p^i}{\partial y_{J_1 i_1}^{\sigma_1} \partial y_{J_2 i_2}^{\sigma_2} ... \partial y_{J_p i_p}^{\sigma_p}}$$

$$\cdot \varepsilon_{i i_1 i_2 ... i_{p-1} i_p l_{p+1} ... l_{n-1}} \varepsilon^{j j_1 j_2 ... j_{p-1} j_p l_{p+1} ... l_{n-1}}$$

where $I_p = J_p j_p$. Set

$$A_{\sigma_1 \sigma_2 ... \sigma_p l_{p+1} l_{p+2} ... l_{n-1}}^{J_1 J_2 \quad J_p} = \frac{1}{(p+1)!} \frac{\partial^p g_p^i}{\partial y_{J_1 k_1}^{\sigma_1} \partial y_{J_2 k_2}^{\sigma_2} ... \partial y_{J_p k_p}^{\sigma_p}} \varepsilon_{i k_1 k_2 ... k_p l_{p+1} l_{p+2} ... l_{n-1}},$$

and for each $p = 1, 2, ..., n-1$

$$\mu_p = \frac{1}{p!(n-p-1)!} A_{\sigma_1 \sigma_2 ... \sigma_p l_{p+1} l_{p+2} ... l_{n-1}}^{J_1 J_2 \quad J_p} dy_{J_1}^{\sigma_1} \wedge dy_{J_2}^{\sigma_2} \wedge ... \wedge dy_{J_p}^{\sigma_p}$$

$$\wedge dx^{l_{p+1}} \wedge dx^{l_{p+2}} \wedge ... \wedge dx^{l_{n-1}}.$$

We extend this definition for $p = 0$, setting

$$\mu_0 = g_0^i \omega_i.$$

Let us now consider the form

$$\mu = \mu_0 + \mu_1 + \mu_2 + ... + \mu_{n-1}.$$

μ is obviously defined on V^{r-1}. Since

$$\varepsilon^{ik_2 k_3 ... k_n} \omega_i = dx^{k_2} \wedge dx^{k_3} \wedge ... \wedge dx^{k_n},$$

we have

$$
\begin{aligned}
h\mu_p &= \frac{1}{p!(n-p-1)!} A^{J_1 J_2}_{\sigma_1 \sigma_2} ..._{\sigma_p l_{p+1} l_{p+2} ... l_{n-1}}^{J_p} y^{\sigma_1}_{J_1 i_1} y^{\sigma_2}_{J_2 i_2} ... y^{\sigma_p}_{J_p i_p} \\
&\quad \cdot dx^{i_1} \wedge dx^{i_2} \wedge ... \wedge dx^{i_p} \wedge dx^{l_{p+1}} \wedge ... \wedge dx^{l_{n-1}} \\
&= (p+1) \frac{1}{(p+1)!} \frac{\partial^p g^i_p}{\partial y^{\sigma_1}_{J_1 i_1} \partial y^{\sigma_2}_{J_2 i_2} ... \partial y^{\sigma_p}_{J_p i_p}} y^{\sigma_1}_{J_1 i_1} y^{\sigma_2}_{J_2 i_2} ... y^{\sigma_p}_{J_p i_p} \omega_i \\
&= g^i_p \omega_i,
\end{aligned}
$$

in accordance with (10). Now we can conclude that

$$h\mu = h\mu_0 + \sum_{p=1}^{n-1} h\mu_p = g^i_0 \omega_i + \sum_{p=1}^{n-1} g^i_p \omega_i = g^i \omega_i = \eta.$$

This completes the proof of Lemma 2. $\square$

4. Lepage equivalents of a lagrangian

We have the following assertion.

Lemma 3. *For any function $f : V^r \to \mathbb{R}$, there exists an n-form Θ, defined on V^{2r-1}, such that (a) $\lambda = h\Theta$, and (b) the form $p_1 d\Theta$ is ω^σ-generated.*

Proof. We search for Θ of the form

$$\Theta = f\omega_0 + (f^i_{\ \sigma}\omega^\sigma + f^{ij_1}_{\ \sigma}\omega^\sigma_{j_1} + f^{ij_1 j_2}_{\ \sigma}\omega^\sigma_{j_1 j_2} + ... + f^{ij_1 j_2 ... j_{r-1}}_{\ \sigma}\omega^\sigma_{j_1 j_2 ... j_{r-1}}) \wedge \omega_i, \quad (12)$$

with undetermined coefficients $f^i_{\ \sigma}, f^{ij_1}_{\ \sigma}, f^{ij_1 j_2}_{\ \sigma}, ..., f^{ij_1 j_2 ... j_{r-1}}_{\ \sigma}$. Note that

$$
\begin{aligned}
\omega^{\sigma_1}_{j_1 j_2 ... j_{r-1}} \wedge \omega_i &= \omega^{\sigma_1}_{j_1 j_2 ... j_{r-1}} \wedge \omega_i \quad \text{sym}(j_1 j_2 ... j_{r-1} i) \\
&+ \frac{r-1}{r} d(\omega^{\sigma_1}_{j_2 j_3 ... j_{r-1}} \wedge \omega_{j_1 i}) \quad \text{sym}(j_1 j_2 ... j_{r-1}).
\end{aligned}
\quad (13)
$$

Denote by $\tilde{f}_\sigma^{ij_1j_2\dots j_{r-1}}$ the expression $f_\sigma^{ij_1j_2\dots j_{r-1}}$, *symmetrized* in the superscripts. Applying formula (13) to the last term in (12), we obtain

$$f_\sigma^{ij_1j_2\dots j_{r-1}}\omega^\sigma_{j_1j_2\dots j_{r-1}} \wedge \omega_i = \tilde{f}_\sigma^{ij_1j_2\dots j_{r-1}}\omega^\sigma_{j_1j_2\dots j_{r-1}} \wedge \omega_i$$
$$+\frac{r-1}{r}f_\sigma^{ij_1j_2\dots j_{r-1}}d(\omega^\sigma_{j_2j_3\dots j_{r-1}} \wedge \omega_{j_1i})$$
$$= \tilde{f}_\sigma^{ij_1j_2\dots j_{r-1}}\omega^\sigma_{j_1j_2\dots j_{r-1}} \wedge \omega_i$$
$$+\frac{r-1}{r}d_k(f_\sigma^{kij_2j_3\dots j_{r-1}} - f_\sigma^{ikj_2j_3\dots j_{r-1}})\omega^\sigma_{j_2j_3\dots j_{r-1}} \wedge \omega_i$$
$$+\frac{r-1}{r}d(f_\sigma^{ij_1j_2\dots j_{r-1}}\omega^\sigma_{j_2j_3\dots j_{r-1}} \wedge \omega_{j_1i})$$
$$-\frac{r-1}{r}pdf_\sigma^{ij_1j_2\dots j_{r-1}} \wedge \omega^\sigma_{j_2j_3\dots j_{r-1}} \wedge \omega_{j_1i}.$$

Then

$$\Theta = f\omega_0 + (f^i_\sigma\omega^\sigma + f^{ij_1}_\sigma\omega^\sigma_{j_1} + f^{ij_1j_2}_\sigma\omega^\sigma_{j_1j_2}$$
$$+\dots + f_\sigma^{ij_1j_2\dots j_{r-2}}\omega^\sigma_{j_1j_2\dots j_{r-2}}) \wedge \omega_i + \tilde{f}_\sigma^{ij_1j_2\dots j_{r-1}}\omega^\sigma_{j_1j_2\dots j_{r-1}} \wedge \omega_i$$
$$+\frac{r-1}{r}d_k(f_\sigma^{kij_1j_2\dots j_{r-2}} - f_\sigma^{ikj_1j_2\dots j_{r-2}})\omega^\sigma_{j_1j_2\dots j_{r-2}} \wedge \omega_i$$
$$+\frac{r-1}{r}d(f_\sigma^{ij_1j_2\dots j_{r-1}}\omega^\sigma_{j_2j_3\dots j_{r-1}} \wedge \omega_{j_1i})$$
$$-\frac{r-1}{r}pdf_\sigma^{ij_1j_2\dots j_{r-1}} \wedge \omega^\sigma_{j_2j_3\dots j_{r-1}} \wedge \omega_{j_1i}.$$

We can apply the same decomposition to the term

$$\left(f_\sigma^{ij_1j_2\dots j_{r-2}} + \frac{r-1}{r}d_k(f_\sigma^{kij_1j_2\dots j_{r-2}} - f_\sigma^{ikj_1j_2\dots j_{r-2}})\right)\omega^\sigma_{j_1j_2\dots j_{r-2}} \wedge \omega_i$$

defined on V^r, etc. After $r-1$ steps we obtain a form on V^{2r-1},

$$\Theta = f\omega_0+(f^i_\sigma\omega^\sigma+\tilde{f}^{ij_1}_\sigma\omega^\sigma_{j_1}+\tilde{f}^{ij_1j_2}_\sigma\omega^\sigma_{j_1j_2}+\dots+\tilde{f}_\sigma^{ij_1j_2\dots j_{r-2}}\omega^\sigma_{j_1j_2\dots j_{r-2}}$$
$$+\tilde{f}_\sigma^{ij_1j_2\dots j_{r-1}}\omega^\sigma_{j_1j_2\dots j_{r-1}}) \wedge \omega_i + d\eta + \mu, \qquad (14)$$

with symmetric coefficients $\tilde{f}^{ij_1}_\sigma, \tilde{f}^{ij_1j_2}_\sigma, \dots, \tilde{f}_\sigma^{ij_1j_2\dots j_{r-2}}, \tilde{f}_\sigma^{ij_1j_2\dots j_{r-1}}$.

From formula (14) we have

$$p_1d\Theta = \left(\frac{\partial f}{\partial y^\sigma}\omega^\sigma+\frac{\partial f}{\partial y^\sigma_{k_1}}\omega^\sigma_{k_1}+\frac{\partial f}{\partial y^\sigma_{k_1k_2}}\omega^\sigma_{k_1k_2}+\dots+\frac{\partial f}{\partial y^\sigma_{k_1k_2\dots k_r}}\omega^\sigma_{k_1k_2\dots k_r}\right) \wedge \omega_0$$
$$-\left(d_if^i_\sigma\omega^\sigma+d_i\tilde{f}^{ij_1}_\sigma\omega^\sigma_{j_1}+d_i\tilde{f}^{ij_1j_2}_\sigma\omega^\sigma_{j_1j_2}+\dots+d_i\tilde{f}_\sigma^{ij_1j_2\dots j_{r-1}}\omega^\sigma_{j_1j_2\dots j_{r-1}}\right) \wedge \omega_0$$
$$-(f^i_\sigma\omega^\sigma_i+\tilde{f}^{ij_1}_\sigma\omega^\sigma_{ij_1}+\tilde{f}^{ij_1j_2}_\sigma\omega^\sigma_{ij_1j_2}+\dots+\tilde{f}_\sigma^{ij_1j_2\dots j_{r-2}}\omega^\sigma_{ij_1j_2\dots j_{r-2}}$$
$$+\tilde{f}_\sigma^{ij_1j_2\dots j_{r-1}}\omega^\sigma_{ij_1j_2\dots j_{r-1}}) \wedge \omega_0.$$

Collecting together the terms with the same elements of the basis of $(n + 1)$-forms, we obtain

$$p_1 d\Theta = \left(\frac{\partial f}{\partial y^\sigma} - d_i \tilde{f}^i_\sigma \right) \omega^\sigma \wedge \omega_0 + \left(\frac{\partial f}{\partial y^\sigma_{j_1}} - d_i \tilde{f}^{ij_1}_\sigma - \tilde{f}^{j_1}_\sigma \right) \omega^\sigma_{j_1} \wedge \omega_0$$

$$+ \left(\frac{\partial f}{\partial y^\sigma_{j_1 j_2}} - d_i \tilde{f}^{ij_1 j_2}_\sigma - \tilde{f}^{j_2 j_1}_\sigma \right) \omega^\sigma_{j_1 j_2} \wedge \omega_0$$

$$+ ... + \left(\frac{\partial f}{\partial y^\sigma_{j_1 j_2 ... j_{r-1}}} - d_i \tilde{f}^{ij_1 j_2 ... j_{r-1}}_\sigma - \tilde{f}^{j_{r-1} j_1 j_2 ... j_{r-2}}_\sigma \right) \omega^\sigma_{j_1 j_2 ... j_{r-1}} \wedge \omega_0$$

$$+ \left(\frac{\partial f}{\partial y^\sigma_{j_1 j_2 ... j_r}} - \tilde{f}^{j_r j_1 j_2 ... j_{r-1}}_\sigma \right) \omega^\sigma_{j_1 j_2 ... j_r} \wedge \omega_0.$$

Consequently, $p_1 d\Theta$ is ω^σ-generated if and only if

$$\frac{\partial f}{\partial y^\sigma_{j_1 j_2 ... j_r}} - \tilde{f}^{j_r j_1 j_2 ... j_{r-1}}_\sigma = 0,$$

$$\frac{\partial f}{\partial y^\sigma_{j_1 j_2 ... j_{r-1}}} - d_i \tilde{f}^{ij_1 j_2 ... j_{r-1}}_\sigma - \tilde{f}^{j_{r-1} j_1 j_2 ... j_{r-2}}_\sigma = 0,$$

$$\frac{\partial f}{\partial y^\sigma_{j_1 j_2 ... j_{r-2}}} - d_i \tilde{f}^{ij_1 j_2 ... j_{r-2}}_\sigma - \tilde{f}^{j_{r-2} j_1 j_2 ... j_{r-3}}_\sigma = 0,$$

$$...$$

$$\frac{\partial f}{\partial y^\sigma_{j_1 j_2}} - d_i \tilde{f}^{ij_1 j_2}_\sigma - \tilde{f}^{j_2 j_1}_\sigma = 0,$$

$$\frac{\partial f}{\partial y^\sigma_{j_1}} - d_i \tilde{f}^{ij_1}_\sigma - \tilde{f}^{j_1}_\sigma = 0.$$

These equations have a unique solution

$$\tilde{f}^{j_r j_1 j_2 ... j_{r-1}}_\sigma = \frac{\partial f}{\partial y^\sigma_{j_1 j_2 ... j_r}},$$

$$\tilde{f}^{j_{r-1} j_1 j_2 ... j_{r-2}}_\sigma = \frac{\partial f}{\partial y^\sigma_{j_1 j_2 ... j_{r-1}}} - d_{i_r} \frac{\partial f}{\partial y^\sigma_{j_1 j_2 ... j_{r-1} i_r}},$$

$$\tilde{f}^{j_{r-2} j_1 j_2 ... j_{r-3}}_\sigma = \frac{\partial f}{\partial y^\sigma_{j_1 j_2 ... j_{r-2}}} - d_{i_{r-1}} \frac{\partial f}{\partial y^\sigma_{j_1 j_2 ... j_{r-2} i_{r-1}}}$$

$$+ d_{i_{r-1}} d_{i_r} \frac{\partial f}{\partial y^\sigma_{j_1 j_2 ... j_{r-2} i_{r-1} i_r}},$$

$$...$$

$$\tilde{f}_\sigma^{j_2 j_1} = \frac{\partial f}{\partial y_{j_1 j_2}^\sigma} - d_{i_3}\frac{\partial f}{\partial y_{j_1 j_2 i_3}^\sigma} + d_{i_3}d_{i_4}\frac{\partial f}{\partial y_{j_1 j_2 i_3 i_4}^\sigma}$$

$$-... + (-1)^{r-2}d_{i_3}d_{i_4}...d_{i_r}\frac{\partial f}{\partial y_{j_1 j_2 i_3 i_4 ... i_r}^\sigma},$$

$$\tilde{f}_\sigma^{j_1} = \frac{\partial f}{\partial y_{j_1}^\sigma} - d_{i_2}\frac{\partial f}{\partial y_{j_1 i_2}^\sigma} + d_{i_2}d_{i_3}\frac{\partial f}{\partial y_{j_1 i_2 i_3}^\sigma}$$

$$-... + (-1)^{r-1}d_{i_2}d_{i_3}...d_{i_r}\frac{\partial f}{\partial y_{j_1 i_2 i_3 i_4 ... i_r}^\sigma}.$$

This proves Lemma 3. □

Any form Θ, satisfying properties (a) and (b) of Lemma 3, is called a *Lepage equivalent* of f. The form

$$\Theta_f = f\omega_0 + (f^i{}_\sigma\omega^\sigma + \tilde{f}_\sigma^{ij_1}\omega_{j_1}^\sigma + \tilde{f}_\sigma^{ij_1 j_2}\omega_{j_1 j_2}^\sigma + ... + \tilde{f}_\sigma^{ij_1 j_2...j_{r-2}}\omega_{j_1 j_2...j_{r-2}}^\sigma$$

$$+\tilde{f}_\sigma^{ij_1 j_2...j_{r-1}}\omega_{j_1 j_2...j_{r-1}}^\sigma) \wedge \omega_i$$

is the *principal Lepage equivalent* of f.

Note that Lepage equivalents Θ of f are defined by prescribing some properties of the exterior derivative $d\Theta$. The meaning of any Θ for the total divergence equation consists in the structure of the form $p_1 d\Theta$. Computing $p_1 d\Theta$, we obtain the *Euler-Lagrange form*

$$p_1 d\Theta = \left(\frac{\partial f}{\partial y^\sigma} - d_i f^i{}_\sigma\right)\omega^\sigma \wedge \omega_0 = E_\sigma(f)\omega^\sigma \wedge \omega_0.$$

5. The fibered homotopy operator

Let $U \subset \mathbb{R}^n$ be an open set, let $W \subset \mathbb{R}^m$ be an open ball with center at the origin, and let $\zeta : U \to U \times W$ be the zero section. We define a mapping $\chi : [0,1] \times U \times W \to U \times W$ by

$$\chi(s,(x^i,y^\sigma)) = (x^i, sy^\sigma).$$

Then

$$\chi^* dx^i = dx^i, \quad \chi^* dy^\sigma = y^\sigma ds + s dy^\sigma. \tag{15}$$

For any k-form ρ on $U \times W$, where $k \geq 1$, consider the pull-back $\chi^*\rho$, which is a k-form on the set $[0,1] \times U \times W$. Obviously, there exists a unique decomposition

$$\chi^*\rho = ds \wedge \rho^{(0)}(s) + \rho'(s) \tag{16}$$

such that the $(k-1)$-form $\rho^{(0)}(s)$ and k-form $\rho'(s)$ do not contain ds. Note that by (15), $\rho'(s)$ arises from ρ by replacing each factor dy^σ by

sdy^σ, and by replacing each coefficient f in ρ by $f \circ \chi$; the factors dx^i remain unchanged. Thus, $\rho'(s)$ obeys

$$\rho'(1) = \rho, \quad \rho'(0) = \pi^*\zeta^*\rho. \tag{17}$$

Define

$$I\rho = \int \rho^{(0)}(s), \tag{18}$$

where the expression on the right means integration of the coefficients in the form $\rho^{(0)}(s)$ over s from 0 to 1. If $f : \times W \to \mathbb{R}$ is a function, we define

$$If = 0.$$

The mapping $\rho \to I\rho$ is called the *fibered homotopy operator*.

We prove the following result.

Lemma 4. *For any differential k-form ρ on $U \times W$,*

$$\rho = Id\rho + dI\rho + \pi^*\zeta^*\rho.$$

Proof. 1. Let $k = 0$. If f is a function, we have by (15)

$$\chi^*df = \left(\frac{\partial f}{\partial x^i} \circ \chi\right)dx^i + \left(\frac{\partial f}{\partial y^\sigma} \circ \chi\right)(y^\sigma ds + sdy^\sigma),$$

and

$$Idf = y^\sigma \int \left(\frac{\partial f}{\partial y^\sigma} \circ \chi\right)ds.$$

Now the identity

$$f - \pi^*\zeta^*f = f \circ \chi|_{s=1} - f \circ \chi|_{s=0} = \int \frac{d(f \circ \chi)}{ds}ds = y^\sigma \int \left(\frac{\partial f}{\partial y^\sigma} \circ \chi\right)ds$$

gives the result.

2. Let $k = 1$. ρ has an expression

$$\rho = A_i dx^i + B_\sigma dy^\sigma.$$

Thus

$$\chi^*\rho = y^\sigma(B_\sigma \circ \chi)ds + (A_i \circ \chi)dx^i + (B_\sigma \circ \chi)sdy^\sigma,$$

and

$$\chi^*d\rho = ds \wedge \left(-d(y^\sigma(B_\sigma \circ \chi)) + \frac{\partial(A_i \circ \chi)}{\partial s}dx^i + \frac{\partial((B_\sigma \circ \chi)s)}{\partial s}dy^\sigma\right)$$
$$+ \left(\frac{\partial(A_i \circ \chi)}{\partial x^j}dx^j + \frac{\partial(A_i \circ \chi)}{\partial y^\nu}dy^\nu\right) \wedge dx^i$$
$$+ \left(\frac{\partial(B_\sigma \circ \chi)}{\partial x^j}dx^j + \frac{\partial(B_\sigma \circ \chi)}{\partial y^\nu}dy^\nu\right) \wedge dy^\sigma,$$

hence

$$I\rho = y^\sigma \int B_\sigma \circ \chi \cdot ds,$$

$$Id\rho = \int \left(\frac{\partial(A_i \circ \chi)}{\partial s} - \frac{\partial(y^\nu \cdot B_\nu \circ \chi)}{\partial x^i} \right) ds \cdot dx^i$$

$$+ \int \left(\frac{\partial((B_\sigma \circ \chi)s)}{\partial s} - \frac{\partial(y^\nu \cdot B_\nu \circ \chi)}{\partial y^\sigma} \right) ds \cdot dy^\sigma,$$

and

$$dI\rho = y^\sigma \int \frac{\partial}{\partial x^i}(B_\sigma \circ \chi)ds \cdot dx^i + \int \frac{\partial}{\partial y^\sigma}(y^\nu \cdot B_\nu \circ \chi)ds \cdot dy^\sigma.$$

Consequently,

$$Id\rho + dI\rho = \int \left(\frac{\partial(A_i \circ \chi)}{\partial s} \right) ds \cdot dx^i + \int \left(\frac{\partial((B_\sigma \circ \chi)s)}{\partial s} \right) ds \cdot dy^\sigma$$

$$= \rho - \pi^*\zeta^*\rho.$$

3. Let $k \geq 2$. Write ρ in the form

$$\rho = dx^i \wedge \Phi_i + dy^\sigma \wedge \Psi_\sigma,$$

and we define differential forms $\Phi_i^{(0)}(s), \Phi_i'(s), \Psi_\sigma^{(0)}(s), \Psi_\sigma'(s)$ by the following decompositions

$$\chi^*\Phi_i = ds \wedge \Phi_i^{(0)}(s) + \Phi_i'(s), \quad \chi^*\Psi_\sigma = ds \wedge \Psi_\sigma^{(0)}(s) + \Psi_\sigma'(s).$$

Then we get

$$\chi^*\rho = ds \wedge (-dx^i \wedge \Phi_i^{(0)}(s) - sdy^\sigma \wedge \Psi_\sigma^{(0)}(s) + y^\sigma\Psi_\sigma'(s))$$

$$+ dx^i \wedge \Phi_i'(s) + sdy^\sigma \wedge \Psi_\sigma'(s).$$

Thus,

$$I\rho = -dx^i \wedge \int \Phi_i^{(0)}(s) - dy^\sigma \wedge \int s\Psi_\sigma^{(0)}(s) + y^\sigma \int \Psi_\sigma'(s)ds.$$

To determine $Id\rho$, we compute $\chi^*d\rho$. We get

$$\chi^*d\rho = d\chi^*\rho = ds \wedge \left(-dx^i \wedge d\Phi_i^{(0)}(s) + dx^i \wedge \frac{\partial\Phi_i'(s)}{\partial s} \right.$$

$$-sdy^\sigma \wedge d\Psi_\sigma^{(0)}(s) - dy^\sigma \wedge \Psi_\sigma'(s)$$

$$\left. -y^\sigma d\Psi_\sigma'(s) + dy^\sigma \wedge \frac{\partial(s\Psi_\sigma'(s))}{\partial s} \right) \qquad (19)$$

$$-dx^i \wedge \left(dx^j \wedge \frac{\partial\Phi_i'(s)}{\partial x^j} + dy^\nu \wedge \frac{\partial\Phi_i'(s)}{\partial y^\nu} \right)$$

$$-dy^\sigma \wedge \left(dx^j \wedge \frac{\partial(s\Psi_\sigma'(s))}{\partial x^j} + dy^\nu \wedge \frac{\partial(s\Psi_\sigma'(s))}{\partial y^\nu} \right),$$

where $\partial\eta(s)/\partial s$ denotes the form, arising from $\eta(s)$ by differentiation of the coefficients with respect to s. Now by (19) and (18),

$$
I d\rho = -dx^i \wedge \int d\Phi_i^{(0)}(s) - dy^\sigma \wedge \int s d\Psi_\sigma^{(0)}(s) - dy^\sigma \wedge \int \Psi_\sigma'(s)
$$
$$
-y^\sigma \int d\Psi_\sigma'(s) + dx^i \wedge \int \frac{\partial\Phi_i'(s)}{\partial s} + dy^\sigma \wedge \int \frac{\partial(s\Psi_\sigma'(s))}{\partial s}.
$$

It is important to notice that the exterior derivatives $d\Phi_i^{(0)}(s), d\Psi_\sigma^{(0)}(s)$, and $d\Psi_\sigma'(s)$ have the meaning of the derivatives with respect to x^i, y^σ (the terms containing ds are canceled; see the definition of I (16), (18)).

Now we easily get

$$
I d\rho + dI\rho = dx^i \wedge \int \frac{\partial\Phi_i'(s)}{\partial s} + dy^\sigma \wedge \int \frac{\partial(s\Psi_\sigma'(s))}{\partial s}.
$$

Remembering that the integral symbol denotes integration of coefficients in the corresponding forms with respect to s from 0 to 1, and using (17), one obtains the final formula

$$
I d\rho + dI\rho = dx^i \wedge \Phi_i + dy^\sigma \wedge \Psi_\sigma - dx^i \wedge \pi^*\zeta^*\Phi_i
$$
$$
= \rho - \pi^*\zeta^*\rho.
$$

$\square$

6. The total divergence equation

We now prove two theorems, describing solutions of the total divergence equation

$$
d_i g^i = f, \tag{20}
$$

where $f : V^r \to \mathbb{R}$ is a given function. By a solution of this equation we mean any system of functions $g = g^i$, defined on V^s for some nonnegative integer s, satisfying condition (20).

Lemma 5. *If the total divergence equation has a solution defined on V^s and $s \geq r + 1$, then it has a solution defined on V^{s-1}.*

Proof. Suppose that we have a solution $g = g^i$ of equation (20), defined on V^s. Since f and g^i do not depend on $y^\nu_{j_1 j_2 \dots j_s j_{s+1}}$, the functions g^i

satisfy

$$\frac{\partial f}{\partial y^\sigma_{i_1 i_2 \ldots i_{s+1}}} = d_i \frac{\partial g^i}{\partial y^\sigma_{i_1 i_2 \ldots i_{s+1}}} + \frac{1}{s+1}\left(\frac{\partial g^{i_1}}{\partial y^\sigma_{i_2 i_3 \ldots i_{s+1}}} + \frac{\partial g^{i_2}}{\partial y^\sigma_{i_1 i_3 \ldots i_{s+1}}} \right.$$
$$+ \frac{\partial g^{i_3}}{\partial y^\sigma_{i_2 i_1 i_4 \ldots i_{s+1}}} + \ldots + \frac{\partial g^{i_{s+1}}}{\partial y^\sigma_{i_2 i_3 \ldots i_s i_1}} \right)$$
$$= \frac{1}{s+1}\left(\frac{\partial g^{i_1}}{\partial y^\sigma_{i_2 i_3 \ldots i_{s+1}}} + \frac{\partial g^{i_2}}{\partial y^\sigma_{i_1 i_3 \ldots i_{s+1}}} + \frac{\partial g^{i_3}}{\partial y^\sigma_{i_2 i_1 i_4 \ldots i_{s+1}}} \right.$$
$$+ \ldots + \frac{\partial g^{i_{s+1}}}{\partial y^\sigma_{i_2 i_3 \ldots i_s i_1}} \right) = 0. \tag{21}$$

Analogously, with the help of (21),

$$\frac{\partial f}{\partial y^\sigma_{i_1 i_2 \ldots i_s}} = d'_i \frac{\partial g^i}{\partial y^\sigma_{i_1 i_2 \ldots i_s}} + \frac{\partial^2 g^i}{\partial y^\nu_{j_1 j_2 \ldots j_s} \partial y^\sigma_{i_1 i_2 \ldots i_s}} y^\nu_{j_1 j_2 \ldots j_s i}$$
$$+ \frac{1}{s}\left(\frac{\partial g^{i_1}}{\partial y^\sigma_{i_2 i_3 \ldots i_s}} + \frac{\partial g^{i_2}}{\partial y^\sigma_{i_1 i_3 \ldots i_s}} + \frac{\partial g^{i_3}}{\partial y^\sigma_{i_2 i_1 i_4 \ldots i_s}} + \ldots + \frac{\partial g^{i_s}}{\partial y^\sigma_{i_2 i_3 \ldots i_{s-1} i_1}} \right)$$
$$= d'_i \frac{\partial g^i}{\partial y^\sigma_{i_1 i_2 \ldots i_s}} + \frac{1}{s}\left(\frac{\partial g^{i_1}}{\partial y^\sigma_{i_2 i_3 \ldots i_s}} + \frac{\partial g^{i_2}}{\partial y^\sigma_{i_1 i_3 \ldots i_s}} + \frac{\partial g^{i_3}}{\partial y^\sigma_{i_2 i_1 i_4 \ldots i_s}} \right.$$
$$+ \ldots + \frac{\partial g^{i_s}}{\partial y^\sigma_{i_2 i_3 \ldots i_{s-1} i_1}} \right) = 0 \tag{22}$$

because $s \geq r + 1$. From (21) and (22) we obtain

$$f = \frac{\partial g^i}{\partial x^i} + \frac{\partial g^{j_1}}{\partial y^\nu} y^\nu_{j_1} + \frac{\partial g^{j_2}}{\partial y^\nu_{j_1}} y^\nu_{j_1 j_2} + \ldots + \frac{\partial g^{j_{s-1}}}{\partial y^\nu_{j_1 j_2 \ldots j_{s-2}}} y^\nu_{j_1 j_2 \ldots j_{s-2} j_{s-1}}$$
$$- d'_i \frac{\partial g^i}{\partial y^\nu_{j_1 j_2 \ldots j_s}} y^\nu_{j_1 j_2 \ldots j_s}. \tag{23}$$

We already know that as a consequence of condition (1),

$$g^i = g^i_0 + g^i_1 + g^i_2 + \ldots + g^i_{n-1}, \tag{24}$$

where g^i_p is a homogeneous polynomial of degree p (proof of Lemma 2, part 3). Note that substituting from (24) to (23), we get the sum of homogeneous polynomials of degree p in $y^\sigma_{l_1 l_2 \ldots l_s}$,

$$\frac{\partial g^i_p}{\partial x^i} + \frac{\partial g^{j_1}_p}{\partial y^\nu} y^\nu_{j_1} + \frac{\partial g^{j_2}_p}{\partial y^\nu_{j_1}} y^\nu_{j_1 j_2} + \ldots + \frac{\partial g^{j_{s-1}}_p}{\partial y^\nu_{j_1 j_2 \ldots j_{s-2}}} y^\nu_{j_1 j_2 \ldots j_{s-2} j_{s-1}} - d'_i \frac{\partial g^i_p}{\partial y^\nu_{j_1 j_2 \ldots j_s}} y^\nu_{j_1 j_2 \ldots j_s}.$$

Then setting $y^\nu_{j_1 j_2 \ldots j_s} = 0$ on both sides, we obtain

$$f = \frac{\partial g^i_0}{\partial x^i} + \frac{\partial g^{j_1}_0}{\partial y^\sigma} y^\sigma_{j_1} + \frac{\partial g^{j_2}_0}{\partial y^\sigma_{j_1}} y^\sigma_{j_1 j_2} + \ldots + \frac{\partial g^{j_{s-1}}_0}{\partial y^\sigma_{j_1 j_2 \ldots j_{s-2}}} y^\sigma_{j_1 j_2 \ldots j_{s-2} j_{s-1}} = d_i g^i_0.$$

Thus, under the hypothesis (b), we have constructed from the solution g^i of the total divergence equation, defined on V^s, a new solution, g_0^i, defined on V^{s-1}. $\qquad\square$

Theorem 1. *Let $f : V^r \to \mathbb{R}$ be a function. The following two conditions are equivalent:*

(a) *The total divergence equation has a solution, defined on V^r.*

(b) *The function f satisfies*

$$E_\sigma(f) = 0.$$

Proof. 1. Suppose that (1) has a solution $g = g^i$. Then from (2), Section 1,

$$\frac{\partial d_i g^i}{\partial y^\sigma} = d_i \frac{\partial g^i}{\partial y^\sigma},$$

and for every $k = 1, 2, ..., r$,

$$\frac{\partial d_i g^i}{\partial y^\sigma_{i_1 i_2 ... i_k}} = d_i \frac{\partial g^i}{\partial y^\sigma_{i_1 i_2 ... i_k}} + \frac{1}{k} \left(\frac{\partial g^{i_1}}{\partial y^\sigma_{i_2 i_3 ... i_k}} + \frac{\partial g^{i_2}}{\partial y^\sigma_{i_1 i_3 ... i_k}} + \frac{\partial g^{i_3}}{\partial y^\sigma_{i_2 i_1 i_4 ... i_k}} \right.$$
$$\left. + ... + \frac{\partial g^{i_k}}{\partial y^\sigma_{i_2 i_3 ... i_{k-1} i_1}} \right).$$

Using these formulas, we can compute the expression $E_\sigma(f) = E_\sigma(d_i g^i)$ in several steps. First, we have

$$E_\sigma(d_i g^i) = d_{i_1} \left(\frac{\partial g^{i_1}}{\partial y^\sigma} - \frac{\partial d_s g^s}{\partial y^\sigma_{i_1}} + d_{i_2} \frac{\partial d_s g^s}{\partial y^\sigma_{i_1 i_2}} \right.$$
$$\left. - ... + (-1)^r d_{i_2} d_{i_3} ... d_{i_r} \frac{\partial d_s g^s}{\partial y^\sigma_{i_1 i_2 ... i_r}} \right)$$
$$= d_{i_1} d_{i_2} \left(-\frac{\partial g^{i_2}}{\partial y^\sigma_{i_1}} + \frac{\partial d_s g^s}{\partial y^\sigma_{i_1 i_2}} - d_{i_3} \frac{\partial d_s g^s}{\partial y^\sigma_{i_1 i_2 i_3}} \right.$$
$$\left. + ... + (-1)^{r-1} d_{i_3} d_{i_4} ... d_{i_{r-1}} \frac{\partial d_s g^s}{\partial y^\sigma_{i_1 i_2 ... i_{r-1}}} + (-1)^r d_{i_3} d_{i_4} ... d_{i_r} \frac{\partial d_s g^s}{\partial y^\sigma_{i_1 i_2 ... i_r}} \right).$$

Second, using symmetrization,

$$E_\sigma(d_i g^i) = d_{i_1} d_{i_2} \left(-\frac{\partial g^{i_2}}{\partial y^\sigma_{i_1}} + d_i \frac{\partial g^i}{\partial y^\sigma_{i_1 i_2}} + \frac{1}{2}\left(\frac{\partial g^{i_1}}{\partial y^\sigma_{i_2}} + \frac{\partial g^{i_2}}{\partial y^\sigma_{i_1}} \right) - d_{i_3}\frac{\partial d_s g^s}{\partial y^\sigma_{i_1 i_2 i_3}} \right.$$

$$\left. + ... + (-1)^{r-1} d_{i_3} d_{i_4}...d_{i_{r-1}} \frac{\partial d_s g^s}{\partial y^\sigma_{i_1 i_2...i_{r-1}}} + (-1)^r d_{i_3} d_{i_4}...d_{i_r}\frac{\partial d_s g^s}{\partial y^\sigma_{i_1 i_2...i_r}} \right)$$

$$= d_{i_1} d_{i_2} d_{i_3} \left(\frac{\partial g^{i_3}}{\partial y^\sigma_{i_1 i_2}} - \frac{\partial d_s g^s}{\partial y^\sigma_{i_1 i_2 i_3}} + ... + (-1)^{r-1} d_{i_4} d_{i_5}...d_{i_{r-1}} \frac{\partial d_s g^s}{\partial y^\sigma_{i_1 i_2...i_{r-1}}} \right.$$

$$\left. + (-1)^r d_{i_4} d_{i_5}...d_{i_r} \frac{\partial d_s g^s}{\partial y^\sigma_{i_1 i_2...i_r}} \right).$$

Third, again with the help of symmetrization,

$$E_\sigma(d_i g^i) = d_{i_1} d_{i_2} d_{i_3} \left(-d_{i_4}\frac{\partial g^{i_4}}{\partial y^\sigma_{i_1 i_2 i_3}} + d_{i_4}\frac{\partial d_s g^s}{\partial y^\sigma_{i_1 i_2 i_3 i_4}} \right.$$

$$\left. - ... + (-1)^{r-1} d_{i_4} d_{i_5}...d_{i_{r-1}} \frac{\partial d_s g^s}{\partial y^\sigma_{i_1 i_2...i_{r-1}}} + (-1)^r d_{i_4} d_{i_5}...d_{i_r} \frac{\partial d_s g^s}{\partial y^\sigma_{i_1 i_2...i_r}} \right)$$

$$= d_{i_1} d_{i_2} d_{i_3} d_{i_4} \left(-\frac{\partial g^{i_4}}{\partial y^\sigma_{i_1 i_2 i_3}} + \frac{\partial d_s g^s}{\partial y^\sigma_{i_1 i_2 i_3 i_4}} \right.$$

$$\left. - ... + (-1)^{r-1} d_{i_5} d_{i_6}...d_{i_{r-1}} \frac{\partial d_s g^s}{\partial y^\sigma_{i_1 i_2...i_{r-1}}} + (-1)^r d_{i_5} d_{i_6}...d_{i_r} \frac{\partial d_s g^s}{\partial y^\sigma_{i_1 i_2...i_r}} \right).$$

We continue this process, and obtain after $r-1$ steps

$$E_\sigma(d_i g^i) = (-1)^r d_{i_1} d_{i_2}...d_{i_{r-1}} d_{i_r} d_i \frac{\partial g^i}{\partial y^\sigma_{i_1 i_2...i_r}}. \tag{25}$$

But since f is defined on V^r, the solution g of equation (1) necessarily satisfies

$$\frac{\partial g^k}{\partial y^\sigma_{i_1 i_2...i_r}} + \frac{\partial g^{i_1}}{\partial y^\sigma_{k i_2 i_3...i_r}} + \frac{\partial g^{i_2}}{\partial y^\sigma_{i_1 k i_3 i_4...i_r}} + ... + \frac{\partial g^{i_{r-1}}}{\partial y^\sigma_{i_1 i_2...i_{r-2} k i_r}} + \frac{\partial g^{i_r}}{\partial y^\sigma_{i_1 i_2...i_{r-1} k}} = 0.$$

Using this formula in (25) we see that condition (b) is satisfied.

2. From now on, we suppose that condition (b) is satisfied. We want to show that there exist functions $g^i : V^r \to \mathbb{R}$ such that $d_i g^i = f$, or, in an explicit form,

$$\frac{\partial g^i}{\partial x^i} + \frac{\partial g^{j_1}}{\partial y^\sigma} y^\sigma_{j_1} + \frac{\partial g^{j_2}}{\partial y^\sigma_{j_1}} y^\sigma_{j_1 j_2} + ... + \frac{\partial g^{j_r}}{\partial y^\sigma_{j_1 j_2...j_{r-1}}} y^\sigma_{j_1 j_2...j_{r-1} j_r} = f,$$

and

$$\frac{\partial g^{i_1}}{\partial y^\sigma_{i_2 i_3 \ldots i_{r+1}}} + \frac{\partial g^{i_2}}{\partial y^\sigma_{i_1 i_3 i_4 \ldots i_{r+1}}} + \frac{\partial g^{i_3}}{\partial y^\sigma_{i_2 i_1 i_4 i_5 \ldots i_{r+1}}}$$
$$+ \ldots + \frac{\partial g^{i_r}}{\partial y^\sigma_{i_2 i_3 \ldots i_{r-1} i_1 i_{r+1}}} + \frac{\partial g^{i_{r+1}}}{\partial y^\sigma_{i_2 i_3 \ldots i_{r-1} i_r i_1}} = 0.$$

Let I be the fibered homotopy operator for differential form on V^{2r}, associated with the projection $\pi^{2r} : V^{2r} \to U$ (Section 5). We have

$$\Theta_f = Id\Theta_f + dI\Theta_f + \Theta_0 = Ip_1 d\Theta_f + Ip_2 d\Theta_f + dI\Theta_f + \Theta_0,$$

where Θ_0 is an n-form, projectable on U. In this formula, $p_1 d\Theta_f = 0$ by hypothesis, and we may suppose that $\Theta_0 = d\vartheta_0$ (on U). Moreover $h\Theta_f = hd(I\Theta_f + \vartheta_0) = f\omega_0$. Defining functions g^i on V^s, where $s \le 2r$, by the condition

$$h(I\Theta_f + \vartheta_0) = g^i \omega_i,$$

we see we have constructed a solution of the total divergence equation $d_i g^i = f$. Explicitly,

$$\frac{\partial g^i}{\partial x^i} + \frac{\partial g^{j_1}}{\partial y^\sigma} y^\sigma_{j_1} + \frac{\partial g^{j_2}}{\partial y^\sigma_{j_1}} y^\sigma_{j_1 j_2} + \ldots + \frac{\partial g^{j_{s+1}}}{\partial y^\sigma_{j_1 j_2 \ldots j_s}} y^\sigma_{j_1 j_2 \ldots j_s j_{s+1}} = f. \qquad (26)$$

Note, however, that in general, we have not yet proved that the total divergence equation has a solution defined on V^r.

If $s \le r$, formula (26) shows that condition (a) holds. If $s \ge r + 1$, we apply Lemma 5 several times, and obtain a solution of equation (26) defined on V^r.

This concludes the proof. □

Combining Theorem 1 and Lemma 2, we can easily describe all solutions of the total divergence equations $d_i g^i = f$ such that the right side f satisfies the *integrability condition* $E_\sigma(f) = 0$. In particular, we show that a convenient description of solutions arises when we interpret them as some differential forms.

Theorem 2. *Let $f : V^r \to \mathbb{R}$ be a function such that $E_\sigma(f) = 0$. The following conditions are equivalent:*

(a) $g = g^i$ is a solution of the total divergence equation $d_i g^i = f$, defined on V^r.

(b) The form $\eta = g^i \omega_i$ has a projectable extension.

(c) g^i *is given by*

$$g^i = \frac{1}{(n-1)!}\varepsilon^{ii_1i_2...i_{n-1}}\left(A_{i_1i_2...i_{n-1}}\right.$$

$$\left. + \sum_{k=1}^{n-1}\frac{(n-1)!}{k!(n-1-k)!}A^{J_1J_2}{}_{\sigma_1\sigma_2}{}^{...J_k}{}_{\sigma_ki_{k+1}i_{k+2}...i_{n-1}}y^{\sigma_1}_{J_1i_1}y^{\sigma_2}_{J_2i_2}...y^{\sigma_k}_{J_ki_k}\right),$$

where $A_{i_1i_2...i_{n-1}}, A^{J_1J_2}{}_{\sigma_1\sigma_2}{}^{...J_k}{}_{\sigma_ki_{k+1}i_{k+2}...i_{n-1}}$ *are arbitrary functions on* V^{r-1}, *antisymmetric in all indices and multi-indices, and* $|J_1|, |J_2|, ..., |J_{n-1}| = r - 1$.

Proof. Assertions (a) and (b) are equivalent by Lemma 2. To show that (c) is equivalent with (b), we consider any $(n-1)$-form on V^{r-1},

$$\mu = \frac{1}{(n-1)!}A_{i_1i_2...i_{n-1}}dx^{i_1}\wedge dx^{i_2}\wedge ...\wedge dx^{i_{n-1}}$$

$$+ \sum_{k=1}^{n-2}\frac{1}{k!(n-1-k)!}A^{J_1J_2}{}_{\sigma_1\sigma_2}{}^{...J_k}{}_{\sigma_ki_{k+1}i_{k+2}...i_{n-1}}dy^{\sigma_1}_{J_1}\wedge dy^{\sigma_2}_{J_2}\wedge ...\wedge dy^{\sigma_k}_{J_k}$$

$$\wedge dx^{i_{k+1}}\wedge dx^{i_{k+2}}\wedge ...\wedge dx^{i_{n-1}}$$

$$+ \frac{1}{(n-1)!}A^{J_1J_2}{}_{\sigma_1\sigma_2}{}^{...J_{n-1}}{}_{\sigma_{n-1}}dy^{\sigma_1}_{J_1}\wedge dy^{\sigma_2}_{J_2}\wedge ...\wedge dy^{\sigma_{n-1}}_{J_{n-1}} + \tau,$$

where τ is a contact form. μ has the horizontal component

$$h\mu = \frac{1}{(n-1)!}\left(A_{i_1i_2...i_{n-1}}\right.$$

$$\left. + \sum_{k=1}^{n-1}\frac{(n-1)!}{k!(n-1-k)!}A^{J_1J_2}{}_{\sigma_1\sigma_2}{}^{...J_k}{}_{\sigma_ki_{k+1}i_{k+2}...i_{n-1}}y^{\sigma_1}_{J_1i_1}y^{\sigma_2}_{J_2i_2}...y^{\sigma_k}_{J_ki_k}\right)$$

$$\cdot dx^{i_1}\wedge dx^{i_2}\wedge ...\wedge dx^{i_k}\wedge dx^{i_{k+1}}\wedge ...\wedge dx^{i_{n-1}}.$$

Transforming $h\mu$ we obtain $h\mu = g^i\omega_i$, where

$$g^i = \frac{1}{(n-1)!}\varepsilon^{ii_1i_2...i_{n-1}}\left(A_{i_1i_2...i_{n-1}}\right.$$

$$\left. + \sum_{k=1}^{n-1}\frac{(n-1)!}{k!(n-1-k)!}A^{J_1J_2}{}_{\sigma_1\sigma_2}{}^{...J_k}{}_{\sigma_ki_{k+1}i_{k+2}...i_{n-1}}y^{\sigma_1}_{J_1i_1}y^{\sigma_2}_{J_2i_2}...y^{\sigma_k}_{J_ki_k}\right).$$

$$\square$$

Note that Theorem 2 together with Lemma 2 show that solutions of the total divergence equation can be interpreted as certain differential

$(n-1)$-forms on V^{r-1}; the correspondence between these two objects is given by formula

$$\eta = g^i \omega_i = \frac{1}{(n-1)!} h_{j_2 j_3 \ldots j_n} dx^{j_2} \wedge dx^{j_3} \wedge \ldots \wedge dx^{j_n},$$

and Lemma 1.

References

[1] I.M. Anderson, Introduction to the variational bicomplex, Contemporary Mathematics 132 Amer. Math. Soc., Providence, RI, 1992, 51-73.

[2] P. Dedecker, W.M. Tulczyjew, Spectral sequences and the inverse problem of the calculus of variations, in: Internat. Colloq. on Diff. Geom. Methods in Math. Phys., Aix-enProvence, 1979; Lecture Notes in Math. 836, Springer, Berlin, 498-503.

[3] M. Francaviglia, M. Palese, R. Vitolo, Superpotentials in variational sequences, in: Differential Geometry and Applications, Proc. Conf., Brno, Czech Republic, 1998; Masaryk University, Brno, 1999, 469-480.

[4] M. Grassi, Local vanishing of characteristic cohomology, Duke Math. J. 102 (2000), 307-328.

[5] D.R. Grigore, The variational sequence on finite jet bundle extensions and the Lagrange formalism, Diff. Geom. Appl. 10 (1999), 43-77.

[6] A. Hakova, O. Krupkova, Variational first-order partial differential equations, J. Differential Equations 191 (2003), 67/68.

[7] I.S. Krasilschik, Geometry of differential equations: A concise Introduction, Acta Appl. Math. 72 (2002), 1-17.

[8] I.S. Krasilschik, V.V. Lychagin, A.M. Vinogradov, *Geometry of Jet Spaces and Nonlinear Partial Differential Equations*, Gordon and Breach, New York, 1986.

[9] M. Krbek, J. Musilova, Representation of the variational sequence by differential forms, Rep. Math. Phys. 51 (2003), 251-258.

[10] D. Krupka, A geometric theory of ordinary first order variational problems in fibered manifolds, I. Critical sections, J. Math. Anal. Appl. 49 (1975) 180-206; II. Invariance, J. Math. Anal. Appl. 49 (1975) 469-476.

[11] D. Krupka, The contact ideal, Diff. Geom. Appl. 5 (1995), 257-276.

[12] D. Krupka, Variational sequences on finite order jet spaces, Differential Geometry and its Applications, Proc. Internat. Conf. on Diff. Geom. and Appl., Brno August 1989, J. Janyska and D. Krupka, Eds., World Scientific, 1990, 236-254.

[13] O. Krupkova, *The Geometry of Ordinary Variational Equations*, Lecture Notes in Math. 1678, Springer, Berlin, 1997.

[14] J.F. Pommaret, Spencer sequence and variational sequence, Acta Appl. Math. 41 (1995) 285-296.

[15] D. Saunders, *The Geometry of Jet Bundles*, London Mathematical Society Lecture Note Series 142, Cambridge University Press, New York, 1989.

[16] F. Takens, A global version of the inverse problem of the calculus of variations, J. Differential Geometry 14 (1979) 543-562.

[17] W.M. Tulczyjew, The Lagrange complex, Bull. Soc. Math. France 105 (1977), 419-431.

[18] A.M. Vinogradov, The C-spectral sequence, Lagrangian formalism, and conservation laws, I. The linear theory, II. The nonlinear theory, J. Math. Anal. Appl. 100 (1984), 1-40, 41-129.

[19] R. Vitolo, Finite order variational bicomplexes, Math. Proc. Cambridge. Phil. Soc. 125 (1999), 321-333.

[20] T. Tsujishita, On variational bicomplexes associated with differential equations, Osaka J. Math. 19 (1982), 311-363.

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