Lobachevskii Journal of Mathematics
http://ljm.ksu.ru
ISSN 1818-9962
Vol. 24, 2006, 55–62

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SOME REMARKS ABOUT STRICTLY PSEUDOCONVEX FUNCTIONS WITH RESPECT TO THE CLARKE-ROCKAFELLAR SUBDIFFERENTIAL

(submitted by A. V. Lapin)

Abstract. Using the notion of radially Clarke-Rockafellar subdifferentiable functions (RCRS-functions), we characterize strictly pseudoconvex functions with respect to the Clarke-Rockafellar subdifferential in two different ways, and we study a maximization problem involving RCRS-strictly pseudoconvex functions over a convex set.

1. Introduction

Generalized convexity has proved to be a good tool in the study of some economic problems and in mathematical programming. Strict pseudoconvexity is a kind of generalized convexity that appeared recently as an important part of the class of pseudoconvex functions. The former class has been characterized by many authors (see for instance [1, 2, 4, 7, 10]). In this paper we will refine these results in section 2, using the Clarke Rockafellar subdifferential. While, in section 3 we give a necessary and sufficient condition for a point to be a maximum of a strictly pseudoconvex function over a convex set.

Let us recall some definitions and well known results in connection with what we shall do in the sequel. By X we mean a Banach space and

Key words and phrases. Strictly pseudoconvex functions, RCRS-functions, strongly RCRS-functions.

2000 Mathematical Subject Classification. 49J52, 49J50.

by X^* its topological dual, while $\langle .,. \rangle$ is the duality pairing between X and X^*. For x and y in X, the closed segment $[x,y]$ is the set $[x,y] = \{\, x+t(y-x);\ t \in [0,1]\,\}$. By $[x,y)$ we denote the set $[x,y] \setminus \{y\}$. Given a lower semi-continuous (l.s.c.) function $f : X \to R \cup \{+\infty\}$ whose domain

$$\mathrm{dom} f \;=\; \big\{\, x \in X;\ f(x) < +\infty \,\big\}$$

is nonempty. The Clarke-Rockafellar generalized directional derivative $f^{\uparrow}(x,v)$ of f at $x \in \mathrm{dom} f$ along the direction v is defined by:

$$f^{\uparrow}(x,v) \;=\; \sup_{\varepsilon>0}\ \limsup_{y \to_f x,\, t \searrow 0}\ \inf_{u \in B(v,\varepsilon)} t^{-1}\big[f(y+tu) - f(y)\big], \tag{1}$$

where by $y \to_f x$, we mean $y \to x$ and $f(y) \to f(x)$. Here, by $B(v,\varepsilon)$ we denote the open ball centered at v with radius ε. The Clarke-Rockafellar subdifferential of f at $x \in \mathrm{dom} f$ is defined by

$$\partial f(x) \;=\; \big\{\, x^* \in X^* :\ \langle x^*, v\rangle \le f^{\uparrow}(x,v) \quad \forall v \in X \,\big\}.$$

We adopt the convention $\partial f(x) = \emptyset$ when $x \notin \mathrm{dom} f$.

A function f is said to be quasiconvex if for any $x, y \in X$ and $\lambda \in [0,1]$ we have

$$f(\lambda x + (1-\lambda)y) \le \max\{\, f(x), f(y)\,\}. \tag{2}$$

f is said to be strictly quasiconvex if the inequality (2) is strict when $x \neq y$. f is said to be pseudoconvex(with respect to the Clarke-Rockafellar subdifferential) if for any x and y in X the following implication holds:

$$\big(\exists x^* \in \partial f(x) :\ \langle x^*, y-x\rangle \ge 0\big) \Longrightarrow \quad f(x) \le f(y). \tag{3}$$

The relation between pseudoconvexity and quasiconvexity has been described in [2, 4, 7, 10] by the following result.

Theorem 1. *Let $f : X \mapsto R \cup \{+\infty\}$ be a l.s.c. function. Consider the propositions:*

i) *f is pseudoconvex.*

ii) *f is quasiconvex and ($0 \in \partial f(x) \Longrightarrow x$ is a global minimum of f).*

Then **i)** *implies* **ii)**. *If moreover, f is radially continuous, then* **ii)** *implies* **i)**.

Generally, in generalized convexity, there is a close link between the kind of convexity of a function and a corresponding kind of monotonicity of its subdifferential. Recall that a multifunction $T : X \to X^*$ is said to be pseudomonotone if for any $x, y \in X$, we have:

$$[\exists x^* \in T(x) :\ \langle x^*, y-x\rangle \ge 0] \Longrightarrow \quad \forall y^* \in T(y) :\ \langle y^*, y-x\rangle \ge 0. \tag{4}$$

We have the following classical result:

Theorem 2. [2, 4, 7, 11] *Let $f : X \to R \cup \{+\infty\}$ be a l.s.c. function. Consider the propositions:*

i) *f is pseudoconvex.*
ii) *∂f is pseudomonotone.*

Then **i)** *implies* **ii)**. *If moreover, f is radially continuous, then* **ii)** *implies* **i)**.

In this paper, we want to characterize strictly pseudoconvex functions with respect to the Clarke-Rockafellar subdifferential in two different ways. For this, we introduce the so what we call radially Clarke-Rockafellar subdifferentiable functions (RCRS-functions).

Let $f : X \mapsto R \cup \{+\infty\}$ be a l.s.c. function. We say that f is radially Clarke-Rockafellar subdifferentiable if for all $x, y \in X$ with $x \neq y$, there is $x_0 \in (x, y)$ such that $\partial f(x_0) \neq \emptyset$. Recall that an extended-real valued function $f : X \mapsto R \cup \{+\infty\}$ is said to be radially continuous if for all $x, y \in X$ f is continuous on $[x, y]$.

2. Characterization of RCRS-strict pseudoconvex functions

In this section, we get analogous results to theorem 1 and theorem 2 for RCRS-strictly pseudoconvex functions.

An extended-real valued function $f : X \mapsto R \cup \{+\infty\}$ is said to be radially non constant if for all $x, y \in X$ with $x \neq y$, $f \not\equiv$ constant on $[x, y]$.

Definition 3. A function $f : X \mapsto R \cup \{+\infty\}$ is said to be strictly pseudoconvex(with respect to the Clarke-Rockafellar subdifferential) if for any different points $x, y \in X$, the following implication holds:

$$\Big(\exists x^* \in \partial f(x) : \langle x^*, y - x\rangle \geq 0\Big) \Longrightarrow \quad f(x) < f(y). \tag{5}$$

We can check immediately that a strict pseudoconvex function is pseudoconvex while the converse is not true in general as we can see for example for the function

$$f(x) = \begin{cases} \sqrt{\mid x \mid -1} & \text{if } x \in]-\infty, -1] \cup [1, +\infty[, \\ 0 & \text{if } x \in [-1, 1]. \end{cases} \tag{6}$$

We can describe the relation between strict pseudoconvexity and strict quasiconvexity via the following result:

Theorem 4. *Let $f : X \mapsto R \cup \{+\infty\}$ be a l.s.c. function such that f is radially Clarke-Rockafellar subdifferentiable. Consider the following assertions:*

i) *f is strictly pseudoconvex.*

ii) *f is strictly quasiconvex and ($0 \in \partial f(x) \Longrightarrow x$ is a strict global minimum of f).*

Then **i)** *implies* **ii)**. *If moreover, f is radially continuous, then* **ii)** *implies* **i)**.

Proof. Let f be a strictly pseudoconvex function, then by theorem 1, f is quasiconvex. Let us prove now that f is strictly quasiconvex. Since f is quasiconvex, then according to Diewert [5], it suffices to prove that f is radially non constant. By the contrary, assume that there exists a closed segment $[x, y]$ $(x \neq y)$ on which f is constant. Let $z \in (x, y)$. Then applying the strict pseudoconvexity property on x and z, we deduce

$$\forall z^* \in \partial f(z) \quad \langle z^*, x - z\rangle < 0.$$

Using the same argument for z and y, we obtain

$$\forall z^* \in \partial f(z) \quad \langle z^*, y - z\rangle < 0.$$

Therefore,

$$\forall z^* \in \partial f(z), \quad \langle z^*, x - y\rangle < 0 \quad \text{and} \quad \langle z^*, x - y\rangle > 0.$$

Consequently, for all $z \in (x, y)$ we have $\partial f(z) = \emptyset$. But this contradicts the fact that f is a RCRS-function. Thus, f is strictly quasiconvex. On the other hand, f is pseudoconvex. Therefore,

$$0 \in \partial f(x) \Longrightarrow x \text{ is a strict global minimum of } f.$$

Conversely, assume that f satisfies the condition **ii)** and f is radially continuous. Then by theorem 1, f is pseudoconvex.

Let us prove now that f is strictly pseudoconvex. Assume by contradiction that there exist $x \neq y$ in X and $x^* \in \partial f(x)$ such that

$$\langle x^*, y - x\rangle \geq 0 \quad \text{and} \quad f(x) \geq f(y).$$

Then, It follows by pseudoconvexity property that

$$f(x) = f(y) \text{ and } \forall z \in [x, y], \quad f(z) \geq f(x) \geq f(y).$$

On the other hand, f is quasiconvex. Therefore,

$$f(z) = f(x) = f(y), \quad \forall z \in [x, y].$$

Consequently, f is not radially non constant on X. But this contradicts the fact that f is strictly quasiconvex. Thus, we achieve the proof.

Analogously to pseudomonotone multioperators, we define strictly pseudomonotone multioperators as follows:

Definition 5. A multioperator $T : X \to X^*$ is said to be strictly pseudomonotone if for any different points x and y in X, the following implication holds:

$$\exists x^* \in T(x) : \langle x^*, y - x\rangle \geq 0 \Longrightarrow \quad \forall y^* \in T(y) : \langle y^*, y - x\rangle > 0. \quad (7)$$

We have also a relation between strict pseudoconvexity of functions and strict monotonicity of their corresponding Clarke-Rockafellar subdifferentials.

Theorem 6. *Let $f : X \mapsto R \cup \{+\infty\}$ be a l.s.c. function such that f is radially Clarke-Rockafellar subdifferentiable. Consider the following assertions*

i) *f is strictly pseudoconvex.*
ii) *∂f is strictly pseudomonotone.*

Then **i)** *implies* **ii)**. *if moreover, f is radially continuous, then* **ii)** *implies* **i)**.

Proof. The first implication can be easily proved, nevertheless we include it here for completeness. Assume that f is strictly pseudoconvex. Let us prove by the contrary that ∂f is strictly pseudomonotone. Suppose that there exist two different points $x, y \in X$, $x^* \in \partial f(x)$ and $y^* \in \partial f(y)$ such that

$$\langle x^*, y - x\rangle \geq 0 \quad \text{and} \quad \langle y^*, x - y\rangle \geq 0.$$

Since f is strictly pseudoconvex, then

$$f(x) < f(y) \quad \text{and} \quad f(x) > f(y).$$

Contradiction. Thus, ∂f is strictly pseudomonotone.

Conversely, assume that f satisfies the condition **ii)** and f is radially continuous. Let us prove that f is strictly pseudoconvex. By the contrary, assume that there exist two different points x and y in X, and $x^* \in \partial f(x)$ such that both inequalities

$$\langle x^*, y - x\rangle \geq 0 \quad \text{and} \quad f(x) \geq f(y)$$

hold. Then

$$\langle x^*, z - x\rangle \geq 0 \qquad \forall z \in [x, y]. \quad (8)$$

By theorem 2, it follows that f is pseudoconvex. Therefore,

$$f(x) \leq f(z) \quad \forall z \in [x, y].$$

By theorem 1, f is quasiconvex. Consequently, we can easily see that f must be constant on $[x, y]$. On the other hand, by (8) and the strict pseudomonotonicity of ∂f, we have:

$$\langle z^*, z - x\rangle > 0, \qquad \forall z \in (x, y)\ \forall z* \in \partial f(z). \tag{9}$$

Pick $z_0 \in (x, y)$ such that $\partial f(z_0) \neq \emptyset$ (such a z_0 exists since f is a RCRS-function). Choose any $z_0^* \in \partial f(z_0)$. Then, $\langle z_0^*, z_0 - x\rangle > 0$. Therefore, $\langle z_0^*, y - z_0\rangle > 0$. Consequently, there is $\varepsilon > 0$ such that

$$\langle z_0^*, y' - z_0\rangle > 0 \qquad \forall y' \in B(y, \varepsilon).$$

By the pseudoconvexity of f, it follows that y is a global minimum of f. Hence, z_0 is also a global minimum of f. Thus, $0 \in \partial f(z_0)$ and this is in contradiction with (9).

3. Maxima of strongly RCRS-strict pseudoconvex functions

In this section, we study a maximization problem over a convex set involving a certain class of RCRS-strictly pseudoconvex functions called class of strongly RCRS-strictly pseudoconvex functions.

Let $f : X \mapsto R \cup \{+\infty\}$ be a l.s.c. function. We say that f is strongly radially Clarke-Rockafellar subdifferentiable if for all $x, y \in X$ with $x \neq y$ and for all $c : f(x) < c < f(y)$, there is $x_0 \in (x, y)$ such that $f(x_0) = c$ and $\partial f(x_0) \neq \emptyset$.

Let C be a nonempty convex set of X. Consider the following maximization problem:

$$(\mathcal{P}) \qquad \max_{x \in C} f(x),$$

where the function f is assumed to be strictly pseudoconvex, l.s.c. and strongly radially Clarke-Rockafellar subdifferentiable.

Theorem 7. *Consider $\bar{x} \in C$ such that*

$$-\infty \leq \inf_C f < f(\bar{x}). \tag{10}$$

Then $\bar{x}$ is a maximum of f over C if and only if for all $x \in C$ such that $f(x) = f(\bar{x})$ and all $x^* \in \partial f(x)$ we have:

$$\langle x^*, y - x\rangle < 0 \qquad \forall y \in C \setminus \{x\}. \tag{11}$$

Proof. Assume that $\bar{x}$ is a solution of the problem $(\mathcal{P})$. Let $x \in X$ such that $f(x) = f(\bar{x})$ and let $x^* \in \partial f(x)$. Then

$$f(y) \leq f(x), \qquad \forall y \in C.$$

Since f is strictly pseudoconvex, then

$$\langle x^*, y - x \rangle < 0, \quad \forall y \in C \setminus \{x\}.$$

Conversely, suppose that there exists $z \in C$ such that $f(z) > f(\bar{x})$. By the hypotheses, there is $z_0 \in C$ such that $f(z_0) < f(\bar{x})$. Since f is strongly radially Clarke-Rockafellar subdifferentiable, then there is some $x_0 \in (z_0, z)$ such that $f(x_0) = f(\bar{x})$ and $\partial f(x_0) \neq \emptyset$. Pick any ${x_0}^* \in \partial f(x_0)$. Then

$$\langle x_0^*, z - x_0 \rangle < 0 \quad \text{and} \quad \langle x_0^*, z_0 - x_0 \rangle < 0.$$

Which is impossible. To prove that (11) holds when $\bar{x}$ is a maximum, we use only the strict pseudoconvexity of f, the other conditions that appear in theorem 7 are needed only to prove that (11) implies that $\bar{x}$ is a maximum. This result is a refinement of both theorem 2.1 of [8] where the function was supposed to be convex continuous and of theorem 4.1 of [7] where the function was assumed to be pseudoconvex and radially continuous.

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Received October 19, 2006