Lobachevskii Journal of Mathematics
http://ljm.ksu.ru
ISSN 1818-9962
Vol. 25, 2007, 3–7

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THE PROBABILITY OF A SUCCESSFUL ALLOCATION OF BALL GROUPS BY BOXES

ABSTRACT. Let $p = p_{Nn}$ be the probability of a successful allocation of n groups of distinguishable balls in N boxes in equiprobable scheme for group allocation of balls with the following assumption: each group contains m balls and each box contains not more than q balls from a same group. If $q = 1$, then we easily calculate p and observe that $p \to e^{-\frac{m(m-1)}{2}\alpha_0}$ as $n, N \to \infty$ such that $\alpha = \frac{n}{N} \to \alpha_0 < \infty$. In the case $2 \leq q$ we also find an explicit formula for the probability and prove that $p \to 1$ as $n, N \to \infty$ such that $\alpha = \frac{n}{N} \leq \alpha' < \infty$.

1. Introduction

Many papers are devoted with problems of the allocation theory (see, for instance, Weiss (1958), Bekéssy (1963, 1964), the monograph by Kolchin, Sevast'yanov, Chistyakov (1978) and references therein, Vatutin and Mikhajlov (1982)). In this paper we intend to study some open problems related to known results by A.N.Timashev. Namely, in the paper by Timashev (2000) the allocation theory with a restriction for the number of balls in boxes is developed. We will study a more general case when there is a restriction for balls from a same group.

More precisely, we consider the following situation. Let n be the number of ball groups and let N be the number of boxes in equiprobable

Key words and phrases. equiprobable scheme for group allocation of particles, generating function, Cauchy integral.

scheme for group allocation of distinguishable balls. We suppose that each ball group contains m balls. Denote by $B_q(n, N, m)$ the following event: *each box contains not more than q balls from a same ball group,* $1 \le q \le m$. The aim of this paper is to find the probability p of the event $B_q(n, N, m)$ and its asymptotic behavior as $n, N \to \infty$ such that $\alpha = \frac{n}{N} \to \alpha_0$. We will obtain some explicit formulas for the probability and prove that

$$1 - p \le \frac{\alpha}{N^{q-1}} C,$$

where C is a constant dependent on m and q only.

2. The main results

The number of allocations of the i-th group is equal to N^m. The number of allocations of the i-th group such that each box contains not more than q balls equals

$$M_m(N, q) = \sum_{\substack{l_1 + \cdots + l_N = m \\ 0 \le l_1, \ldots, l_N \le q}} \frac{m!}{l_1! \ldots l_N!}.$$

Therefore the probability of the event such that for the allocation of i-th group in each box it appears not more than q balls

$$p_i = \frac{M_m(N, q)}{N^m} \quad \text{and} \quad p = \prod_{i=1}^{n} p_i = \left(\frac{M_m(N, q)}{N^m} \right)^n.$$

If $q \ge m$ then it is clear that $M_m(N, q) = N^m$ and, consequently, $p = 1$. In particular, if $m = 1$ or $m = q = 2$, then $p = 1$.

Therefore, we only have to consider the case when $m \ge 2$ and $q \le m-1$.

Theorem 1. *If $m \ge 2$ and $1 \le q \le m - 1$, then*

$$p = \left(1 - \frac{A}{q!N}\right)^n, \tag{1}$$

where

$$A = \sum_{\nu=q}^{m-1} \frac{1}{N^{\nu-1}} \left. \frac{d^\nu (z^q (1 + \frac{z}{1!} + \ldots \frac{z^q}{q!})^{N-1})}{dz^\nu} \right|_{z=0}.$$

Moreover

$$1 - p \le \frac{\alpha}{N^{q-1}} C, \tag{2}$$

where

$$C = \frac{1}{q!} \sum_{\nu=q}^{m-1} \frac{\nu!}{(\nu - q)!}.$$

Proof. We will use the representation of $M_m(N,q)$ as a Cauchy integral (see Timashev (2000), formula (4)):

$$M_m(N,q) = \frac{m!}{2\pi i} \oint\limits_C \frac{\left(1+\frac{z}{1!}+\dots\frac{z^q}{q!}\right)^N}{z^{m+1}} dz,$$

where C is a positively oriented circle with center at the point $z=0$. Therefore, we obtain

$$p = \left(\frac{m!}{2\pi i} \oint\limits_C \frac{[f(z)]^N}{N^m z^{m+1}} dz \right)^n,$$

where

$$f(z) = 1 + \frac{z}{1!} + \dots \frac{z^q}{q!}.$$

Denote

$$a(m,N,q) = \left. \frac{d^m f^N(z)}{dz^m} \right|_{z=0}.$$

It is clear that

$$p = \left(\frac{a(m,N,q)}{N^m} \right)^{N\alpha}.$$

Since

$$f'(z) = f(z) - \frac{z^q}{q!},$$

by mathematical induction on m, we obtain

$$a(m,N,q) = N \left. \frac{d^{m-1}(f^{N-1}(z) f'(z))}{dz^{m-1}} \right|_{z=0} =$$

$$= N \left. \frac{d^{m-1} f^N(z)}{dz^{m-1}} \right|_{z=0} - \frac{N}{q!} \left. \frac{d^{m-1}(z^q f^{N-1}(z))}{dz^{m-1}} \right|_{z=0} =$$

$$= N^m - \frac{1}{q!} \sum_{\nu=q}^{m-1} N^{m-\nu} \left. \frac{d^\nu (z^q f^{N-1}(z))}{dz^\nu} \right|_{z=0}.$$

Denoting

$$A = \sum_{\nu=q}^{m-1} \frac{1}{N^{\nu-1}} \left. \frac{d^\nu (z^q f^{N-1}(z))}{dz^\nu} \right|_{z=0}$$

we have (1).

Obviously, $A \geq 0$. Moreover, for all $N, \nu, q \in \mathbf{N}$, $\nu \geq q$ we obtain

$$\left. \frac{d^\nu (z^q f^{N-1}(z))}{dz^\nu} \right|_{z=0} \leq \left. \frac{d^\nu (z^q e^{(N-1)z})}{dz^\nu} \right|_{z=0} = \frac{(N-1)^{\nu-q} \nu!}{(\nu-q)!}.$$

From this it follows that

$$0 \le A \le \sum_{\nu=q}^{m-1} \frac{\nu!}{(\nu-q)!} \frac{(N-1)^{\nu-q}}{N^{\nu-1}} < \frac{1}{N^{q-1}} \sum_{\nu=q}^{m-1} \frac{\nu!}{(\nu-q)!}.$$

Consequently,

$$1-p = 1 - \left(1 - \frac{A}{q!N}\right)^{N\alpha} \le \frac{A}{q!N} N\alpha \le \frac{\alpha}{N^{q-1}} C.$$

This completes the proof of Theorem 1.

Theorem 2. *Suppose that $m \ge 2$ and $1 \le q \le m-1$.*

(i) If $q = 1$, then

$$p = \left(1 - \frac{1}{N}\right)^n \left(1 - \frac{2}{N}\right)^n \dots \left(1 - \frac{m-1}{N}\right)^n$$

and

$$p \to e^{-\frac{m(m-1)}{2}\alpha_0} \text{ as } n, N \to \infty \text{ such that } \frac{n}{N} \to \alpha_0.$$

(ii) If $2 \le q \le m-1$, then

$$p \to 1 \text{ as } n, N \to \infty \text{ such that } \alpha = \frac{n}{N} \le \alpha' < \infty.$$

Proof. (i) We easily have $M_m(N,1) = N(N-1)\dots(N-m+1)$ and

$$p = \left(\frac{N}{N}\frac{N-1}{N} \dots \frac{N-m+1}{N}\right)^n = \left(1 - \frac{1}{N}\right)^n \left(1 - \frac{2}{N}\right)^n \dots \left(1 - \frac{m-1}{N}\right)^n.$$

Therefore, if $n, N \to \infty$ such that $\frac{n}{N} \to \alpha_0$, then one has

$$p \to e^{-(1+2+\dots+m-1)\alpha_0} = e^{-\frac{m(m-1)}{2}\alpha_0}.$$

In the case $2 \le q \le m-1$ we have $\frac{\alpha}{N^{q-1}}C \to 0$ as $n, N \to \infty$ such that $\alpha \le \alpha'$. By (2) this implies (ii).

The proof of Theorem 2 is complete.

3. Some remarks

Let $q = 3$ and n_k, N_k, $k \in \mathbf{N}$ be natural numbers with the properties: $N_k < N_{k+1}$, $k \in \mathbf{N}$, and $n_k, N_k \to \infty$ as $k \to \infty$ such that $\alpha_{N_k} = \frac{n_k}{N_k} \le \alpha' < \infty$. Suppose that the events $B_3(n_k, N_k)$ are defined on the same probability space $(\Omega,\ \mathrm{A}, \mathbf{P})$. Using (2) with $q = 3$, from (1) with $p = p_{N_k n_k}$ we obtain

$$\sum_{k=1}^{\infty}(1 - p_{N_k n_k}) = \sum_{k=1}^{\infty}\left(1 - \left(1 - \frac{A_{N_k}}{3!N_k}\right)^{N_k \alpha_{N_k}}\right) \leq \sum_{k=1}^{\infty}\frac{A_{N_k}}{3!N_k}N_k\alpha_{N_k} \leq$$

$$\frac{\alpha'}{6}\sum_{k=1}^{\infty}A_{N_k} \leq \frac{\alpha'}{6}\left(\sum_{\nu=3}^{m-1}\frac{\nu!}{(\nu-3)!}\right)\sum_{k=1}^{\infty}\frac{1}{N_k^2} < \infty.$$

Therefore for almost all sequences of allocations into N_k boxes of n_k groups containing m balls there exists $k_0 \in \mathbf{N}$ dependent on $\omega \in \Omega$ such that for all $k > k_0$ and for each allocation each box contains not more than 3 balls from a same group.

In the case $q = 1$ this is not true. The case $q = 2$ presents an open problem.

Acknowledgement

The authors thank the Russian fund of basic research for financial support by the grant 05-01-00523 for F.G. Avkhadiev.

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Received January 24, 2007