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**CHARACTERIZATION OF ULTRA SEPARATION
AXIOMS VIA $(1, 2)\alpha$ -KERNEL**

(submitted by M. A. Malakhaltsev)

ABSTRACT. In this paper, we introduce the concept of weakly-ultra-separation of two sets in a bitopological space using $(1, 2)\alpha$ -open sets. The $(1, 2)\alpha$ -closure and the $(1, 2)\alpha$ -kernel are defined in terms of this weakly-ultra-separation. We also investigate the properties of some weak separation axioms like ultra- T_0 , ultra- T_1 , and ultra- R_0 .

1. INTRODUCTION

In 1962, Kelly initiated the study of a triple (X, τ_1, τ_2) , where X is a non empty set and τ_1, τ_2 are topologies on X . The notion of $(1, 2)\alpha$ -open sets [7] in a bitopological space was introduced in 1991. In this paper, we define that a set A is weakly-ultra-separated from B if there exists a $(1, 2)\alpha$ -open set G containing A such that $G \cap B = \emptyset$. Using this concept, we define the $(1, 2)\alpha$ -closure and the $(1, 2)\alpha$ -kernel. We also define the $(1, 2)\alpha$ -derived set and the $(1, 2)\alpha$ -shell of a set A of a bitopological space X . We also offer some new characteristics of the low separation axioms defined and developed in [5] and [10] with respect to $(1, 2)\alpha$ -open sets.

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2. PRELIMINARIES

In this section, let us recall some definitions which are useful in the following sequel.

Definition 2.1. A subset A of a topological space (X, τ_1) is called an α -open set [9] if $A \subset \tau_1\text{-int}(\tau_1\text{-cl}(\tau_1\text{-int}(A)))$, where $\tau_1\text{-int}(A)$ and $\tau_1\text{-cl}(A)$ represent the interior and closure of A with respect to τ_1 .

Definition 2.2. A topological space is α -symmetric [2] if for $x, y \in X$, $x \in \alpha(\text{cl}\{y\})$ implies $y \in \alpha(\text{cl}\{x\})$.

Hereafter throughout this paper (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) denote bitopological spaces on which no separation axioms are assumed unless explicitly stated otherwise.

Definition 2.3. A subset A of X is called [7]

- (i) $\tau_1\tau_2$ -open if $A \in \tau_1 \cup \tau_2$,
- (ii) $\tau_1\tau_2$ -closed if $A^c \in \tau_1 \cup \tau_2$.

Definition 2.4. Let A be a subset of a space X . Then $\tau_1\tau_2$ -closure [7] of A is denoted by $\tau_1\tau_2\text{-cl}(A)$ and is defined as the intersection of all $\tau_1\tau_2$ -closed sets containing A .

Definition 2.5. A subset A of X is called $(1, 2)\alpha$ -open [7] if

$$A \subseteq \tau_1\text{-int}(\tau_1\tau_2\text{-cl}(\tau_1\text{-int}(A))).$$

The set of all $(1, 2)\alpha$ -open sets is denoted by $(1, 2)\alpha O(X)$. The complement of a $(1, 2)\alpha$ -open set is called a $(1, 2)\alpha$ -closed set. The $(1, 2)\alpha$ -closure of a set A of X is denoted by $(1, 2)\alpha\text{-cl}(A)$ and is defined as the intersection of all $(1, 2)\alpha$ -closed sets containing A . The family of $(1, 2)\alpha$ -closed sets is denoted by $(1, 2)\alpha CL(X)$.

Definition 2.6. A function $f : X \rightarrow Y$ is called a $(1, 2)\alpha$ -irresolute map [10] if the inverse image of every $(1, 2)\alpha$ -open set in Y is a $(1, 2)\alpha$ -open set in X .

Definition 2.7. A subset A of X is called a $(1, 2)\alpha g$ -closed set [10] if $(1, 2)\alpha\text{cl}(A) \subseteq U$, where $A \subseteq U$ and $U \in (1, 2)\alpha O(X)$.

Definition 2.8. (i) A bitopological space X is called an ultra- T_0 space [10] if and only if for each $x, y \in X$ such that $x \neq y$, there exists a $(1, 2)\alpha$ -open set containing x but not y or a $(1, 2)\alpha$ -open set containing y but not x .

(ii) A bitopological space X is called an ultra- T_1 [10] space if and only if for each $x, y \in X$ such that $x \neq y$, there exists a $(1, 2)\alpha$ -open set containing x but not y and a $(1, 2)\alpha$ -open set containing y but not x .

(iii) A subset A of X is called a $(1, 2)\alpha$ -difference set [10] (briefly $(1, 2)\alpha D$ -set) if there exist two $(1, 2)\alpha$ -open sets U_1 and U_2 such that $A = U_1/U_2$ and $U_1 \neq X$.

(iv) A bitopological space X is called an ultra- D_1 [10] space if for each $x, y \in X$ and $x \neq y$ there exists two $(1, 2)\alpha D$ -sets G_1 and G_2 such that $x \in G_1, x \notin G_2$ and $y \in G_2, y \notin G_1$.

Definition 2.9. A bitopological space X is said to be an ultra- R_0 space [5] if every $(1, 2)\alpha$ -open set G contains the $(1, 2)\alpha$ -closure of each of its singleton.

Definition 2.10. Let X be bitopological space and $x \in X$. Then a subset N_x of X is called a $(1, 2)\alpha$ -nbd ($(1, 2)\alpha$ -neighborhood) of X [10] if there exists a $(1, 2)\alpha$ -open set G such that $x \in G \subseteq N_x$.

3. $(1, 2)\alpha$ -KERNEL AND $(1, 2)\alpha$ -CLOSURE

In this section, we define the operator $(1, 2)\alpha$ -kernel and find its properties.

Definition 3.1. Let A be a non empty subset of a space X . Then $(1, 2)\alpha$ -kernel of A is denoted by $(1, 2)\alpha$ -ker(A) and is defined as $(1, 2)\alpha$ -ker(A) = $\cap\{G \in (1, 2)\alpha O(X)/A \subseteq G\}$.

Definition 3.2. Let $x \in X$. Then the $(1, 2)\alpha$ -kernel of x is denoted by $(1, 2)\alpha$ -ker($\{x\}$) = $\cap\{G \in (1, 2)\alpha O(X)/x \in G\}$.

Lemma 3.3. Let X be a bitopological space. Then for any non-empty subset A of X , $(1, 2)\alpha$ -ker(A) = $\{x \in X/(1, 2)\alpha$ -cl($\{x\}) \cap A \neq \emptyset\}$.

Proof. Let $x \in (1, 2)\alpha$ -ker(A). Suppose $(1, 2)\alpha$ -cl($\{x\}) \cap A = \emptyset$. Then $A \subseteq X - (1, 2)\alpha$ -cl($\{x\})$ and $X - (1, 2)\alpha$ -cl($\{x\})$ is a $(1, 2)\alpha$ -open set containing A but not x , which is a contradiction.

Conversely, let $x \notin (1, 2)\alpha$ -ker($\{A\}$) and $(1, 2)\alpha$ -cl($\{x\}) \cap A \neq \emptyset$. Then there exists a $(1, 2)\alpha$ -open set D containing A but not x and a $y \in (1, 2)\alpha$ -cl($\{x\}) \cap A$. Hence we get a $(1, 2)\alpha$ -nbd of y , say, D with $x \notin D$, which is a contradiction. Hence $x \in (1, 2)\alpha$ -ker(A).

Definition 3.4. In a space X , a set A is said to be weakly-ultra-separated from a set B if there exists a $(1, 2)\alpha$ -open set G such that $A \subset G$ and $G \cap B = \emptyset$ or $A \cap (1, 2)\alpha$ -cl(B) = \emptyset .

Lemma 3.5. *In view of the lemma 3.3 and definition of 3.4, let us have the followings for $x, y \in X$ of a bitopological space,*

- (i) $(1, 2)\alpha\text{-cl}(\{x\}) = \{y : y \text{ is not weakly-ultra-separated from } x\}$ and
- (ii) $(1, 2)\alpha\text{-ker}(\{x\}) = \{y \text{ is not weakly-ultra-separated from } y\}$.

Definition 3.6. *For any point x of a space X ,*

(i) *the derived set of x is denoted by $(1, 2)\alpha\text{-d}(\{x\})$ and is defined to be the set*

$$(1, 2)\alpha\text{-d}(\{x\}) = (1, 2)\alpha\text{-cl}(\{x\}) - \{x\} = \{y : y \neq x \text{ and } y \text{ is not weakly-ultra-separated from } x\},$$

(ii) *the shell of a singleton set $\{x\}$ is denoted by $(1, 2)\alpha\text{-shl}(\{x\})$ and is defined to be the set*

$$(1, 2)\alpha\text{-shl}(\{x\}) = (1, 2)\alpha\text{-ker}(\{x\}) - \{x\} = \{y : y \neq x \text{ and } x \text{ is not weakly-ultra-separated from } y\}.$$

Definition 3.7. *Let X be a bitopological space. Then we define*

- (i) $(1, 2)\alpha - N - D = \{x : x \in X \text{ and } (1, 2)\alpha - d(\{x\}) = \emptyset\}$,
- (ii) $(1, 2)\alpha - N - shl = \{x : x \in X \text{ and } (1, 2)\alpha - shl(\{x\}) = \emptyset\}$,
- (iii) $(1, 2)\alpha\text{-} \langle x \rangle = (1, 2)\alpha\text{-cl}(\{x\}) \cap (1, 2)\alpha\text{-ker}(\{x\})$.

Theorem 3.8. *Let $x, y \in X$. Then the following conditions hold good:*

- (i) $y \in (1, 2)\alpha\text{-ker}(\{x\})$ *if and only if* $x \in (1, 2)\alpha\text{-cl}(\{y\})$,
- (ii) $y \in (1, 2)\alpha\text{-shl}(\{x\})$ *if and only if* $x \in (1, 2)\alpha\text{-d}(\{y\})$,
- (iii) $y \in (1, 2)\alpha\text{-cl}(\{x\})$ *implies* $(1, 2)\alpha\text{-cl}(\{y\}) \subseteq (1, 2)\alpha\text{-cl}(\{x\})$ *and*
- (iv) $y \in (1, 2)\alpha\text{-ker}(\{x\})$ *implies* $(1, 2)\alpha\text{-ker}(\{y\}) \subseteq (1, 2)\alpha\text{-ker}(\{x\})$.

Proof. The proof of (i) and (ii) are obvious from Remark 3.5.

(iii) Let $z \in (1, 2)\alpha\text{-cl}(\{y\})$. Then z is not weakly-ultra-separated from y . So there exists a $(1, 2)\alpha$ -open set G containing z such that $G \cap \{y\} \neq \emptyset$. Hence $y \in G$ and by assumption $G \cap \{x\} \neq \emptyset$. Hence z is not weakly-ultra-separated from x . So $z \in (1, 2)\alpha\text{-cl}(\{x\})$. Therefore $(1, 2)\alpha\text{-cl}(\{y\}) \subseteq (1, 2)\alpha\text{-cl}(\{x\})$.

(iv) Let $z \in (1, 2)\alpha\text{-ker}(\{y\})$. Then y is not weakly-ultra-separated from z . So $y \in (1, 2)\alpha\text{-cl}(\{z\})$. Hence $(1, 2)\alpha\text{-cl}(\{y\}) \subseteq (1, 2)\alpha\text{-cl}(\{z\})$. By assumption $y \in (1, 2)\alpha\text{-ker}(\{x\})$ and then $x \in (1, 2)\alpha\text{-cl}(\{y\})$. So $(1, 2)\alpha\text{-cl}(\{x\}) \subseteq (1, 2)\alpha\text{-cl}(\{y\})$. Ultimately

$$(1, 2)\alpha\text{-cl}(\{x\}) \subseteq (1, 2)\alpha\text{-cl}(\{z\}).$$

Hence $x \in (1, 2)\alpha\text{-cl}(\{z\})$, that is $z \in (1, 2)\alpha\text{-ker}(\{x\})$. Therefore $(1, 2)\alpha\text{-ker}(\{y\}) \subseteq (1, 2)\alpha\text{-ker}(\{x\})$.

Let us recall that a subset A of X is called a degenerate set if A is either a null set or a singleton set.

Theorem 3.9. *Let $x, y \in X$. Then,*

(i) *for every $x \in X$, $(1, 2)\alpha$ -shl($\{x\}$) is degenerate if and only if for all $x, y \in X$, $x \neq y$, $(1, 2)\alpha$ -d($\{x\}) \cap (1, 2)\alpha$ -d($\{y\}) = \emptyset$,*

(ii) *for every $x \in X$, $(1, 2)\alpha$ -d($\{x\}$) is degenerate if and only if for every $x, y \in X$, $x \neq y$, $(1, 2)\alpha$ -shl($\{x\}) \cap (1, 2)\alpha$ -shl($\{y\}) = \emptyset$.*

Proof. Let $(1, 2)\alpha$ -d($\{x\}) \cap (1, 2)\alpha$ -d($\{y\}) \neq \emptyset$. Then there exists a $z \in X$ such that $z \in (1, 2)\alpha$ -d($\{x\})$ and $z \in (1, 2)\alpha$ -d($\{y\})$. Then $z \neq y \neq x$ and $z \in (1, 2)\alpha$ -cl($\{x\})$ and $z \in (1, 2)\alpha$ -cl($\{y\})$, that is $x, y \in (1, 2)\alpha$ -ker($\{z\}$). Hence $(1, 2)\alpha$ -ker($\{z\})$ and so $(1, 2)\alpha$ -shl($\{z\})$ is not a degenerate set.

Conversely, let $x, y \in (1, 2)\alpha$ -shl($\{z\})$. Then we get $x \neq z, x \in (1, 2)\alpha$ -ker($\{z\})$ and $y \neq z$ and $y \in (1, 2)\alpha$ -ker($\{z\})$ and hence z is an element of both $(1, 2)\alpha$ -cl($\{x\})$ and $(1, 2)\alpha$ -cl($\{y\})$, which is a contradiction.

The proof of (ii) is simple and hence omitted.

Theorem 3.10. *If $y \in (1, 2)\alpha$ - $\langle x \rangle$, then $(1, 2)\alpha$ - $\langle x \rangle = (1, 2)\alpha$ - $\langle y \rangle$.*

Proof. If $y \in (1, 2)\alpha$ - $\langle x \rangle$, then $y \in (1, 2)\alpha$ -cl($\{x\}) \cap (1, 2)\alpha$ -ker($\{x\})$. Hence $y \in (1, 2)\alpha$ -cl($\{x\})$ and $y \in (1, 2)\alpha$ -ker($\{x\})$ and so we have $(1, 2)\alpha$ -cl($\{y\}) \subseteq (1, 2)\alpha$ -cl($\{x\})$ and $(1, 2)\alpha$ -ker($\{y\}) \subseteq (1, 2)\alpha$ -ker($\{x\})$. Then

$$(1, 2)\alpha$$
-cl($\{y\}) \cap (1, 2)\alpha$ -ker($\{y\}) \subseteq (1, 2)\alpha$ -cl($\{x\}) \cap (1, 2)\alpha$ -ker($\{x\})$.

Hence $(1, 2)\alpha$ - $\langle y \rangle \subseteq (1, 2)\alpha$ - $\langle x \rangle$. The fact that $y \in (1, 2)\alpha$ -cl($\{x\})$ implies $x \in (1, 2)\alpha$ -ker($\{y\})$ and $y \in (1, 2)\alpha$ -ker($\{x\})$ implies $x \in (1, 2)\alpha$ -cl($\{y\})$. Then we have that $(1, 2)\alpha$ - $\langle x \rangle \subseteq (1, 2)\alpha$ - $\langle y \rangle$. So $(1, 2)\alpha$ - $\langle x \rangle = (1, 2)\alpha$ - $\langle y \rangle$.

Theorem 3.11. *For all $x, y \in X$, either $(1, 2)\alpha$ - $\langle x \rangle \cap (1, 2)\alpha$ - $\langle y \rangle = \emptyset$ or $(1, 2)\alpha$ - $\langle x \rangle = (1, 2)\alpha$ - $\langle y \rangle$.*

Proof. If $(1, 2)\alpha$ - $\langle x \rangle \cap (1, 2)\alpha$ - $\langle y \rangle \neq \emptyset$, then there exists $z \in X$ such that $z \in (1, 2)\alpha$ - $\langle x \rangle$ and $z \in (1, 2)\alpha$ - $\langle y \rangle$. So by Theorem 3.10, $(1, 2)\alpha$ - $\langle z \rangle = (1, 2)\alpha$ - $\langle x \rangle = (1, 2)\alpha$ - $\langle y \rangle$. Hence the result.

Theorem 3.12. *For any two points $x, y \in X$, the following statements are equivalent.*

- (i) $(1, 2)\alpha$ -ker($\{x\}) \neq (1, 2)\alpha$ -ker($\{y\})$, and
- (ii) $(1, 2)\alpha$ -cl($\{x\}) \neq (1, 2)\alpha$ -cl($\{y\})$.

Proof. (i) \Rightarrow (ii) Let us assume $(1, 2)\alpha\text{-ker}(\{x\}) \neq (1, 2)\alpha\text{-ker}(\{y\})$. Then there exists a $z \in (1, 2)\alpha\text{-ker}(\{x\})$ but $z \notin (1, 2)\alpha\text{-ker}(\{y\})$. As $z \in (1, 2)\alpha\text{-ker}(\{x\})$, $x \in (1, 2)\alpha\text{-cl}(\{z\})$ and $(1, 2)\alpha\text{-cl}(\{x\}) \subset (1, 2)\alpha\text{-cl}(\{z\})$. Also we have taken $z \notin (1, 2)\alpha\text{-ker}(\{y\})$, by lemma 3.3, $(1, 2)\alpha\text{-cl}(\{z\}) \cap \{y\} = \emptyset$, so $(1, 2)\alpha\text{-cl}(\{x\}) \cap \{y\} = \emptyset$ and so y is weakly-ultra-separated from x and hence we get that $y \notin (1, 2)\alpha\text{-cl}(\{x\})$. Hence $(1, 2)\alpha\text{-cl}(\{y\}) \neq (1, 2)\alpha\text{-cl}(\{x\})$.

(ii) \Rightarrow (i) Suppose $(1, 2)\alpha\text{-cl}(\{x\}) \neq (1, 2)\alpha\text{-cl}(\{y\})$. Then there exists a point $z \in (1, 2)\alpha\text{-cl}(\{x\})$ but $z \notin (1, 2)\alpha\text{-cl}(\{y\})$. So, we get a $(1, 2)\alpha$ -open set containing z and x but not y . That is $y \notin (1, 2)\alpha\text{-ker}(\{x\})$. Hence $(1, 2)\alpha\text{-ker}(\{y\}) \neq (1, 2)\alpha\text{-ker}(\{x\})$.

4. ULTRA- T_i ($i = 0,1$) AND ULTRA- D_1 SPACES

In this section, some of the properties of ultra- T_i ($i = 0,1$) and ultra- D_1 spaces are derived by means of weakly-ultra-separation.

Theorem 4.1. *A space X is ultra- T_0 if and only if any of the following conditions holds good:*

- (i) *For arbitrary $x, y \in X, x \neq y$, either x is weakly-ultra-separated from y or y is weakly-ultra-separated from x .*
- (ii) *$y \in (1, 2)\alpha\text{-cl}(\{x\})$ implies $x \notin (1, 2)\alpha\text{-cl}(\{y\})$.*
- (iii) *For all $x, y \in X$ if $x \neq y$, then $(1, 2)\alpha\text{-cl}(\{x\}) \neq (1, 2)\alpha\text{-cl}(\{y\})$.*

Proof. (i) Obvious from the definitions of ultra- T_0 and weakly-ultra-separation.

(ii) By assumption, $y \in (1, 2)\alpha\text{-cl}(\{x\})$ and so y is not weakly-ultra-separated from x . As X is ultra- T_0 , x should be weakly-ultra-separated from y , that is $x \notin (1, 2)\alpha\text{-cl}(\{y\})$.

(iii) If X is ultra- T_0 , then for all $x, y \in X$ and $x \neq y$, $(1, 2)\alpha\text{-cl}(\{x\}) \neq (1, 2)\alpha\text{-cl}(\{y\})$ as evidenced by (ii). Now let us prove the converse. Let $(1, 2)\alpha\text{-cl}(\{x\}) \neq (1, 2)\alpha\text{-cl}(\{y\})$. Then there exists a $z \in X$, such that $z \in (1, 2)\alpha\text{-cl}(\{x\})$ and $z \notin (1, 2)\alpha\text{-cl}(\{y\})$. If x is not ultra-weakly separated from y , then $x \in (1, 2)\alpha\text{-cl}(\{y\})$. So $(1, 2)\alpha\text{-cl}(\{x\}) \subseteq (1, 2)\alpha\text{-cl}(\{y\})$. Then $z \in (1, 2)\alpha\text{-cl}(\{y\})$, which is a contradiction.

Corollary 4.2. *A space X is ultra- T_0 if and only if one of the following conditions hold good:*

- (i) *For $x, y \in X, y \in (1, 2)\alpha\text{-ker}(\{x\})$ implies $x \notin (1, 2)\alpha\text{-ker}(\{y\})$.*
- (ii) *For all $x, y \in X$, if $x \neq y$, $(1, 2)\alpha\text{-ker}(\{x\}) \neq (1, 2)\alpha\text{-ker}(\{y\})$.*

Theorem 4.3. *A space X is ultra- T_0 if and only if $[(1, 2)\alpha\text{-cl}(\{x\}) \cap \{y\}] \cap [(1, 2)\alpha\text{-cl}(\{y\}) \cap \{x\}]$ is degenerate.*

Proof. Let X is ultra- T_0 . Then we have any one of the two cases viz, x is weakly-ultra-separated from y and y is weakly-ultra-separated from x .

Case (i) :- If x is weakly-ultra-separated from y , then we have $\{x\} \cap (1, 2)_\alpha\text{-cl}(\{y\}) = \emptyset$ and $\{y\} \cap (1, 2)_\alpha\text{-cl}(\{x\})$ is a degenerate set.

Case (ii) :- If y is weakly-ultra-separated from x , then we have $\{y\} \cap (1, 2)_\alpha\text{-cl}(\{x\}) = \emptyset$ and $\{x\} \cap (1, 2)_\alpha\text{-cl}(\{y\})$ is a degenerate set. Hence $[\{x\} \cap (1, 2)_\alpha\text{-cl}(\{y\})] \cap [\{y\} \cap (1, 2)_\alpha\text{-cl}(\{x\})]$ is a degenerate set.

Conversely, suppose $[\{x\} \cap (1, 2)_\alpha\text{-cl}(\{y\})] \cap [\{y\} \cap (1, 2)_\alpha\text{-cl}(\{x\})]$ is a degenerate set. Then it is either \emptyset or a singleton set. If it is \emptyset , then there is nothing to prove. If it is a singleton set, its value is either $\{x\}$ or $\{y\}$. If it is $\{x\}$, then y is weakly-ultra-separated from x . If it is $\{y\}$, then x is weakly-ultra-separated from y . Hence X is ultra- T_0 .

Theorem 4.4. *A space is ultra- T_0 if and only if*

$$(1, 2)_\alpha\text{-d}(\{x\}) \cap (1, 2)_\alpha\text{-shl}(\{x\}) = \emptyset.$$

Proof. Let X be ultra- T_0 . Suppose we have

$$(1, 2)_\alpha\text{-d}(\{x\}) \cap (1, 2)_\alpha\text{-shl}(\{x\}) \neq \emptyset.$$

Then let $z \in (1, 2)_\alpha\text{-d}(\{x\})$ and $z \in (1, 2)_\alpha\text{-shl}(\{x\})$. Then $z \neq x$ and $z \in (1, 2)_\alpha\text{-cl}(\{x\})$ and $z \in (1, 2)_\alpha\text{-ker}(\{x\})$. Then z is not weakly-ultra-separated from x and also x is not weakly-ultra-separated from z , which is a contradiction.

Conversely, let $(1, 2)_\alpha\text{-d}(\{x\}) \cap (1, 2)_\alpha\text{-shl}(\{x\}) = \emptyset$. Then there exists a $z \neq x, z \in (1, 2)_\alpha\text{-cl}(\{x\})$ and $z \notin (1, 2)_\alpha\text{-ker}(\{x\})$. Hence if we have a z , which is not weakly-ultra-separate from x , then x is weakly-ultra-separated from z .

Corollary 4.5. *If X is ultra- T_0 , then for any $x \in X, (1, 2)_\alpha\text{-}\langle x \rangle = \{x\}$.*

Theorem 4.6. *A space X is ultra- T_1 if and only if one of the following conditions hold good:*

- (i) *For arbitrary $x, y \in X, x \neq y, x$ is weakly-ultra-separated form y .*
- (ii) *For every $x \in X, (1, 2)_\alpha\text{-cl}(\{x\}) = \{x\}$.*
- (iii) *For every $x \in X, (1, 2)_\alpha\text{-d}(\{x\}) = \emptyset$ or $(1, 2)_\alpha\text{-N-D} = X$.*
- (iv) *For every $x \in X (1, 2)_\alpha\text{-ker}(\{x\}) = \{x\}$.*
- (v) *For every $x \in X, (1, 2)_\alpha\text{-shl}(\{x\}) = \emptyset$ or $(1, 2)_\alpha\text{-N-shl} = X$.*
- (vi) *For every $x, y \in X, x \neq y (1, 2)_\alpha\text{-cl}(\{x\}) \cap (1, 2)_\alpha\text{-cl}(\{y\}) = \emptyset$.*
- (vii) *For every arbitrary $x, y \in X, x \neq y, we have (1, 2)_\alpha\text{-ker}(\{x\}) \cap (1, 2)_\alpha\text{-ker}(\{y\}) = \emptyset$.*

Proof.

(i) This is just a reformulation of the definition of ultra- T_1 .

(ii) If x is weakly-ultra-separated from y , then for $y \neq x$, we have $y \notin (1, 2)\alpha-cl(\{x\})$, and hence $x \notin (1, 2)\alpha-ker(\{y\})$. Therefore we get that $(1, 2)\alpha-ker(\{y\}) = \{y\}$. Its converse is just a reformulation of the above proof.

(iii) (iv) and (v) are obvious.

(vi) As X is ultra- T_1 , $(1, 2)\alpha-cl(\{x\}) = \{x\}$ and $(1, 2)\alpha-cl(\{y\}) = \{y\}$ so, when $x \neq y$, $(1, 2)\alpha-cl(\{x\}) \cap (1, 2)\alpha-cl(\{y\}) = \emptyset$.

(vii) Obvious from (vi).

Definition 4.7. For a bitopological space (X, τ_1, τ_2) ,

(i) a point $x \in X$ is called a $(1, 2)\alpha$ -neat point if it has X as its only $(1, 2)\alpha$ -nbd, and

(ii) a bitopological space X is $(1, 2)\alpha$ -symmetric if $\{x\} \in (1, 2)\alpha-cl(\{y\})$ implies $y \in (1, 2)\alpha-cl(\{x\})$.

Theorem 4.8. For a ultra- T_0 space, the following are equivalent.

(i) X is ultra- D_1 .

(ii) X has no $(1, 2)\alpha$ -neat point.

Proof. (i) \Rightarrow (ii) Assume X is ultra- D_1 . Then [by 2.8 (iii) and (iv)] each point x of X is contained in a $(1, 2)\alpha$ D-set S , where $S = U/V$ and so $x \in U$ with $U \neq X$. This implies that x is not a $(1, 2)\alpha$ -neat point

(ii) \Rightarrow (i) If X is ultra- T_0 , then for any two distinct points x, y of X , there exists a $(1, 2)\alpha$ -open set $U \neq X$ containing the point x but not y . Thus we have set a $U \neq X$ as a $(1, 2)\alpha$ -D-set. Given that X has no $(1, 2)\alpha$ -neat point. Then any $y \in X$ is not a $(1, 2)\alpha$ -neat point and so there exists a $(1, 2)\alpha$ -nbd V of y such that $V \neq X$. Thus $y \in V/U$ but $x \notin V/U$ and V/U is a $(1, 2)\alpha$ D-set. Hence X is an ultra- D_1 space.

Theorem 4.9. A bitopological space X is $(1, 2)\alpha$ -symmetric if and only if $\{x\}$ is $(1, 2)\alpha$ g-closed for each $x \in X$.

Proof. Let $\{y\}$ is $(1, 2)\alpha$ g-closed. Assume $\{x\} \in (1, 2)\alpha-cl(\{y\})$ and $\{y\} \notin (1, 2)\alpha-cl(\{x\})$. Hence $\{y\} \subseteq (X - (1, 2)\alpha-cl(\{x\}))$. As $\{y\}$ is $(1, 2)\alpha$ g-closed, $(1, 2)\alpha-cl(\{y\}) \subseteq X - (1, 2)\alpha-cl(\{x\})$, which is a contradiction to the fact that $x \in (1, 2)\alpha-cl(\{y\})$. Conversely, let $x \in (1, 2)\alpha-cl(\{y\})$ and $y \in (1, 2)\alpha-cl(\{x\})$. Suppose that $x \in E$, E is $(1, 2)\alpha$ -open and $(1, 2)\alpha-cl(\{x\})$ is not a subset of E . Then $(1, 2)\alpha-cl(\{x\}) \subseteq X - E$ and so $(1, 2)\alpha-cl(\{x\}) \cap (X - E) \neq \emptyset$. So let $y \in (1, 2)\alpha-cl(\{x\}) \cap (X - E)$ but by assumption $\{x\} \in (1, 2)\alpha-cl(\{y\}) \subseteq X - E$ and hence $x \in X - E$, which is a contradiction.

Corollary 4.10. *An ultra- T_1 space is a $(1, 2)\alpha$ -symmetric space.*

Proof. If X is an ultra- T_1 space, then (by theorem 4.9 of [10]) every singleton set is $(1, 2)\alpha$ -closed. Again by remark 3.3 of [10], every $(1, 2)\alpha$ -closed set is $(1, 2)\alpha$ g-closed. Hence X is $(1, 2)\alpha$ -symmetric.

Theorem 4.11. *Let $f : X \rightarrow Y$ be a $(1, 2)\alpha$ -irresolute, surjective function. If S is a $(1, 2)\alpha$ D-set in Y , then the inverse image of S is a $(1, 2)\alpha$ D-set in X .*

Proof. Let S be a $(1, 2)\alpha$ D-set in Y . Then there exist two $(1, 2)\alpha$ -open sets U_1 and U_2 such that $S = U_1/U_2$ and $U_1 \neq Y$. As f is $(1, 2)\alpha$ -irresolute, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are $(1, 2)\alpha$ -open sets in X and $f^{-1}(U_1) \neq X$. Then $f^{-1}(S) = f^{-1}(U_1)/f^{-1}(U_2)$ is a $(1, 2)\alpha$ D-set in X .

Theorem 4.12. *Let $f : X \rightarrow Y$ be a $(1, 2)\alpha$ -irresolute and bijective function. If Y is an ultra- D_1 space then X is also an ultra D_1 space.*

Proof. Suppose Y is an ultra- D_1 space and x and y be any pair of distinct points in X . Since f is injective and Y is ultra- D_1 , there exist $(1, 2)\alpha$ D-sets G_x and G_y of Y containing $f(x)$ and $f(y)$ respectively such that $f(x) \notin G_y$ and $f(y) \notin G_x$. By theorem 4.11, $f^{-1}(G_x)$ and $f^{-1}(G_y)$ are $(1, 2)\alpha$ D-sets in X containing x and y respectively. Hence X is ultra- D_1 .

Theorem 4.13. *A bitopological space X is ultra- D_1 if and only if for each pair of distinct points $x \neq y \in X$, there exists a $(1, 2)\alpha$ -irresolute, surjective function $f : X \rightarrow Y$, where Y is ultra- D_1 such that $f(x)$ and $f(y)$ are distinct.*

Proof. Necessity: Defining f as the identity function we can prove this part.

Sufficiency: Let x, y be distinct points of X . Assume f is a $(1, 2)\alpha$ -irresolute, surjective function and Y is ultra- D_1 . Then for any $f(x) \neq f(y)$ there exists $(1, 2)\alpha$ D-sets G_x and G_y such that $f(y) \in G_y$ and $f(x) \notin G_y$. By Theorem 4.11, $f^{-1}(G_x)$ and $f^{-1}(G_y)$ are two disjoint $(1, 2)\alpha$ D-sets containing x and y respectively. Therefore the space is ultra- D_1 .

5. MORE ON ULTRA- R_0 SPACE

In this chapter, we derive some properties of ultra- R_0 spaces.

Theorem 5.1. *A space X is ultra- R_0 if and only if for any $x, y \in X$, $(1, 2)\alpha$ -cl($\{x\}$) \neq $(1, 2)\alpha$ -cl($\{y\}$) implies $(1, 2)\alpha$ -cl($\{x\}) \cap (1, 2)\alpha$ -cl($\{y\}) = \emptyset$.*

Proof. Necessity: Let X be an ultra- R_0 space and if $x \neq y$, then $(1, 2)\alpha-cl(\{x\}) \neq (1, 2)\alpha-cl(\{y\})$. Then there exists a $z \in X$ such that $z \in (1, 2)\alpha-cl(\{x\})$ and $z \notin (1, 2)\alpha-cl(\{y\})$. Hence we get a $(1, 2)\alpha$ -open set G containing z and x but not y . Therefore $x \notin (1, 2)\alpha-cl(\{y\})$. Then $x \in X - (1, 2)\alpha-cl(\{y\})$, which is a $(1, 2)\alpha$ -open set. Since X is ultra- R_0 , $(1, 2)\alpha-cl(\{x\}) \subseteq X - (1, 2)\alpha-cl(\{y\})$ and so $(1, 2)\alpha-cl(\{x\}) \cap (1, 2)\alpha-cl(\{y\}) = \emptyset$.

Sufficiency : Let V be a $(1, 2)\alpha$ -open set and $x \in V$. Let $y \notin V$. Then $y \in X - V$. Since $x \neq y$ and $x \notin (1, 2)\alpha-cl(\{y\})$, we get $(1, 2)\alpha-cl(\{x\}) \neq (1, 2)\alpha-cl(\{y\})$. Then by assumption, $(1, 2)\alpha-cl(\{x\}) \cap (1, 2)\alpha-cl(\{y\}) = \emptyset$. Hence $y \notin (1, 2)\alpha-cl(\{x\})$ and so $(1, 2)\alpha-cl(\{x\}) \subset V$. Therefore X is an ultra- R_0 space.

Theorem 5.2. *A space X is ultra- R_0 if and only if for any two points $x, y \in X$, $(1, 2)\alpha-ker(\{x\}) \neq (1, 2)\alpha-ker(\{y\})$ implies $(1, 2)\alpha-ker(\{x\}) \cap (1, 2)\alpha-ker(\{y\}) = \emptyset$.*

Proof. If $(1, 2)\alpha-ker(\{x\}) \neq (1, 2)\alpha-ker(\{y\})$, then by theorem 3.12, $(1, 2)\alpha-cl(\{x\}) \neq (1, 2)\alpha-cl(\{y\})$. We need to prove that $(1, 2)\alpha-ker(\{x\}) \cap (1, 2)\alpha-ker(\{y\}) = \emptyset$. If not, let there exist a z in X such that $z \in (1, 2)\alpha-ker(\{x\})$ and $z \in (1, 2)\alpha-ker(\{y\})$. Then $x \in (1, 2)\alpha-cl(\{z\})$ and $y \in (1, 2)\alpha-cl(\{z\})$. Then by theorem 5.1, we have $(1, 2)\alpha-cl(\{z\}) = (1, 2)\alpha-cl(\{x\}) = (1, 2)\alpha-cl(\{y\})$, which is a contradiction. Hence we get that $(1, 2)\alpha-ker(\{x\}) \cap (1, 2)\alpha-ker(\{y\}) = \emptyset$.

Conversely, let X be a space such that for any two distinct points x and y in X , $(1, 2)\alpha-ker(\{x\}) \neq (1, 2)\alpha-ker(\{y\})$ implies that $(1, 2)\alpha-ker(\{x\}) \cap (1, 2)\alpha-ker(\{y\}) = \emptyset$. By theorem 3.12, we get that $(1, 2)\alpha-cl(\{x\}) \neq (1, 2)\alpha-cl(\{y\})$. If $(1, 2)\alpha-cl(\{x\}) \cap (1, 2)\alpha-cl(\{y\}) = \emptyset$, then by theorem 5.1, X is ultra- R_0 . If not, let there be a z in X such that $z \in (1, 2)\alpha-cl(\{x\})$ and $z \in (1, 2)\alpha-cl(\{y\})$. Then $x \in (1, 2)\alpha-ker(\{z\})$ and $y \in (1, 2)\alpha-ker(\{z\})$. So we obtain that

$$(1, 2)\alpha-ker(\{x\}) = (1, 2)\alpha-ker(\{z\}) = (1, 2)\alpha-ker(\{y\}),$$

which is a contradiction. Therefore $(1, 2)\alpha-cl(\{x\}) \cap (1, 2)\alpha-cl(\{y\}) = \emptyset$. Hence X is ultra- R_0 .

Theorem 5.3. *For a space X , the following statements are equivalent:*

- (i) X is ultra- R_0 .
- (ii) For any $A \neq \emptyset$ and $G \in (1, 2)\alpha O(X)$ such that $A \cap G \neq \emptyset$ there exists a $F \in (1, 2)\alpha CL(X)$ such that $F \cap G \neq \emptyset$ and $F \subset G$.
- (iii) For any $G \in (1, 2)\alpha O(X)$, $G = \cup\{F \in (1, 2)\alpha CL(X)/F \subset G\}$.
- (iv) For any $(1, 2)\alpha$ -closed set F , $F = \cap\{G \in (1, 2)\alpha O(X)/F \subset G\}$.

(v) For any $x \in X$, $(1, 2)\alpha\text{-cl}(\{x\}) \subset (1, 2)\alpha\text{-ker}(\{x\})$.

Proof. (i) \Rightarrow (ii) Assume X is ultra- R_0 . Let A be a non empty subset of X and G be a $(1, 2)\alpha$ -open set such that $A \cap G \neq \emptyset$. So there exists a $x \in A \cap G$. Then $x \in A$ and $x \in G$. As X is ultra- R_0 , $(1, 2)\alpha\text{-cl}(\{x\}) \subset G$. Take $F = (1, 2)\alpha\text{-ker}(\{x\})$ and then $F \subset G$ and $A \cap G \neq \emptyset$.

(ii) \Rightarrow (iii) Let $G \in (1, 2)\alpha O(X)$ and $F \in (1, 2)\alpha\text{-CL}(\{X\})$ such that $F \subset G$. Then $\bigcup\{F/F \in (1, 2)\alpha\text{-CL}(\{X\}) \text{ and } F \subset G\} \subset G$. To prove the other inclusion, let $x \in G$. Then by assumption, there exists a $(1, 2)\alpha$ -closed set F such that $x \in F$ and $F \subset G$. Hence

$$G \subset \bigcup\{F \in (1, 2)\alpha\text{-CL}(\{X\}) \text{ and } F \subset G\}.$$

Hence we get the result.

(iii) \Rightarrow (iv) Obvious.

(iv) \Rightarrow (v) Let $x \in X$ and there exists a $y \in X$ such that $y \notin (1, 2)\alpha\text{-ker}(\{x\})$. Then there exists a $(1, 2)\alpha$ -open set V containing x but not y . That is $(1, 2)\alpha\text{-cl}(\{y\}) \cap V = \emptyset$, which implies that $(1, 2)\alpha\text{-cl}(\{y\}) \subset X - V$. Again, by our assumption, $\bigcap\{G \in (1, 2)\alpha O(X) / (1, 2)\alpha\text{-cl}(\{y\}) \subset G\} \cap V = \emptyset$. Hence there exists a $(1, 2)\alpha$ -open set G such that $x \notin G$ and $(1, 2)\alpha\text{-cl}(\{y\}) \subset G$. Therefore $(1, 2)\alpha\text{-cl}(\{x\}) \cap G = \emptyset$ and $y \notin (1, 2)\alpha\text{-cl}(\{x\})$. Consequently, $(1, 2)\alpha\text{-cl}(\{x\}) \subset (1, 2)\alpha\text{-ker}(\{x\})$.

(v) \Rightarrow (i) Let $G \in (1, 2)\alpha O(X)$ and $x \in G$. Suppose that $y \in (1, 2)\alpha\text{-ker}(\{x\})$. Then $(1, 2)\alpha\text{-cl}(\{y\}) \cap \{x\} \neq \emptyset$. So $\{x\} \in (1, 2)\alpha\text{-cl}(\{y\})$ and $y \in G$ which implies that $(1, 2)\alpha\text{-cl}(\{x\}) \subset (1, 2)\alpha\text{-ker}(\{x\}) \subset G$. So X is ultra- R_0 .

Theorem 5.4. For a bitopological space X , the following properties are equivalent:

- (i) X is ultra- R_0 .
- (ii) $(1, 2)\alpha\text{-cl}(\{x\}) = (1, 2)\alpha\text{-ker}(\{x\})$ for all $x \in X$.

Proof. Suppose that X is an ultra- R_0 space. Then by theorem 5.3, $(1, 2)\alpha\text{-cl}(\{x\}) \subset (1, 2)\alpha\text{-ker}(\{x\})$ for each $x \in X$. Assume that $y \in (1, 2)\alpha\text{-ker}(\{x\})$. Then $x \in (1, 2)\alpha\text{-cl}(\{y\})$ and so $(1, 2)\alpha\text{-cl}(\{x\}) = (1, 2)\alpha\text{-cl}(\{y\})$. Therefore $y \in (1, 2)\alpha\text{-cl}(\{x\})$ and so $(1, 2)\alpha\text{-ker}(\{x\}) \subset (1, 2)\alpha\text{-cl}(\{x\})$. Thus $(1, 2)\alpha\text{-ker}(\{x\}) = (1, 2)\alpha\text{-cl}(\{x\})$.

(ii) \Rightarrow (i) is obvious by Theorem 5.2.

Theorem 5.5. For a space X , the following properties are equivalent:

- (i) X is ultra- R_0 .
- (ii) $x \in (1, 2)\alpha\text{-cl}(\{y\})$ if and only if $y \in (1, 2)\alpha\text{-cl}(\{x\})$, for any points x and y in X .

Proof. (i) \Rightarrow (ii) Assume X is ultra- R_0 . Let $x \in (1, 2)\alpha\text{-cl}(\{y\})$ and D be any $(1, 2)\alpha$ -open set such that $y \in D$. As X is ultra- R_0 , $(1, 2)\alpha\text{-cl}(\{y\}) \subset D$ and hence $x \in D$. Therefore every $(1, 2)\alpha$ -open set containing y contains x . Hence $y \in (1, 2)\alpha\text{-cl}(\{x\})$

(ii) \Rightarrow (i) Let U be a $(1, 2)\alpha$ -open set and $x \in U$. If $y \notin U$, then $x \notin (1, 2)\alpha\text{-cl}(\{y\})$ and hence $y \notin (1, 2)\alpha\text{-cl}(\{x\})$. This implies that $(1, 2)\alpha\text{-cl}(\{x\}) \subset U$. Hence X is ultra- R_0 .

Theorem 5.6. *For a space X , the following properties are equivalent:*

- (i) X is ultra- R_0 .
- (ii) If F is $(1, 2)\alpha$ -closed, then $F = (1, 2)\alpha\text{-ker}(\{F\})$.
- (iii) If F is $(1, 2)\alpha$ -closed, then $(1, 2)\alpha\text{-ker}(\{x\}) \subset F$, where $x \in F$.
- (iv) If $x \in X$ then $(1, 2)\alpha\text{-ker}(\{x\}) \subset (1, 2)\alpha\text{-cl}(\{x\})$.

Proof. (i) \Rightarrow (ii) Let F be a $(1, 2)\alpha$ -closed set and $x \notin F$. Then $X - F$ is a $(1, 2)\alpha$ -open set containing x . Since X is ultra- R_0 , $(1, 2)\alpha\text{-cl}(\{x\}) \subset X - F$. Thus $(1, 2)\alpha\text{-cl}(\{x\}) \cap F = \emptyset$. Hence, by lemma 3.3, $x \notin (1, 2)\alpha\text{-ker}(\{F\})$ and so $(1, 2)\alpha\text{-ker}(\{F\}) \subset F$. Also by the definition of kernel of a set, $F \subset (1, 2)\alpha\text{-ker}(\{F\})$. Hence $F = (1, 2)\alpha\text{-ker}(\{F\})$.

(ii) \Rightarrow (iii) Let $x \in F$. So $(1, 2)\alpha\text{-ker}(\{x\}) \subset (1, 2)\alpha\text{-ker}(\{F\})$. As F is $(1, 2)\alpha$ -closed, $(1, 2)\alpha\text{-ker}(\{F\}) = F$.

(iii) \Rightarrow (iv) Since $x \in (1, 2)\alpha\text{-cl}(\{x\})$ and $(1, 2)\alpha\text{-cl}(\{x\})$ is a $(1, 2)\alpha$ -closed set, by (iii) $(1, 2)\alpha\text{-ker}(\{x\}) \subset (1, 2)\alpha\text{-cl}(\{x\})$.

(iv) \Rightarrow (i) Let $x \in (1, 2)\alpha\text{-cl}(\{y\})$. Then $y \in (1, 2)\alpha\text{-ker}(\{x\})$ and $(1, 2)\alpha\text{-cl}(\{y\})$ is a $(1, 2)\alpha$ -closed set. By assumption, we have $y \in (1, 2)\alpha\text{-ker}(\{x\}) \subset (1, 2)\alpha\text{-cl}(\{x\})$, which implies that $y \in (1, 2)\alpha\text{-cl}(\{x\})$. Hence, by Theorem 5.5, X is ultra- R_0 .

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