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UNIQUENESS OF SOLUTIONS TO A CLASS OF STRONGLY DEGENERATE PARABOLIC EQUATION

(submitted by M. A. Malakhaltsev)

ABSTRACT. In this paper, by virtue of Holmgren's approach, we show the uniqueness of the bounded solutions to a class of parabolic equation with two kinds degeneracies at the same time under some necessary conditions on the growth of the convection and sources.

1. INTRODUCTION

This paper concerns the uniqueness of the bounded solutions to the initial-boundary value problem of the strongly degenerate parabolic equation

$$\frac{\partial \sigma(u)}{\partial t} = \frac{\partial^2 A(u)}{\partial x^2} + \frac{\partial B(u)}{\partial x} + f(u, x, t), \qquad (x, t) \in Q_T, \qquad (1.1)$$

$$A(u(0, t)) = g_1(t), \quad A(u(1, t)) = g_2(t), \qquad t \in (0, T), \qquad (1.2)$$

$$\sigma(u(x, 0)) = u_0(x), \qquad\qquad x \in (0, 1), \qquad (1.3)$$

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where $Q_T = (0,1) \times (0,T)$,

$$\sigma(u) = \int_0^u c(s)ds, \quad A(u) = \int_0^u a(s)ds, \quad B(u) = \int_0^u b(s)ds,$$

and $c(s) \geq 0, a(s) \geq 0, b(s), f(s,x,t), g_i(t)(i = 1,2), u_0(x)$ are suitably smooth functions.

The equation (1.1) can be used to describe a variety of diffusion phenomena appeared widely in nature(see [1]). It is degenerate at the sets $E = \{s \in \mathbb{R}; c(s) = 0\}$ or $F = \{s \in \mathbb{R}; a(s) = 0\}$. Generally speaking, the equation (1.1) is a classical parabolic-hyperbolic equation if $c(s) \neq 0$ and when $a(s) \neq 0$ an elliptical-parabolic equation, whose degeneracy appears respectively in the sets $\{s; a(s) = 0\}$ and $\{s; c(s) = 0\}$. As we know, the equation (1.1) with only one kind of degeneracy, especially for the case $A(s) = s$ and $\sigma(s) = s$ respectively was studied in many papers, see [1–5] and the references therein. In this paper, we investigate the equation (1.1) with two degeneracies at the same time, the sets $E = \{s \in \mathbb{R}; c(s) = 0\}$ and $F = \{s \in \mathbb{R}; a(s) = 0\}$ are allowed to have a infinite points.

In [6], the authors considered the equation (1.1) with $f(u,x,t) = 0$, they proved the uniqueness of the bounded solutions under the assumption that $|b(s)|^2 \leq h(s)c(s)$ with $h(s)$ is a given continuous function.

The purpose of the present paper is to generalize the result obtained by the authors in [6] to a more general case, i.e., we establish the uniqueness of the bounded solutions of the problem (1.1)–(1.3) under the assumptions that $|b(s)|^{p_1} \leq h_1(s)c(s)$, $|f'_s(s,x,t)|^{p_2} \leq h_2(s)c(s)$, where $1 < p_1 \leq 2$, $1 < p_2 \leq 2$ and $h_i(s)(i = 1,2)$ are given continuous function. It is easy to see that our result also improves the condition in [6].

The main theorem on the uniqueness of the bounded solutions and its proof will be given in the next section.

2. MAIN THEOREM AND ITS PROOF

The bounded solution of the problem (1.1)–(1.3) is defined as follows.

Definition A function $u \in L^\infty(Q_T)$ is called a bounded solution of the problem (1.1)–(1.3), if the following integral equality holds

$$\iint_{Q_T} \left(\sigma(u)\frac{\partial\varphi}{\partial t} + A(u)\frac{\partial^2\varphi}{\partial x^2} - B(u)\frac{\partial\varphi}{\partial x} + f(u,x,t)\varphi\right)dx\,dt$$

$$+ \int_0^T g_1(t)\frac{\partial\varphi}{\partial x}(0,t)dt - \int_0^T g_2(t)\frac{\partial\varphi}{\partial x}(1,t)dt + \int_0^1 u_0(x)\varphi(x,0)dx = 0$$

for any test function $\varphi \in C^\infty(Q_T)$ with $\varphi(0,t) = \varphi(1,t) = \varphi(x,T) = 0$.

The following theorem is our main result in this paper.

Theorem 2.1. *Assume that the set $E = \{s; a(s) = 0\}$ has no interior point, there are $p_1, p_2 \in \mathbb{R} : 1 < p_1 \leq 2, \quad 1 < p_2 \leq 2$ and continuous, bounded functions $h_1(s) \geq 0, h_2(s,x,t) \geq 0$, such that $|b(s)|^{p_1} \leq c(s)h_1(s), |f'_s(s,x,t)|^{p_2} \leq c(s)h_2(s,x,t)$. Then the problem (1.1)–(1.3) admits at most one bounded solution.*

The proof of Theorem 2.1 will be completed by the reduction to absurdity. Let $u_1, u_2 \in L^\infty(Q_T)$ be two bounded solutions of the problem (1.1)–(1.3). It only needs to show $u_1 = u_2$ a.e. on Q_T. We firstly show $\sigma(u_1) = \sigma(u_2)$ a.e. on Q_T by the Holmgren's method which used in [3], [6].

By the definition of bounded solution, we have

$$\iint_{Q_T} (u_1 - u_2)\left(\tilde{\sigma}\frac{\partial\varphi}{\partial t} + \tilde{A}\frac{\partial^2\varphi}{\partial x^2} - \tilde{B}\frac{\partial\varphi}{\partial x} + \tilde{f}\varphi\right)dx\,dt = 0.$$

for any function $\varphi \in C^2(Q_T)$ with $\varphi(0,t) = \varphi(1,t) = \varphi(x,T) = 0$, where

$$\tilde{\sigma} = \tilde{\sigma}(u_1, u_2) = \int_0^1 c(\theta u_1 + (1-\theta)u_2)d\theta,$$

$$\tilde{A} = \tilde{A}(u_1, u_2) = \int_0^1 a(\theta u_1 + (1-\theta)u_2)d\theta,$$

$$\tilde{B} = \tilde{B}(u_1, u_2) = \int_0^1 b(\theta u_1 + (1-\theta)u_2)d\theta,$$

$$\tilde{f} = \tilde{f}(u_1, u_2, x, t) = \int_0^1 f'_u(\theta u_1 + (1-\theta)u_2, x, t)d\theta.$$

If for any $g \in C_0^\infty(Q_T)$, the adjoint problem

$$\tilde{\sigma}\frac{\partial\varphi}{\partial t} + \tilde{A}\frac{\partial^2\varphi}{\partial x^2} - \tilde{B}\frac{\partial\varphi}{\partial x} + \tilde{f}\varphi = \tilde{\sigma}g,$$
$$\varphi(0,t) = \varphi(1,t) = 0,$$
$$\varphi(x,T) = 0$$

admits a solution $\varphi \in C^\infty(\overline{Q}_T)$, then we have

$$\iint_{Q_T} (\sigma(u_1) - \sigma(u_2))g\,dx\,dt = 0.$$

Then the arbitrariness of g implying that $\sigma(u_1) = \sigma(u_2)$. But we see that the smooth solution of the above problem may not exist since that

the coefficients of it are not smooth enough. Thus, we will consider the approximation of the above adjoint problem.

Let $\tilde{\sigma}_\varepsilon > 0$ and $\tilde{A}_\varepsilon > 0$ be a C^∞ approximation of $\tilde{\sigma}$ and $\tilde{A}$ respectively, such that

$$\lim_{\varepsilon \to 0} \tilde{\sigma}_\varepsilon = \tilde{\sigma}, \quad \lim_{\varepsilon \to 0} \tilde{A}_\varepsilon = \tilde{A}, \quad a.e. \quad x \in Q_T,$$

$$\tilde{\sigma}_\varepsilon \leq C, \quad \tilde{A}_\varepsilon \leq C.$$

For sufficiently small $\eta > 0, \delta > 0, \gamma > 0$, let

$$\lambda_{\eta\delta\gamma B} = \begin{cases} 0, & \text{if} \quad (x,t) \in G_\delta, \\ (\eta + \tilde{A})^{-1/2}(\gamma + \tilde{\sigma})^{-1/p_1}\tilde{B}, & \text{if} \quad (x,t) \in F_\delta, \end{cases}$$

$$\lambda_{\eta\delta\gamma f} = \begin{cases} 0, & \text{if} \quad (x,t) \in G_\delta, \\ (\eta + \tilde{A})^{-1/2}(\gamma + \tilde{\sigma})^{-1/p_2}\tilde{f}, & \text{if} \quad (x,t) \in F_\delta, \end{cases}$$

where $G_\delta = \{(x,t) \in Q_T, |u_1 - u_2| < \delta\}, F_\delta = \{(x,t) \in Q_T, |u_1 - u_2| \geq \delta\}$. Clearly, the assumptions in Theorem 2.1 imply that

$$\tilde{B}^{p_1} = \left(\int_0^1 b(\theta u_1 + (1-\theta)u_2)d\theta \right)^{p_1} \leq C\tilde{\sigma}, \tag{2.1}$$

$$\tilde{f}^{p_2} = \left(\int_0^1 f'_u(\theta u_1 + (1-\theta)u_2, x, t)d\theta \right)^{p_2} \leq C\tilde{\sigma}, \tag{2.2}$$

and furthermore,

$$\tilde{B} \leq C(\tilde{\sigma})^{1/p_1} \leq C(\gamma + \tilde{\sigma})^{1/p_1}, \quad \tilde{f} \leq C(\tilde{\sigma})^{1/p_2} \leq C(\gamma + \tilde{\sigma})^{1/p_2}.$$

Here and in the sequel, we use C to denote a universal constant independent of δ, η, γ and ε, $K(\delta)$ a constant depending only on δ, which may take different values on different occasions.

Since $A(s)$ is strictly increasing and $u_1, u_2 \in L^\infty(Q_T)$, there must be constants $L(\delta) > 0, K(\delta) > 0$ depending on δ, but independent of η and γ, such that

$$\tilde{A} = \frac{A(u_1) - A(u_2)}{u_1 - u_2} \geq L(\delta), \text{ whenever } (x,t) \in F_\delta,$$

$$|\lambda_{\eta\delta\gamma B}| \leq K(\delta), \quad |\lambda_{\eta\delta\gamma f}| \leq K(\delta).$$

Let $\lambda^\varepsilon_{\eta\delta\gamma B}$ and $\lambda^\varepsilon_{\eta\delta\gamma f}$ be a C^∞ approximation of $\lambda_{\eta\delta\gamma B}$ and $\lambda_{\eta\delta\gamma f}$ respectively, such that

$$\lim_{\varepsilon \to 0} \lambda^\varepsilon_{\eta\delta\gamma B} = \lambda_{\eta\delta\gamma B}, \quad \lim_{\varepsilon \to 0} \lambda^\varepsilon_{\eta\delta\gamma f} = \lambda_{\eta\delta\gamma f}, \text{ a.e. in } Q_T,$$

$$|\lambda^\varepsilon_{\eta\delta\gamma B}| \leq K(\delta), \quad |\lambda^\varepsilon_{\eta\delta\gamma f}| \leq K(\delta). \tag{2.3}$$

Denote
$$\tilde{B}_\varepsilon = \lambda^\varepsilon_{\eta\delta\gamma B}(\eta + \tilde{A}_\varepsilon)^{1/2}(\gamma + \tilde{\sigma}_\varepsilon)^{1/p_1},$$
$$\tilde{f}_\varepsilon = \lambda^\varepsilon_{\eta\delta\gamma f}(\eta + \tilde{A}_\varepsilon)^{1/2}(\gamma + \tilde{\sigma}_\varepsilon)^{1/p_2}.$$

For any given $g \in C_0^\infty(Q_T)$, consider the approximate adjoint problem

$$\frac{\partial\varphi}{\partial t} + \frac{\eta + \tilde{A}_\varepsilon}{\gamma + \tilde{\sigma}_\varepsilon}\frac{\partial^2\varphi}{\partial x^2} - \frac{\tilde{B}_\varepsilon}{\gamma + \tilde{\sigma}_\varepsilon}\frac{\partial\varphi}{\partial x} + \frac{\tilde{f}_\varepsilon}{\gamma + \tilde{\sigma}_\varepsilon}\varphi = g, \qquad (2.4)$$

$$\varphi(0, t) = \varphi(1, t) = 0, \qquad (2.5)$$

$$\varphi(x, T) = 0. \qquad (2.6)$$

It is easy to see that the problem (2.4)–(2.6) admits a smooth solution φ by the classical theory of parabolic equations. We give some useful estimates on φ as follows.

Lemma 2.1. *The solution φ of the problem (2.4)–(2.6) satisfies*

$$\sup_{Q_T} |\varphi(x, t)| \leq C, \qquad (2.7)$$

$$\iint_{Q_T} \frac{\eta + \tilde{A}_\varepsilon}{\gamma + \tilde{\sigma}_\varepsilon}\Big(\frac{\partial^2\varphi}{\partial x^2}\Big)^2 dx\, dt \leq K(\delta)\eta^{-1} + K(\delta), \qquad (2.8)$$

$$\iint_{Q_T} \Big(\frac{\partial\varphi}{\partial x}\Big)^2 dx\, dt \leq K(\delta)\eta^{-1} + K(\delta). \qquad (2.9)$$

Proof. The inequality (2.7) follows from the maximum principle. To prove (2.8) and (2.9), multiply (2.4) by $\dfrac{\partial^2\varphi}{\partial x^2}$ and integrate over Q_T. Integrating by parts and using (2.5), (2.6), we have

$$\frac{1}{2}\int_0^1 \Big(\frac{\varphi(x, 0)}{\partial x}\Big)^2 dx\, dt + \iint_{Q_T} \frac{\eta + \tilde{A}_\varepsilon}{\gamma + \tilde{\sigma}_\varepsilon}\Big(\frac{\partial^2\varphi}{\partial x^2}\Big)^2 dx\, dt$$

$$- \iint_{Q_T} \frac{\tilde{B}_\varepsilon}{\gamma + \tilde{\sigma}_\varepsilon}\frac{\partial\varphi}{\partial x}\frac{\partial^2\varphi}{\partial x^2}dx\, dt + \iint_{Q_T} \frac{\tilde{f}_\varepsilon}{\gamma + \tilde{\sigma}_\varepsilon}\varphi\frac{\partial^2\varphi}{\partial x^2}\varphi dx\, dt$$

$$= \iint_{Q_T} g\frac{\partial^2\varphi}{\partial x^2}dx\, dt. \qquad (2.10)$$

Using Young's inequality and (2.3), we obtain

$$\iint_{Q_T} \frac{\tilde{B}_\varepsilon}{\gamma + \tilde{\sigma}_\varepsilon}\frac{\partial\varphi}{\partial x}\frac{\partial^2\varphi}{\partial x^2}dx\, dt$$

$$= \iint_{Q_T} \frac{(\eta + \tilde{A}_\varepsilon)^{1/2}}{(\gamma + \tilde{\sigma}_\varepsilon)^{1/2}}\frac{\partial^2\varphi}{\partial x^2}\lambda^\varepsilon_{\eta\delta\gamma B}(\gamma + \tilde{\sigma}_\varepsilon)^{1/p_1 - 1/2}\frac{\partial\varphi}{\partial x}dx\, dt$$

$$\leq \frac{1}{4}\iint_{Q_T} \frac{\eta + \tilde{A}_\varepsilon}{\gamma + \tilde{\sigma}_\varepsilon}\Big(\frac{\partial^2\varphi}{\partial x^2}\Big)^2 dx\, dt + K(\delta)\iint_{Q_T} \Big(\frac{\partial\varphi}{\partial x}\Big)^2 dx\, dt, \qquad (2.11)$$

$$\iint_{Q_T} \Big(\frac{\partial\varphi}{\partial x}\Big)^2 dx\,dt = -\iint_{Q_T} \varphi\frac{\partial^2\varphi}{\partial x^2}dx\,dt$$

$$\leq \beta\iint_{Q_T} \frac{\eta+\tilde{A}_\varepsilon}{\gamma+\tilde{\sigma}_\varepsilon}\Big(\frac{\partial^2\varphi}{\partial x^2}\Big)^2 dx\,dt + C\beta^{-1}\eta^{-1}(\beta>0), \qquad (2.12)$$

$$\iint_{Q_T} g\frac{\partial^2\varphi}{\partial x^2}dx\,dt \leq \frac{1}{8}\iint_{Q_T} \frac{\eta+\tilde{A}_\varepsilon}{\gamma+\tilde{\sigma}_\varepsilon}\Big(\frac{\partial^2\varphi}{\partial x^2}\Big)^2 dx\,dt + 8C\eta^{-1}, \qquad (2.13)$$

$$\iint_{Q_T} \frac{\tilde{f}_\varepsilon}{\gamma+\tilde{\sigma}_\varepsilon}\varphi\frac{\partial^2\varphi}{\partial x^2}dx\,dt$$

$$= \iint_{Q_T} \frac{(\eta+\tilde{A}_\varepsilon)^{1/2}}{(\gamma+\tilde{\sigma}_\varepsilon)^{1/2}}\frac{\partial^2\varphi}{\partial x^2}\lambda^\varepsilon_{\eta\delta\gamma f}(\gamma+\tilde{\sigma}_\varepsilon)^{1/p_2-1/2}\varphi dx\,dt$$

$$\leq \frac{1}{8}\iint_{Q_T} \frac{\eta+\tilde{A}_\varepsilon}{\gamma+\tilde{\sigma}_\varepsilon}\Big(\frac{\partial^2\varphi}{\partial x^2}\Big)^2 dx\,dt + K(\delta). \qquad (2.14)$$

Let $\beta = 1/(4K(\delta))$ in (2.12). Then from (2.10)-(2.14) we obtain (2.8) immediately. The inequality (2.9) follows from (2.8) and (2.12)($\beta=1$). The proof is complete.

Lemma 2.2. *The solution φ of the problem (2.4)–(2.6) satisfies*

$$\sup_{0<t<T}\int_0^1 \Big|\frac{\partial\varphi(x,t)}{\partial x}\Big|dx \leq C, \quad \forall t\in(0,T).$$

   Proof.   For small $\beta>0$, let

$$sgn_\beta(s) = \begin{cases} 1, & s\geq\beta, \\ s/\beta, & |s|<\beta, \\ -1, & s\leq-\beta, \end{cases} \qquad I_\beta(s) = \int_0^s sgn_\beta(\theta)d\theta.$$

Differentiate (2.4) with respect to x, multiply the resulting equality by $sgn_\beta\Big(\frac{\partial\varphi}{\partial x}\Big)$ and integrate over $S_t = (0,1)\times(t,T)$. Then integrating by parts and by (2.6) we have

$$\int_0^1 I_\beta\Big(\frac{\partial\varphi(x,t)}{\partial x}\Big)dx = -\iint_{S_t} \frac{\eta+\tilde{A}_\varepsilon}{\gamma+\tilde{\sigma}_\varepsilon}\Big(\frac{\partial^2\varphi}{\partial x^2}\Big)^2 sgn'_\beta\Big(\frac{\partial\varphi}{\partial x}\Big)dx\,dt$$

$$+\iint_{S_t} \frac{\tilde{B}_\varepsilon}{\gamma+\tilde{\sigma}_\varepsilon}\frac{\partial\varphi}{\partial x}\frac{\partial^2\varphi}{\partial x^2}sgn'_\beta\Big(\frac{\partial\varphi}{\partial x}\Big)dx\,dt - \iint_{S_t} \frac{\tilde{f}_\varepsilon}{\gamma+\tilde{\sigma}_\varepsilon}\varphi\frac{\partial^2\varphi}{\partial x^2}sgn'_\beta\Big(\frac{\partial\varphi}{\partial x}\Big)dx\,dt$$

$$+\int_t^T \Big(\frac{\eta+\tilde{A}_\varepsilon}{\gamma+\tilde{\sigma}_\varepsilon}\frac{\partial^2\varphi}{\partial x^2} - \frac{\tilde{B}_\varepsilon}{\gamma+\tilde{\sigma}_\varepsilon}\frac{\partial\varphi}{\partial x} + \frac{\tilde{f}_\varepsilon}{\gamma+\tilde{\sigma}_\varepsilon}\varphi\Big)sgn_\beta\Big(\frac{\partial\varphi}{\partial x}\Big)\Big|_{x=0}^{x=1}dt$$

$$-\iint_{S_t} \frac{\partial g}{\partial x}sgn_\beta\Big(\frac{\partial\varphi}{\partial x}\Big)dx\,dt. \quad (2.15)$$

Notice that the first term on the right side is non-positive, the second and third term tends to 0 while $\beta \to 0$ respectively, the last term is bounded. By the boundary condition (2.5) and $g \in C_0^\infty(Q_T)$, we see that

$$\int_t^T \Big(\frac{\eta + \tilde{A}_\varepsilon}{\gamma + \tilde{\sigma}_\varepsilon}\frac{\partial^2 \varphi}{\partial x^2} - \frac{\tilde{B}_\varepsilon}{\gamma + \tilde{\sigma}_\varepsilon}\frac{\partial \varphi}{\partial x} + \frac{\tilde{f}_\varepsilon}{\gamma + \tilde{\sigma}_\varepsilon}\varphi\Big) sgn_\beta\Big(\frac{\partial \varphi}{\partial x}\Big)\Big|_{x=0}^{x=1} dt$$

$$= \int_t^T \Big(g - \frac{\partial \varphi}{\partial t}\Big) sgn_\beta\Big(\frac{\partial \varphi}{\partial x}\Big)\Big|_{x=0}^{x=1} dt = 0.$$

Therefore, letting $\beta \to 0$ in (2.15) gives

$$\sup_{0<t<T} \int_0^1 \Big|\frac{\partial \varphi(x,t)}{\partial x}\Big| dx \le C.$$

Lemma 2.3. *The solution φ of the problem (2.4)–(2.6) satisfies*

$$\iint_{Q_T} \Big(\frac{\partial \varphi}{\partial t}\Big)^2 dx\,dt \le K(\delta)\eta^{-1}\gamma^{-1} + K(\delta)\gamma^{-1} + C. \qquad (2.16)$$

Proof. Multiplying (2.4) by $\frac{\partial \varphi}{\partial t}$ and integrating it over Q_T yield

$$\iint_{Q_T} \Big(\frac{\partial \varphi}{\partial t}\Big)^2 dx\,dt + \iint_{Q_T} \frac{\eta + \tilde{A}_\varepsilon}{\gamma + \tilde{\sigma}_\varepsilon}\frac{\partial^2 \varphi}{\partial x^2}\frac{\partial \varphi}{\partial t}dx\,dt - \iint_{Q_T} \frac{\tilde{B}_\varepsilon}{\gamma + \tilde{\sigma}_\varepsilon}\frac{\partial \varphi}{\partial x}\frac{\partial \varphi}{\partial t}dx\,dt$$

$$+ \iint_{Q_T} \frac{\tilde{f}_\varepsilon}{\gamma + \tilde{\sigma}_\varepsilon}\varphi\frac{\partial \varphi}{\partial t}dx\,dt = \iint_{Q_T} g\frac{\partial \varphi}{\partial t}dx\,dt. \qquad (2.17)$$

Using Young's inequality, we obtain

$$\iint_{Q_T} \frac{\eta + \tilde{A}_\varepsilon}{\gamma + \tilde{\sigma}_\varepsilon}\frac{\partial^2 \varphi}{\partial x^2}\frac{\partial \varphi}{\partial t}dx\,dt$$

$$\le \frac{1}{4}\iint_{Q_T} \Big(\frac{\partial \varphi}{\partial t}\Big)^2 dx\,dt + \frac{\sup(\eta + \tilde{A}_\varepsilon)}{\gamma}\iint_{Q_T} \frac{\eta + \tilde{A}_\varepsilon}{\gamma + \tilde{\sigma}_\varepsilon}\Big(\frac{\partial^2 \varphi}{\partial x^2}\Big)^2 dx\,dt$$

$$\le \frac{1}{4}\iint_{Q_T} \Big(\frac{\partial \varphi}{\partial t}\Big)^2 dx\,dt + K(\delta)\eta^{-1}\gamma^{-1} + K(\delta)\gamma^{-1}, \qquad (2.18)$$

$$\iint_{Q_T} \frac{\tilde{B}_\varepsilon}{\gamma + \tilde{\sigma}_\varepsilon}\frac{\partial \varphi}{\partial x}\frac{\partial \varphi}{\partial t}dx\,dt$$

$$\le \frac{\sup(\eta + \tilde{A}_\varepsilon)\sup(\gamma + \tilde{\sigma}_\varepsilon)^{2/p_1-1}}{\gamma}\iint_{Q_T} (\lambda_{\eta\delta\gamma B}^\varepsilon)^2\Big(\frac{\partial \varphi}{\partial x}\Big)^2 dx\,dt$$

$$+ \frac{1}{4}\iint_{Q_T} \Big(\frac{\partial \varphi}{\partial t}\Big)^2 dx\,dt$$

$$\le \frac{1}{4}\iint_{Q_T} \Big(\frac{\partial \varphi}{\partial t}\Big)^2 dx\,dt + K(\delta)\eta^{-1}\gamma^{-1} + K(\delta)\gamma^{-1}, \qquad (2.19)$$

$$\iint_{Q_T} \frac{\tilde{f}_\varepsilon}{\gamma + \tilde{\sigma}_\varepsilon} \varphi \frac{\partial \varphi}{\partial t} dx\, dt$$

$$\leq \frac{\sup(\eta + \tilde{A}_\varepsilon) \sup(\gamma + \tilde{\sigma}_\varepsilon)^{2/p_2 - 1}}{\gamma} \iint_{Q_T} (\lambda^\varepsilon_{\eta \delta \gamma f})^2 \left(\frac{\partial \varphi}{\partial x}\right)^2 dx\, dt$$

$$+ \frac{1}{4} \iint_{Q_T} \left(\frac{\partial \varphi}{\partial t}\right)^2 dx\, dt$$

$$\leq \frac{1}{4} \iint_{Q_T} \left(\frac{\partial \varphi}{\partial t}\right)^2 dx\, dt + K(\delta)\gamma^{-1}, \tag{2.20}$$

$$\left| \iint_{Q_T} g \frac{\partial \varphi}{\partial t} dx\, dt \right| = \left| \iint_{Q_T} \varphi \frac{\partial g}{\partial t} dx\, dt \right| \leq C. \tag{2.21}$$

By (2.17)–(2.21) we obtain (2.16). The proof is complete.

Lemma 2.4. *Under the assumptions in Theorem 2.1, if u_1 and u_2 are bounded solutions of the problem (1.1)–(1.3), then $\sigma(u_1) = \sigma(u_2)$ a.e. on Q_T.*

Proof.   For any given $g \in C_0^\infty(Q_T)$, let φ be a solution of (2.4)–(2.6), then

$$\iint_{Q_T} \Big(\sigma(u_1) - \sigma(u_2)\Big) g dx\, dt = \iint_{Q_T} (u_1 - u_2)\tilde{\sigma} g dx\, dt$$

$$= \iint_{Q_T} (u_1 - u_2)(\tilde{\sigma} - \tilde{\sigma}_\varepsilon) g dx\, dt + \iint_{Q_T} (u_1 - u_2)(\gamma + \tilde{\sigma}_\varepsilon) g dx dt$$

$$- \iint_{Q_T} (u_1 - u_2)\gamma g dx\, dt = I_1 + I_2 + I_3. \tag{2.22}$$

It is not difficult to see that

$$|I_1| = \left| \iint_{Q_T} (u_1 - u_2)(\tilde{\sigma} - \tilde{\sigma}_\varepsilon) g dx\, dt \right| \leq C \left(\iint_{Q_T} (\tilde{\sigma} - \tilde{\sigma}_\varepsilon)^2 dx\, dt \right)^{1/2},$$
$$\tag{2.23}$$

$$|I_3| = \left| \iint_{Q_T} (u_1 - u_2)\gamma g dx\, dt \right| \leq C\gamma. \tag{2.24}$$

Hence, $\lim\limits_{\varepsilon \to 0} I_1 = 0, \lim\limits_{\gamma \to 0} I_3 = 0$. As indicated above, from the definition of bounded solution, we have

$$\iint_{Q_T} (u_1 - u_2)\Big(\tilde{\sigma}\frac{\partial \varphi}{\partial t} + \tilde{A}\frac{\partial^2 \varphi}{\partial x^2} - \tilde{B}\frac{\partial \varphi}{\partial x} + \tilde{f}\varphi\Big) dx\, dt = 0.$$

Combining with the equation (2.4) yields

$$I_2 = \iint_{Q_T} (u_1 - u_2)(\gamma + \tilde{\sigma}_\varepsilon)g dx\, dt$$

$$= \iint_{Q_T} (u_1 - u_2)\Big((\gamma + \tilde{\sigma}_\varepsilon)\frac{\partial\varphi}{\partial t} + (\tilde{A}_\varepsilon + \eta)\frac{\partial^2\varphi}{\partial x^2} - \tilde{B}_\varepsilon\frac{\partial\varphi}{\partial x} + \tilde{f}_\varepsilon\varphi\Big)dx\, dt$$

$$= \iint_{Q_T} (u_1 - u_2)\gamma\frac{\partial\varphi}{\partial t}dx\, dt + \iint_{Q_T} (u_1 - u_2)(\tilde{\sigma}_\varepsilon - \tilde{\sigma})\frac{\partial\varphi}{\partial t}dx\, dt$$

$$+ \iint_{Q_T} (u_1 - u_2)(\tilde{A}_\varepsilon - \tilde{A})\frac{\partial^2\varphi}{\partial x^2}dx\, dt + \iint_{Q_T} (u_1 - u_2)\eta\frac{\partial^2\varphi}{\partial x^2}dx\, dt$$

$$- \iint_{Q_T} (u_1 - u_2)(\tilde{B}_\varepsilon - \tilde{B})\frac{\partial\varphi}{\partial x}dx\, dt + \iint_{Q_T} (u_1 - u_2)(\tilde{f}_\varepsilon - \tilde{f})\varphi dx\, dt$$

$$= I_{21} + I_{22} + I_{23} + I_{24} + I_{25} + I_{26}. \tag{2.5}$$

In the following, we estimate all terms on the right of the above inequality.

$$|I_{21}| \le C\gamma\Big(\iint_{Q_T} (\frac{\partial\varphi}{\partial t})^2 dx\, dt\Big)^{1/2} \le C\gamma\Big(K(\delta)\eta^{-1}\gamma^{-1} + K(\delta)\gamma^{-1} + C\Big)^{1/2},$$

$$|I_{22}| \le C\Big(\iint_{Q_T} (\tilde{\sigma}_\varepsilon - \tilde{\sigma})^2 dx\, dt\Big)^{1/2}\Big(\iint_{Q_T} (\frac{\partial\varphi}{\partial t})^2 dx\, dt\Big)^{1/2}$$

$$\le C\Big(K(\delta)\eta^{-1}\gamma^{-1} + K(\delta)\gamma^{-1} + C\Big)^{1/2}\Big(\iint_{Q_T} (\tilde{\sigma}_\varepsilon - \tilde{\sigma})^2 dx\, dt\Big)^{1/2},$$

$$|I_{23}| \le C\Big(\iint_{Q_T} (\tilde{A}_\varepsilon - \tilde{A})^2 dx\, dt\Big)^{1/2}\Big(\iint_{Q_T} (\frac{\partial^2\varphi}{\partial x^2})^2 dx\, dt\Big)^{1/2}$$

$$\le C\eta^{-1}\Big(K(\delta) + K(\delta)\eta\Big)^{1/2}\Big(\iint_{Q_T} (\tilde{A}_\varepsilon - \tilde{A})^2 dx\, dt\Big)^{1/2}.$$

For any fixed $\beta > 0$, F_β, G_δ are defined just as F_δ, G_δ. By the Cauchy inequality, we have

$$|I_{24}| \le \Big|\iint_{F_\beta} (u_1 - u_2)\eta\frac{\partial^2\varphi}{\partial x^2}dx\, dt\Big| + \Big|\iint_{G_\beta} (u_1 - u_2)\eta\frac{\partial^2\varphi}{\partial x^2}dx\, dt\Big|$$

$$\le C\eta\sup_{F_\beta}\frac{|\gamma + \tilde{\sigma}_\varepsilon|^{1/2}}{\tilde{A}_\varepsilon^{1/2}}\left(\iint_{F_\beta} \frac{\eta + \tilde{A}_\varepsilon}{\gamma + \tilde{\sigma}_\varepsilon}\Big(\frac{\partial^2\varphi}{\partial x^2}\Big)^2 dx\, dt\right)^{1/2}$$

$$+ C\eta\beta\Big(\iint_{Q_T} (\frac{\partial^2\varphi}{\partial x^2})^2 dx\, dt\Big)^{1/2}$$

$$\le C\eta\sup_{F_\beta}|\gamma + \tilde{\sigma}_\varepsilon|^{1/2}\frac{\Big(K(\delta)\eta^{-1} + K(\delta)\Big)^{1/2}}{L(\beta)^{1/2}} + C\beta(K(\delta) + K(\delta)\eta)^{1/2}.$$

Let $\beta = \delta K(\delta)^{-1/2}$ and $\eta \to 0, \delta \to 0$. We see that I_{24} tends to 0.

$$
\begin{aligned}
|I_{25}| \leq & \Big| \iint_{G_\delta} (u_1 - u_2) \tilde{B}_\varepsilon \frac{\partial \varphi}{\partial x} dx\, dt \Big| + \Big| \iint_{G_\delta} (u_1 - u_2) \tilde{B} \frac{\partial \varphi}{\partial x} dx\, dt \Big| \\
& + \Big| \iint_{F_\delta} (u_1 - u_2)(\tilde{B}_\varepsilon - \tilde{B}) \frac{\partial \varphi}{\partial x} dx\, dt \Big| \\
\leq & C \Big(\iint_{G_\delta} (\lambda_{\eta\delta\gamma B}^\varepsilon)^2 dx\, dt \Big)^{1/2} \Big(\iint_{Q_T} (\frac{\partial \varphi}{\partial x})^2 dx\, dt \Big)^{1/2} + C\delta \\
& + C \Big(\iint_{F_\delta} (\tilde{B}_\varepsilon - \tilde{B})^2 dx\, dt \Big)^{1/2} \Big(\iint_{Q_T} (\frac{\partial \varphi}{\partial x})^2 dx\, dt \Big)^{1/2} \\
\leq & C \Big(K(\delta)\eta^{-1} + K(\delta) \Big)^{1/2} \Big(\iint_{G_\delta} (\lambda_{\eta\delta\gamma B}^\varepsilon)^2 dx\, dt \Big)^{1/2} + C\delta \\
& + C \Big(K(\delta)\eta^{-1} + K(\delta) \Big)^{1/2} \iint_{F_\delta} (\tilde{B}_\varepsilon - \tilde{B})^2 dx\, dt \Big)^{1/2}.
\end{aligned}
$$

Since $\lim_{\varepsilon \to 0} \lambda_{\eta\delta\gamma B}^\varepsilon = 0$ a.e. on G_δ, $\lim_{\varepsilon \to 0} \tilde{B}_\varepsilon = \tilde{B}$ a.e. on F_δ. Then $\overline{\lim}_{\varepsilon \to 0} |I_{25}| \leq C\delta$. Similar to the analysis on I_{25}, it yields $\overline{\lim}_{\varepsilon \to 0} |I_{26}| \leq C\delta$.

From the above inequalities, let $\varepsilon \to 0, \gamma \to 0, \eta \to 0, \delta \to 0$ in turn in (2.25). Then $I_2 \to 0$. Thus combining with (2.23), (2.24), it is seen from (2.22) that

$$
\iint_{Q_T} \Big(\sigma(u_1) - \sigma(u_2) \Big) g dx\, dt = 0.
$$

Because of the arbitrariness of g, we obtain $\sigma(u_1) = \sigma(u_2)$ a.e. on Q_T. The proof is complete.

Proof of Theorem 2.1. By the definition of bounded solutions, we have

$$
\iint_{Q_T} (u_1 - u_2) \Big(\tilde{\sigma} \frac{\partial \varphi}{\partial t} + \tilde{A} \frac{\partial^2 \varphi}{\partial x^2} - \tilde{B} \frac{\partial \varphi}{\partial x} + \tilde{f}\varphi \Big) dx\, dt = 0.
$$

According to Lemma 2.4, it yields

$$
\begin{aligned}
\iint_{Q_T} & (u_1 - u_2) \tilde{A} \frac{\partial^2 \varphi}{\partial x^2} dx\, dt \\
& = \iint_{Q_T} (u_1 - u_2) \tilde{B} \frac{\partial \varphi}{\partial x} dx\, dt - \iint_{Q_T} (u_1 - u_2) \tilde{f}\varphi dx\, dt \\
& \hspace{8cm} = I_1 - I_2.
\end{aligned}
$$

Using Young's inequality and Hölder inequality, we have

$$\left|I_1\right| \le \iint_{Q_T} \left|(u_1 - u_2)^{1/p_1} \tilde{B}(u_1 - u_2)^{1/q_1} \frac{\partial \varphi}{\partial x}\right| dx\, dt$$

$$\le \left(\iint_{Q_T} |(u_1 - u_2)\tilde{B}^{p_1}| dx\, dt \right)^{1/p_1} \left(\iint_{Q_T} |(u_1 - u_2)(\frac{\partial \varphi}{\partial x})^{q_1}| dx\, dt \right)^{1/q_1},$$

$$\left|I_2\right| \le \left(\iint_{Q_T} |(u_1 - u_2)\tilde{f}^{p_2}| dx\, dt \right)^{1/p_2} \left(\iint_{Q_T} |(u_1 - u_2)\varphi^{q_2}| dx\, dt \right)^{1/q_2},$$

where q_1, q_2 satisfy $1/p_1 + 1/q_1 = 1$ and $1/p_2 + 1/q_2 = 1$. From (2.1) and (2.2), we obtain

$$\left|I_1\right| \le C \left(\iint_{Q_T} |(u_1 - u_2)\tilde{\sigma}| dx\, dt \right)^{1/p_1}$$

$$\times \left(\iint_{Q_T} |(u_1 - u_2)(\frac{\partial \varphi}{\partial x})^{q_1}| dx\, dt \right)^{1/q_1} = 0,$$

$$\left|I_2\right| \le C \left(\iint_{Q_T} |(u_1 - u_2)\tilde{\sigma}| dx\, dt \right)^{1/p_2} \left(\iint_{Q_T} |(u_1 - u_2)\varphi^2| dx\, dt \right)^{1/q_2} = 0.$$

Obviously, we obtain

$$\iint_{Q_T} \left(A(u_1) - A(u_2) \right) \frac{\partial^2 \varphi}{\partial x^2} dx\, dt = \iint_{Q_T} (u_1 - u_2)\tilde{A}\frac{\partial^2 \varphi}{\partial x^2} dx\, dt = 0,$$

Since the function φ is arbitrary, we have $A(u_1) = A(u_2)$ a.e. on Q_T. Furthermore, owing to that $A(s)$ is a strictly increasing function, it has been shown that $u_1 = u_2$ a.e. on Q_T. The proof is complete.

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