Lobachevskii Journal of Mathematics
http://ljm.ksu.ru
ISSN 1818-9962
Vol. 26, 2007, 5–15

H. Azanchiler

A CHARACTERIZATION OF THE BASES OF LINE-SPLITTING MATROIDS

(submitted by M. M. Arslanov)

Abstract. In [1] the author extended n-line splitting from graphs to binary matroids and characterized the circuits of the result matroid, i.e. line-splitting matroid (es-splitting). In this paper, we characterize dependent, independent and base sets in line-splitting matroid M_X^e. Moreover, we determine rank function of M_X^e.

1. Introduction

Fleischner [2] introduced the idea of splitting a vertex of degree at least three in a connected graph and Raghunathan, Shikare and Waphare [4] extended the splitting operation from graphs to binary matroids. Shikare, Azadi and Waphare [6] further generalized this operation and also in [7] extended the n-point splitting operation from graphs to a binary matroid. Moreover, in [5] Shikare and Azadi determined the base of splitting matroids and the author in [1] extended the n-line splitting operation [8] from graphs to the binary matroids by the following way.

Definition 1.1. Let M be a binary matroid on a set S and X be a subset of $S, e \in X$. Suppose that A is a matrix over $GF(2)$, that represents the matroid M. Let A_X^e be the matrix that is obtained by adjoining an extra row to A with this row being zero everywhere except in the columns corresponding to the elements of X, where it takes the value 1 and then adjoining two columns a and γ to the resulting matrix such that the column a is zero everywhere except in the last row (new row), where it takes the value 1, and γ is a sum of two column vectors corresponding to a and e.

Let M_X^e be the vector matroid of the matrix A_X^e. We say that M_X^e has been obtained from M by splitting e and X in M. The transition from M to M_X^e is called splitting of M with respect to e and X. For convenience, we say that M_X^e is an element-set splitting (es-splitting) matroid.

Proposition 1.1 ([1])**.** *Let $M = (S, \mathcal{C})$ be a binary matroid, $X \subseteq S$, $e \in X$, and $a, \gamma \notin S$. Then $M_X^e = (S \cup \{a, \gamma\}, \mathcal{C}_X^e)$, where $\mathcal{C}_X^e = \mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \{\Delta\}$ with $\Delta = \{e, a, \gamma\}$ and*

$$\mathcal{C}_0 = \{C \in \mathcal{C} \mid C \textit{ contains an even number of elements of } X\};$$

$$\begin{aligned}\mathcal{C}_1 = &\textit{the set of minimal members of } \{C_1 \cup C_2 \mid C_1, C_2 \in \mathcal{C}, C_1 \cap C_2 = \emptyset \\ &\textit{and each } C_1 \textit{ and } C_2 \textit{ contains an odd number of element of } X \\ &\textit{such that } C_1 \cup C_2 \textit{ contains no member of } \mathcal{C}_0\};\end{aligned}$$

$$\mathcal{C}_2 = \{C \cup \{a\} \mid C \in \mathcal{C} \textit{ and } C \textit{ contains an odd number of elements of } X\}.$$

$$\mathcal{C}_3 = \mathcal{C}_{31} \cup \mathcal{C}_{32} \cup \mathcal{C}_{33},$$

where

$$\begin{aligned}\mathcal{C}_{31} = &\{C \cup \{e, \gamma\}, \mid C \in \mathcal{C}, e \notin C \textit{ and } C \textit{ contains an odd number} \\ &\textit{of elements of } X\},\end{aligned}$$

$$\begin{aligned}\mathcal{C}_{32} = &\{(C \setminus e) \cup \{\gamma\} \mid C \in \mathcal{C}, e \in C \textit{ and } C \textit{ contains an odd number} \\ &\textit{of elements of } X\},\end{aligned}$$

$$\begin{aligned}\mathcal{C}_{33} = &\{(C \setminus e) \cup \{a, \gamma\} \mid C \in \mathcal{C}, e \in C \textit{ and } C \setminus e \textit{ contains an odd number} \\ &\textit{of elements of } X\}.\end{aligned}$$

The terminology from matroid theory which we use can be obtained from Oxely [3].

2. Independent Set in M_X^e

Next theorem characterize the dependent set in es-splitting matroid M_X^e.

Theorem 2.1. Let M be a binary matroid on a set S and D be a dependent set in M. Then D is dependent in M_X^e if and only if D does not contain precisely one circuit of M containing an odd number of elements of X.

Proof. Let D be a dependent set in M and suppose D does not contain precisely one circuit of M containing an odd number of elements of X. Then we have the following two cases:

(i) D contains a circuit C of M with even number of elements of X. Then C is a circuit of M_X^e and is contained in D. Therefore, D is dependent set in M_X^e.

(ii) D contains at least two circuits, say C and C', with odd number of elements of X. Then

$$C \Delta C' = C_1 \cup C_2 \cup \ldots \cup C_m.$$

If for any of the $1 \leq i \leq m$, C_i contains an even number of elements of X, then it is a circuit in M_X^e and is contained in D. Suppose there is no such C_i. Let C_j and C_k be two circuits each of which contains an odd number of elements of X. If $C_j \cup C_k$ contains a member of $\mathcal{C}_0$, say C'', then $C'' \subseteq D$ and we are done. Otherwise $C_j \cup C_k$, or a minimal member of $\mathcal{C}_1$, contained in it, is a circuit of M_X^e contained in D.

Conversely, let D be a dependent set of M which is also dependent in M_X^e. Since $D \subseteq S$, a or γ or both do not belong to D. Suppose C is a circuit of M_X^e contained in D. Then C contains an even number of elements of X. We have two cases:

(i) C is a circuit of M containing an even number of elements of X.

(ii) C is a disjoint union of two circuits of M each of which contains an odd number of elements of X.

In both cases D cannot contain precisely one circuit of M containing an odd number of elements of X. □

Lemma 2.2. Every independent set in M is independent in M_X^e.

Remark 2.3. Converse of the lemma is not true. By Theorem 2.1, every circuit of M containing an odd number of element of X is a independent set of M_X^e.

The next theorem characterizes the independent sets of M_X^e.

Theorem 2.4. Let $I \subseteq S \cup \{a, \gamma\}$. Then I is independent in M_X^e if and only if one of the following conditions hold.

(1) $I = I_1 \cup J$, where I_1 is an independent set in M and $J \in \{\phi, \{a\}\}$.

(2) $I = I_1 \cup \{\gamma\}$, where I_1 is an independent set in M and no circuit of M is contained in $I_1 \cup \{e\}$.

(3) $I = (I_1 \setminus \{e\}) \cup \{a, \gamma\}$, where I_1 is an independent set in M containing e.

(4) $I = (\cup_{i=1}^m C_i) \cup J$, where $J \in \{\phi, \{\gamma\}\}$, each C_i contains an odd number of element of $X, C_i \cap C_j \neq \phi, i \neq j, i, j = 1, 2, \ldots, m$, I

contains no member of $\mathcal{C}_0$, and $I \cup \{e\}$ contains no circuit of M other than C_i for $i = 1, 2, \ldots, m$.

Proof. (1) Let I_1 be an independent set in M. By Lemma 2.2, I_1 is independent in M_X^e. Thus, $I = I_1 \cup J$, where $J = \phi$, is independent in M_X^e. Further, by Definition 1.1 of $A_X^e, I_1 \cup \{a\}$ is an independent set in M_X^e.

(2) We show that $I_1 \cup \{\gamma\}$ is independent in M_X^e, where I and γ satisfy conditions in (2). On the contrary, suppose $I_1 \cup \{\gamma\}$ is dependent in M_X^e and let C' be a circuit of M_X^e contained in $I_1 \cup \{\gamma\}$. We have the following cases:

(i) $C' \in \mathcal{C}_0$ or $\mathcal{C}_1$. Then $C' \subseteq I_1$ and we know C' is a circuit or contains a circuit of M. This shows that I_1 is dependent in M, a contradiction.

(ii) $C' \in \mathcal{C}_2$. Then $C' = C \cup \{a\}$, where C is a cocircuit of M. But then $C \subseteq I_1$, a contradiction.

(iii) $C' \in \mathcal{C}_3$. Then $C' = C_1 \cup \{e, \gamma\}$ or $C' = (C_2 \setminus \{e\}) \cup \{\gamma\}$ or $C' = (C_3 \setminus \{e\}) \cup \{a, \gamma\}$, where C_1, C_2, C_3 are circuits in M and C_1, C_2 and $C_3 \setminus \{e\}$ each contain an odd number of elements of X. Consequently, $C_1 \cup \{e, \gamma\} \subseteq I_1 \cup \{\gamma\}$ implies that $C_1 \subseteq I_1$. $(C_2 \setminus \{e\}) \cup \{\gamma\} \subseteq I_1 \cup \{\gamma\}$ implies $C_2 \setminus \{e\} \subseteq I_1$; and $(C_3 \setminus \{e\}) \cup \{a, \gamma\} \subseteq I_1 \cup \{\gamma\}$ implies $C_3 \setminus \{e\} \subseteq I_1$, contradictions to hypotheses in (2).

(3) Let I_1 be an independent set in M containing e. We show that $(I_1 \setminus \{e\}) \cup \{a, \gamma\}$ is independent in M_X^e. On the contrary, suppose $(I_1 \setminus \{e\}) \cup \{a, \gamma\}$ is dependent in M_X^e. By similar argument as in (2), we get contradictions.

(4) Let $C_1, C_2, \ldots$ and C_m be circuits in M, where each C_i contains an odd number of elements of X and $C_i \cap C_j \neq \phi$ for $i \neq j, i, j = 1, 2, \ldots, m$. Clearly each C_i is independent in M_X^e and $I_1 = \cup_{i=1}^m C_i$ is independent in M_X^e. Further, $\gamma \notin \cup_{i=1}^m C_i$ and by hypothesis, $I = (\cup_{i=1}^m C_i) \cup \{\gamma\}$ contains no circuit of $\mathcal{C}_0$ and $I \cup \{e\}$ contains no circuit of M other than C_i for $i = 1, 2, \ldots, m$. Therefore, I is a independent set in M_X^e.

Conversely, let $I \subseteq S \cup \{a, \gamma\}$ be an independent set in M_X^e. We have the following cases:

(I) Let $I \cap \{a, \gamma\} = \phi$. Then $I \subseteq S$ and we have two subcases:

(i) I be independent in M. Then $I_1 = I$.

(ii) I be dependent set in M. Let $C_1, C_2, \ldots, C_m$ be the circuits of M, contained in I. Then each C_i must contain an odd number of elements of X and $C_i \cap C_j \neq \phi$ for $i \neq j$. If $I - (C_1 \cup C_2 \cup \ldots \cup C_m) = \phi$, then $I = C_1 \cup C_2 \cup \ldots \cup C_m$ is independent in M_X^e such that I does not contain a member of $\mathcal{C}_0$ and $I \cup \{e\}$ does not contain any circuit of M

other than C_i for $i = 1, 2, \ldots, m$. Thus, I is of type (4), where $J = \phi$. If $I - (C_1 \cup C_2 \cup \ldots \cup C_m) \neq \phi$ then $I = (C_1 \cup C_2 \cup \ldots \cup C_m) \cup Y$, where $Y \subseteq S \cup \{a, \gamma\}, Y \cap \{a, \gamma\} = \phi$, so $Y \subseteq S$. But Y does not contain a circuit of M with even number of elements of X, and also Y does not contain any C_i for $i = 1, 2, \ldots, m$, thus $Y = \phi$; a contradiction.

(II) Suppose $I \cap \{a, \gamma\} \neq \phi$. We have the following cases:

(i) $a \in I$ and $\gamma \notin I$. Then $I - \{a\}$ is independent in M, if $I - \{a\} = I_1$, then $I = I_1 \cup \{a\}$.

(ii) Let $a \notin I$ and $\gamma \in I$. We show that $I - \{\gamma\}$ is independent in M. On the contrary, suppose $I - \{\gamma\}$ contains a circuit say C of M. Then $C \subseteq (I - \{\gamma\}) \cup \{e\}$ and $(C \setminus \{e\}) \cup \{\gamma\} \subseteq I$. But $(C \setminus \{e\}) \cup \{\gamma\}$ is a circuit of M_X^e contained in I, a contradiction. So, if $I - \{\gamma\} = I_1$ then $I = I_1 \cup \{\gamma\}$. Now, suppose $I - \{\gamma\}$ contains more than one circuit of M, say $C_1, C_2, \ldots, C_m$, where each C_i contains an odd number of elements of X and $C_i \cap C_j \neq \phi$ for $i \neq j, i, j = 1, 2, \ldots, m$. Thus $C_1 \cup C_2 \cup \ldots \cup C_m \subseteq I - \{\gamma\}$ and $\cup_{i=1}^m C_i \subseteq (I - \{\gamma\}) \cup \{e\}$. Consequently, $((\cup_{i=1}^m C_i) \setminus \{e\}) \cup \{\gamma\} \subseteq I$. For $1 \leq j \leq m, (C_j \setminus \{e\}) \cup \{\gamma\}$ is a circuit of M_X^e, contained in $((\cup_{i=1}^m C_i) \setminus \{e\}) \cup \{\gamma\}$, that is, in I; a contradiction. So $I = (\cup_{i=1}^m C_i) \cup \{\gamma\}$ is a independent set in M_X^e. Moreover, $I \cup \{e\}$ contains no circuit of M other than $C_i, i = 1, 2, \ldots, m$. Thus I is of type (4), where $j = \{\gamma\}$.

(iii) Let $a, \gamma \in I$. Then we show that $I - \{a, \gamma\}$ is independent in M. On the contrary, suppose $I - \{a, \gamma\}$ contains a circuit say C of M. Thus $C \subseteq (I - \{a, \gamma\}) \cup \{e\}$ and $(C \setminus \{e\}) \cup \{a, \gamma\} \subseteq I$. But $(C \setminus \{e\}) \cup \{a, \gamma\}$ is a circuit of M_X^e, where C contains an odd number of elements of X. If C contains an even number of elements of X, then $C \subseteq I$; a contradiction. We conclude that $I_1 = (I \mid \{a, \gamma\}) \cup \{e\}$ is independent in M and $I = (I_1 \setminus \{e\}) \cup \{a, \gamma\}$. I is of type (3). This completes the proof of the theorem. $\square$

3. Bases in M_X^e

In the next theorem, we characterize the bases of the matroid M_X^e in terms of the bases of M.

Theorem 3.1. Let $\mathcal{B}$ be a collection of bases of M. A subset B' of $S \cup \{a, \gamma\}$ is a base of M_X^e if and only if one of the following conditions hold:

(1) $B' = B \cup \{a\}$
(2) $B' = B \cup \{\gamma\}$, where $B \in \mathcal{B}$ and no circuit C of M containing e contains an odd number of elements of X such that $C \setminus \{e\} \subseteq B$.

(3) $B' = (B\backslash\{e\})\cup\{a,\gamma\}$, where $B \in \mathcal{B}$, no circuit C of M containing e contains an odd number of elements of X such that $C\backslash\{e\} \subseteq B$.

(4) $B' = B \cup \{z\}$, where $B \in \mathcal{B}$, $z \in S - B$ and the fundamental circuit of M contained in $B \cup \{z\}$ contains an odd number of elements of X.

Proof. (1) Let B be a base of M. Then B is independent in M and, by Lemma 2.2, B is independent in M_X^e. Further, by Theorem 2.4, $B \cup \{a\}$ is independent in M_X^e. Then

$$r'(B \cup \{a\}) = |B \cup \{a\}| = |B| + 1 = r(M) + 1 = r'(M_X^e).$$

Thus, $B' = B \cup \{a\}$ is a base of M_X^e.

(2) Let $B \cup \{\gamma\}$ satisfies the conditions in (2). We show that $B \cup \{\gamma\}$ is independent in M_X^e. On the contrary, suppose $B \cup \{\gamma\}$ is dependent in M_X^e and C' is a circuit of M_X^e contained in $B \cup \{\gamma\}$. If $C' \in \mathcal{C}_0$ or $\mathcal{C}_1$, then $C' \subseteq B$ and this leads to a contradiction. If $C' = C \cup \{a\}$, where C is a circuit in M and $a \notin C$, then $C \cup \{a\} \subseteq B \cup \{\gamma\}$ implies $C \cup \{a\} \subseteq B$ and again $C \subseteq B$; a contradiction. If $C' \in \mathcal{C}_2$, then $C' = C_1 \cup \{e,\gamma\}, C' = (C_2 \setminus \{e\}) \cup \{\gamma\}$ or $C' = (C_3 \setminus \{e\}) \cup \{a,\gamma\}$. If $C_1 \cup \{e,\gamma\} \subseteq B \cup \{\gamma\}$, then $C_1 \cup \{e\} \subseteq B$, that is, $C_1 \subseteq B$. If $(C_2 \setminus \{e\}) \cup \{\gamma\} \subseteq B \cup \{\gamma\}$, then $C_2 \setminus \{e\} \subseteq B$, that is, $C_2 \subseteq B$. If $(C_3 \setminus \{e\}) \cup \{a,\gamma\} \subseteq B \cup \{\gamma\}$, then $(C_3 \setminus \{e\}) \cup \{a\} \subseteq B$ or $C_3 \subseteq \{e\}$, contradictions to hypothesis in (2). Further,

$$r'(B \cup \{\gamma\}) = |\ B \cup \{\gamma\}\ | = |B| + 1 = r(M) + 1 = r'(M_X^e).$$

This shows that $B \cup \{\gamma\}$ is a base of M_X^e.

(3) Let $(B\backslash\{e\})\cup\{a,\gamma\}$ satisfies the conditions in (3). By the argument similar to one as given above, we show that it is a independent subset of M_X^e. Moreover,

$$\begin{aligned} r'(B \setminus \{e\}) \cup \{a,\gamma\} &= |\ (B \setminus \{e\}) \cup \{a,\gamma\}\ | = |\ B\ | + 1 \\ &= r(M) + 1 = r'(M_X^e). \end{aligned}$$

Thus, $(B \setminus \{e\}) \cup \{a,\gamma\}$ is a base of M_X^e.

(4) Let $B \cup \{z\}$, where $z \in S - B$, satisfies the condition given in (4). By Theorem 4.2.7, $B \cup \{z\}$ is independent in M_X^e and so

$$r'(B \cup \{z\}) = r'(M_X^e).$$

Therefore $B \cup \{z\}$ is a base for M_X^e.

Conversely, let B' be a base for M_X^e. We consider the following cases:

(I) Let $a \in B'$ and $\gamma \notin B'$. Then $B' - \{a\}$ is independent in M_X^e. We show that $B' - \{a\}$ is also independent in M. On the contrary, suppose $B' - \{a\}$ contains a circuit C of M. We have two subcases.

(i) C contains an even number of elements of X. Then $C \subseteq B'$; a contradiction.

(ii) C contains an odd number of elements of X. Then $C \subseteq B' - \{a\}$ and $C \cup \{a\} \subseteq B'$; a contradiction, because $C \cup \{a\}$ is a circuit of M_X^e. Next,

$$r(B' - \{a\}) = \mid B' - \{a\} \mid = \mid B' \mid - 1 = r'(M_X^e) - 1 = r(M).$$

Therefore, $B' - \{a\}$ is a base for M.

(II) Let $a \notin B'$ and $\gamma \in B'$. We show that $B' - \{\gamma\}$ is a base for M. Firstly, we prove that $B' - \{\gamma\}$ is independent in M. On the contrary, suppose $B' - \{\gamma\}$ is dependent in M. Let C be a circuit of M, contained in $B' - \{\gamma\}$. We have two subcases:

(i) Let C contains an even number of elements of X. Then C is a circuit of M_X^e and $C \subseteq B' - \{\gamma\}$. But $C \subseteq B'$, is a contradiction.

(ii) Let C contains an odd number of elements of X. Then $C \subseteq B' - \{\gamma\}$ and so $C \setminus \{e\} \subseteq B' - \{\gamma\}$. This implies that $(C \setminus \{e\}) \cup \{\gamma\} \subseteq B'$ which is a contradiction, since $(C \setminus \{e\}) \cup \{\gamma\}$ is a circuit of M_X^e. Secondly, $B' - \{\gamma\}$ is maximal independent in M, follows from the fact that $r(B' - \{\gamma\}) = r(M)$.

(III) Let $a, \gamma \in B'$. We show that $(B' \setminus \{a, \gamma\}) \cup \{e\}$ is a base for M. Clearly $e \notin B'$, for $e \in B'$ implies that $\{e, a, \gamma\} \subseteq B'$ and this is a contradiction.

Firstly, we show that $(B' - \{a, \gamma\}) \cup \{e\}$ is independent in M. On the contrary, suppose it is dependent in M and let C be a circuit of M contained in $(B' \setminus \{a, \gamma\}) \cup \{e\}$. We have two subcases:

(i) C contains an even number of elements of X. Then C is a circuit of M_X^e and $C \subseteq (B' \setminus \{a, \gamma\}) \cup \{e\}$. Thus $(C \setminus \{e\}) \cup \{a, \gamma\} \subseteq B'$; a contradiction because $(C \setminus \{e\}) \cup \{a, \gamma\}$ is a circuit of M_X^e.

(ii) C contains an odd number of elements of X and $C \subseteq (B' \setminus \{a, \gamma\}) \cup \{e\}$. Hence $(C \setminus \{e\}) \cup \{\gamma\} \subseteq B'$. But $(C \setminus \{e\}) \cup \{\gamma\}$ is a circuit of M_X^e, so we get a contradiction. Further, $(B' \setminus \{a, \gamma\}) \cup \{e\}$ is maximal independent in M, since

$$\begin{aligned} r((B' \setminus \{a, \gamma\}) \cup \{e\}) &= \mid (B' \setminus \{a, \gamma\}) \cup \{e\} \mid \\ &= \mid B' \mid - 1 = r'(M_X^e) - 1 = r(M). \end{aligned}$$

(IV) Let $a, \gamma \notin B'$. Then $B' \subseteq S$ and B' is not independent in M since $r'(B') = r'(M_X^e) = r(M)+1$. Thus B' is dependent in M. So there is a circuit C of M contained in B'. If C contains an even number of elements of X, then C is a circuit of M_X^e and we get a contradiction. So C must contain an odd number of elements of X and suppose x_i be an element of X contained in C. Then $B = B' - \{x_i\}$ is a base of M. □

4. Rank Function of M_X^e

Lemma 4.1. Let M be a binary matroid on S and M_X^e be a es-splitting of M with ground set $S \cup \{a, \gamma\}$. Let r and r' be the rank functions of M and M_X^e, respectively. Then for $Z \subseteq S$ the following properties hold:

(1) $r'(Z \cup \{a\}) = r(Z) + 1$

(2) $r'(Z \cup \{a, \gamma\}) = \begin{cases} r(Z)+1 & \text{if} \quad e \in Z \\ r(Z)+2 & \text{if} \quad e \notin Z \end{cases}$

Proof. (1) Let T be a base for A in M. Then $r(Z) = \mid T \mid$. We show that $T \cup \{a\}$ is a base for $Z \cup \{a\}$ in M_X^e. On the contrary, suppose $T \cup \{a\}$ is dependent in M_X^e and C is circuit of M_X^e contained in $T \cup \{a\}$. We consider the following cases:

(i) $C \in \mathcal{C}_0$ or $\mathcal{C}_1$. Then $C \subseteq T \cup \{a\}$ and hence $C \subseteq T$; a contradiction.

(ii) $C \in \mathcal{C}_2$. Then $C = C_1 \cup \{a\}$, where C_1 is a circuit in M. Consequently $C_1 \subseteq T$ gives a contradiction.

(iii) Let $C \in \mathcal{C}_3$. Then there is a circuit of M, say C_1 with $C_1 \subseteq T$, a contradiction.

Now, we prove that $T \cup \{a\}$ is a maximal independent set in M_X^e. On the contrary, suppose $T \cup \{a\} \cup \{z\}$ is maximal independent in M_X^e, where $z \in Z - (T \cup \{a\})$. Then $T \cup \{z\} \cup \{a\} \subseteq Z \cup \{a\}$. Thus $T \cup \{z\} \subseteq Z$, a contradiction. Now,

$$r'(Z \cup \{a\}) = \mid T \cup \{a\} \mid = \mid T \mid + 1 = r(Z) + 1.$$

By the same argument as above, we can show that

$$r'(Z \cup \{\gamma\}) = r(Z) + 1.$$

(2) Let T be a base for Z in M. Then $r(Z) = \mid T \mid$. We have the following two cases:

(I) Let $e \in T$. Then we claim that $(T \setminus \{e\}) \cup \{a, \gamma\}$ is a base for $Z \cup \{a, \gamma\}$ in M_X^e. On the contrary, suppose that it is dependent set of M_X^e and contains a circuit C of M_X^e. We have the following subcases:

(i) Let $C \in \mathcal{C}_0$ or $\mathcal{C}_1$. Then $C \subseteq (T \setminus \{e\}) \cup \{a, \gamma\}$ implies that $C \subseteq T \setminus \{e\} \subseteq T$; a contradiction.

(ii) Let $C \in \mathcal{C}_2$ and $C = C_1 \cup \{a\}$, where C_1 is a circuit of M. Then $C_1 \cup \{a\} \subseteq (T \setminus \{e\}) \cup \{a, \gamma\}$ and $C_1 \subseteq T \setminus \{e\} \subseteq T$; a contradiction.

(iii) Let $C \in \mathcal{C}_3$ and $C = (C_1 \setminus e) \cup \{a, \gamma\}, e \in C_1$, where C_1 is a circuit in M. Clearly $C_1 \subseteq T$ again; a contradiction. Now, we show that $(T \setminus \{e\}) \cup \{a, \gamma\}$ is maximal independent in M_X^e. On the contrary, suppose $(T \setminus \{e\}) \cup \{a, \gamma\} \cup \{z\}$ for $z \in Z$ is maximal. Then $(T \setminus \{e\}) \cup \{z\} \subseteq Z \setminus \{e\} \subseteq T \cup \{z\} \subseteq Z$. Consequently, $T \cup \{z\} \subseteq Z$; a contradiction. If $e \in T$, then $(T \setminus \{e\}) \cup \{a, \gamma\}$ is a base for $Z \cup \{a, \gamma\}$ in M_X^e and hence

$$r'(Z \cup \{a, \gamma\}) = |\ T \setminus \{e\}\ | + |\ \{e, \gamma\}\ | = |\ T\ | + 1 = r(Z) + 1.$$

(II) Let $e \notin T$. Then we show that $T \cup \{a, \gamma\}$ is a base for $Z \cup \{a, \gamma\}$ in M_X^e. We prove that $T \cup \{a, \gamma\}$ is maximal independent in M_X^e. On the contrary, suppose it is dependent in M_X^e. By the same argument as in case (I), we obtain a contradiction. Thus $T \cup \{a, \gamma\}$ is base of $Z \cup \{a, \gamma\}$. Finally,

$$\begin{aligned} r'(Z \cup \{a, \gamma\}) &= |\ T \cup \{a, \gamma\}\ | = |\ T\ | + |\ \{a, \gamma\}\ | \\ &= |\ T\ | + 2 = r(Z) + 2. \end{aligned}$$

This completes the proof. $\square$

Lemma 4.2. If $Z \subseteq S$, then

$$r'(Z) = \begin{cases} r(Z) + 1 & \text{if } Z \text{ contains a circuit of } M \text{, containing} \\ & \text{an odd number of elements of } X; \\ r(Z) & \text{otherwise.} \end{cases}$$

Proof. Let $Z \subseteq S$. We have the following cases:

(1) Suppose Z does not contain any circuit of M. Then Z is independent in M, and by Lemma 2.2, it is independent in M_X^e and hence $r'(Z) = |\ Z\ | = r(Z)$.

(2) Suppose Z does not contain a circuit of M containing an odd number of elements of X. Suppose Z contains a circuit say C, containing an even number of elements of X. Then C is a circuit in M_X^e and Z is dependent in M, as well as in M_X^e. Consequently, a base of Z in M is also a base of it in M_X^e. Thus, $r'(Z) = r(Z)$.

(3) Let Z contains a circuit of M, say C containing an odd number of elements of X. For $\alpha \in C$, the set $C - \{\alpha\}$ is independent in $M|Z$. Now, we extend $C - \{\alpha\}$ to a base T of Z. Let $T = (C - \{\alpha\}) \cup \{\beta_1, \beta_2, \ldots, \beta_k\}$, where $\beta_i \in Z, i = 1, 2, \ldots, k$, is a base of Z in M. Then

$$r(Z) = |\ T\ | = |\ C\ | - 1 + k. \qquad (*)$$

On the other hand, C is independent in M_X^e and $T' = C \cup \{\beta_1, \beta_2, \ldots \beta_k\}$ is independent in M_X^e. If T' is not a base of Z in M_X^e, then $T'' = C \cup \{\beta_1, \beta_2, \ldots, \beta_k, \delta\}$ for some $\delta \in Z$ is independent subset of Z in M_X^e. Now $C \cup \{\beta_1, \beta_2, \ldots, \beta_k\}$ is a dependent subset of Z in M. Let C' be a circuit of M contained in it. We have the following cases:

(i) C' contains an even number of elements of X. Then C' is a circuit of M_X^e with $C' \subseteq C \cup \{\beta_1, \beta_2, \ldots, \beta_k\}$ a, contradiction.

(ii) C' contains an odd number of element of X. Consider

$$C \Delta C' = C_1' \cup C_2' \cup \ldots \cup C_m', \qquad (**)$$

where C_i' are circuits of M and $C_i' \cap C_j' = \phi, i \neq j$ and $i, j = 1, 2, \ldots, m$. If some C_i' contains an even number of elements of X, then it leads to a contradiction. If not, consider the circuits C_j', C_k' from (**). Then either $C_j' \cup C_k'$ is a circuit or a subset of it, is a circuit of M_X^e contained in T''. We conclude that T' must be a maximal independent subset of Z in M_X^e. Now,

$$r'(Z) = |\ T'\ | = |\ C \cup \{\beta_1, \beta_2, \ldots, \beta_k\}| = |C| + k. \qquad (***)$$

From $(*)$ and $(***)$, we deduce that $r'(Z) = r(Z) + 1$. This completes the proof. ☐

Corollary 4.3. Let $M = (S, r)$ and $M_X^e = (S \cup \{a, \gamma\}, r')$ be matroids with usual meaning. Let $J = \{a, \gamma\}$. If $Y \subseteq S \cup \{a, \gamma\}$, then

$$r'(Y) = \begin{cases} r(Y) + 1, & \text{if } |\ Y \cap J\ | = 0 \text{ and } Y \text{ contains a circuit} \\ & \text{of } M \text{ containing an odd number} \\ & \text{of elements of } X; \\ r(Y), & \text{if } |\ Y \cap J\ | = 0 \text{ and } Y \text{ does not contain} \\ & \text{any circuit of } M \text{ containing odd} \\ & \text{number of elements of } X; \\ r(Y \cap S) + 1, & \text{if } |\ Y \cap J\ | = 1 \text{ or } |\ Y \cap J\ | = 2, e \in Y; \\ & \text{and} \\ r(Y \cap S) + 2, & \text{if } |\ Y \cap J\ | = 2 \text{ and } e \notin Y \end{cases}$$

The proof follows from Lemmas 4.1 and 4.2.

Corollary 4.4.4. Let $M = (S, r)$ be a binary matroid. Let r_1, r_2, r_3 be the rank functions of the matroids M_X, M_X' and M_X^e, respectively. Then $r_3(Y) = r_2(Y) = r_1(Y)$ for $Y \subseteq S$.

Proof. It is known [3] that for a matroid M on S with $T \subseteq S$ and $X \subseteq S - T$,

$$r_{M \setminus T}(X) = r_M(X), \qquad (*)$$

where r_M is a rank function of M. We have

$$M_X^e \setminus \{\gamma\} = M'_X \text{ and } M'_X \setminus \{a\} = M_X.$$

Thus from (*), it follows that, $r_3(Y) = r_2(Y)$ and $r_2(Y) = r_1(Y)$ for $Y \subseteq S$. $\square$

References

[1] Azanchiler H., Extension of line-splitting operation from graphs to binary matroids, *Lobachevskii Journal of Mathematics*, (**24**) (2006), 3-12.

[2] Fleischner H., *Eulerian Graphs and Related Topics*, Part 1, Vol. 1, North Holland, Amsterdam (1990).

[3] Oxley J. G., *Matroid Theory*, Oxford University Press, New York, (1992).

[4] Raghunathan T. T., Shikare M. M. and Waphare B. N., *Splitting in a binary matroid*, Discrete Mathematics, **184** (1997), 261-271.

[5] Shikare M. M., Azadi G., *Determination of the bases of a splitting matroid*, European Journal of Combinatorics, **24** (2003), 45-52.

[6] Shikare, M. M., Azadi G. and Waphare B. N., *Generalized splitting operation and its applications to binary matroids* (preprint).

[7] Shikare, M. M. and Azadi, G., *Element splitting operation for binary matroids* (preprint).

[8] Slater P. J., *Soldering and point splitting*, J. Combinatorial Theory, **24(3)** (1978), 338-343.

Department of Mathematics, University of Urmia, Urmia - 57135 (Iran)

E-mail address: azanchiler@yahoo.com

Received March 28, 2007