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Xun Ge

SPACES WITH A LOCALLY COUNTABLE SN-NETWORK

(submitted by M. A. Malakhaltsev)

Abstract. In this paper, we discuss a class of spaces with a locally countable sn-network. We give some characterizations of this class and investigate variance and inverse invariance of this class under certain mappings.

1. Introduction

sn-networks is a class of important networks between weak-bases and cs-networks. In past years, spaces with a locally countable weak-base and spaces with a locally countable cs-network had been investigated and many interesting results had been obtained ([17, 21, 22, 23, 24, 30, 31, 32, 33]). In this paper, we will discuss spaces with a locally countable sn-network. We give some characterizations of this class and investigate variance and inverse invariance of this class under certain mappings.

Throughout this paper, all spaces are assumed to be regular T_1 and all mappings are continuous and onto. $\mathbb{N}$, ω and ω_1 denote the set of all natural numbers, the first infinite ordinal and the first uncountable ordinal respectively. For a set D, $|D|$ denotes the cardinal of D. $\{x_n\}$ denotes a sequence, where the n-th term is x_n. Let X be a space and $P \subset X$. A sequence $\{x_n\}$ converging to x in X is eventually in P if $\{x_n : n > k\} \bigcup \{x\} \subset P$ for some $k \in \mathbb{N}$; is frequently in P if $\{x_{n_k}\}$ is eventually in P for some subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Let $\mathcal{P}$ be a

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family of subsets of X and f be a mapping defined on X. Then $f(\mathcal{P}) = \{f(P) : P \in \mathcal{P}\}$, $\bigcup\mathcal{P}$ and $\bigcap\mathcal{P}$ denote the union $\bigcup\{P : P \in \mathcal{P}\}$ and the intersection $\bigcap\{P : P \in \mathcal{P}\}$ respectively. Let $\mathcal{P}_x$ be a family of subsets of X. $\mathcal{P}_x$ is a network at x in X, if $x \in \bigcap\mathcal{P}_x$ and for each neighborhood U of x in X there is $P \in \mathcal{P}_x$ such that $P \subset U$. We refer the reader to [7] for notations and terminology not explicitly given here.

2. Spaces with a Locally Countable *sn*-Network

Definition 2.1. [8, 9]. Let X be a space and let $x \in X$.

(1) $P \subset X$ is called a sequential neighborhood of x if each sequence $\{x_n\}$ converging to x is eventually in P.

(2) A subset U of X is called sequentially open if U is a sequential neighborhood of each of its points; a subset F of X is called sequentially closed if $X - F$ is sequentially open.

(3) X is called a sequential space if each sequentially open subset of X is open in X, equivalently, if each sequentially closed subset of X is closed in X.

(4) X is called a k-space if for each $A \subset X$, A is closed in X if and only if $A\bigcap K$ is closed in K for each compact subset K of X.

Remark 2.2. (1) P is a sequential neighborhood of x if and only if each sequence $\{x_n\}$ converging to x is frequently in P.

(2) The intersection of finite sequential neighborhoods of x is a sequential neighborhood of x.

(3) the sequential spaces are the k-spaces.

Definition 2.3. Let $\mathcal{P}$ be a cover of a space X.

(1) $\mathcal{P}$ is called a k-network of X [27], if whenever $K \subset U$ with K compact in X and U open in X, there is a finite $\mathcal{F} \subset \mathcal{P}$ such that $K \subset \bigcup\mathcal{F} \subset U$.

(2) $\mathcal{P}$ is called a cs^*-network of X [10], if each convergent sequence S converging to a point $x \in U$ with U open in X, then S is frequently in $P \subset U$ for some $P \in \mathcal{P}$.

(3) $\mathcal{P}$ is called a cs-network of X [15], if each convergent sequence S converging to a point $x \in U$ with U open in X, then S is eventually in $P \subset U$ for some $P \in \mathcal{P}$.

Definition 2.4. Let $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ be a cover of a space X, such that for each $x \in X$, the following conditions (a) and (b) are satisfied.

(a) $\mathcal{P}_x$ is a network at x in X.

(b) If $U, V \in \mathcal{P}_x$, then $W \subset U\bigcap V$ for some $W \in \mathcal{P}_x$.

(1) $\mathcal{P}$ is called a weak-base of X [1], if for $G \subset X$, G is open in X if and only if for each $x \in G$ there is $P \in \mathcal{P}_x$ such that $P \subset G$, where $\mathcal{P}_x$ is called a weak neighborhood base at x in X.

(2) $\mathcal{P}$ is called an sn-network of X [11], if each element of $\mathcal{P}_x$ is a sequential neighborhood of x for each $x \in X$, where $\mathcal{P}_x$ is called an sn-network at x in X.

Remark 2.5. [23]. (1) weak-bases $\Longrightarrow$ sn-networks $\Longrightarrow$ cs-networks $\Longrightarrow$ cs^*-networks.

(2) In a sequential space, weak-bases $\Longleftrightarrow$ sn-networks.

(3) sn-networks are called universal cs-networks in [20].

Example 2.6. In a k-space, sn-networks$\not\Longrightarrow$ weak-bases.

Proof. Let X be the Stone-Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$. Then X is compact, and so it is a k-space. Since each convergent sequence in $\beta\mathbb{N}$ is trivial, $\mathcal{P} = \{\{x\} : x \in X\}$ is an sn-network of X. It is clear that $\mathcal{P}$ is not a weak-base of X. $\square$

Definition 2.7. (1) A space X is called g-metrizable [8] (respectively sn-metrizable [12], $\aleph$ [27]) if X has a σ-locally finite weak-base (respectively sn-network, k-network).

(2) A space X is called g-second countable [29] (respectively sn-second countable [13], $\aleph_0$ [26]) if X has a countable weak-base (respectively sn-network, k-network).

(3) A space X is called g-first countable [1] (respectively sn-first countable [12]), if there is a countable weak neighborhood base (respectively sn-network) at x in X for each $x \in X$.

Remark 2.8. (1) g-first countable $\Longleftrightarrow$ sequential and sn-first countable.

(2) If X has a point countable weak-base (respectively sn-network), then X is g-first countable (respectively sn-first countable). So each g-metrizable (respectively sn-metrizable) space is g-first countable (respectively sn-first countable).

(3) g-metrizable (respectively g-second countable) $\Longleftrightarrow$ k- and sn-metrizable (respectively k- and sn-second countable).

(4) X is an $\aleph_0$-space $\Longleftrightarrow$ X has a countable cs-network $\Longleftrightarrow$ X has a countable cs^*-network.

(5) sn-first countable is called universally csf-countable in [20].

The following lemma is obtained by combining [19, Theorem 2.8.6] and [22, Corollary 5.1.13].

Lemma 2.9. *The following are equivalent for a space X.*

(1) *X has a locally countable cs-network.*

(2) *X has a locally countable cs*-network.*

(3) *X has a locally countable k-network.*

Now we give some set-theoretical axioms.

Set-Theoretical Axioms 2.10.

(1) CH (Continuum Hypothesis): $2^\omega = \omega_1$.

(2) MA (Martin's Axiom): Let k be a cardinal.

(i) A space X is called k^+-Baire if for each family $\{G_\alpha : \alpha \in A\}$ consisting of open dense subsets of X, $\bigcap\{G_\alpha : \alpha \in A\} \neq \emptyset$, where $|A| < k^+$.

(ii) A space X is called *ccc* if each disjoint family consisting of open subsets of X is countable.

(iii) $MA(k)$: Each compact, *ccc* space is k^+-Baire.

(iv) MA: For each k, $MA(k)$ holds, where $\omega < k < 2^\omega$.

(3) TOP (Thinning-out Principle): Let $(P, \leq)$ be a partial ordered set.

(i) $Q \subset P$ is called centered if whenever finite $q_1, q_2, \ldots, q_n \in Q$, there is $p \in P$ such that $p \leq q_i$ for all $i = 1, 2, \ldots, n$.

(ii) A family $\{B_\alpha : \alpha < \omega_1\}$ is called cofinally centered on a set A if for each uncountable $C \subset A$ there is an $\alpha < \omega_1$, such that $\{B_\beta \bigcap C : \beta \geq \alpha\}$ is centered.

(iii) TOP: If Z, B are uncountable subsets of ω_1 and $\{S_\alpha : \alpha \in B\}$ is a family cofinally centered on Z with each $S_\alpha \subset \alpha$, then there is an uncountable $Y \subset Z$ such that $(Y \bigcap \alpha) - S_\alpha$ is finite for all $\alpha \in B$.

Recall a space is an S-space if it is hereditarily separable and not hereditarily Lindelöf.

Lemma 2.11. [28, Theorem 7.2.3]. *Under $MA + \neg CH + TOP$, there are no S-spaces.*

Theorem 2.12. *Under $MA+\neg CH+TOP$, the following are equivalent for a space X.*

(1) *X has a locally countable sn-network.*

(2) *X is an sn-first countable space with a locally countable cs-network (respectively k-network, cs*-network).*

(3) *X is a locally sn-second countable space with a σ-locally countable sn-network*

(4) *X is a locally $\aleph_0$-space with a σ-locally countable sn-network.*

(5) *X is a locally hereditarily separable space with a σ-locally countable sn-network.*

(6) *X is a locally (hereditarily) Lindelöf space with a σ-locally countable sn-network.*

Proof. (1) $\Longrightarrow$ (2). Note that a space with a locally countable sn-network is sn-first countable. So (1) $\Longrightarrow$ (2) by Remark 2.5(1) and Lemma 2.9.

(2) $\Longrightarrow$ (1). By Lemma 2.9, let $\mathcal{P}$ be a locally countable cs-network of X. We can assume that $\mathcal{P}$ is closed under finite intersections. For each $x \in X$, let $\{B_n(x) : n \in \mathbb{N}\}$ be a countable sn-network at x in X, and let $\mathcal{P}_x = \{P \in \mathcal{P} : B_n(x) \subset P \; for \; some \; n \in \mathbb{N}\}$. Then each element of $\mathcal{P}_x$ is a sequential neighborhood of X. Put $\mathcal{P}' = \bigcup\{\mathcal{P}_x : x \in X\}$, then $\mathcal{P}' \subset \mathcal{P}$ is locally countable. It suffices to prove that $\mathcal{P}_x$ is a network at x in X for each $x \in X$. If not, there is an open neighborhood U of x such that $P \not\subset U$ for each $P \in \mathcal{P}_x$. Let $\{P \in \mathcal{P} : x \in P \subset U\} = \{P_m(x) : m \in \mathbb{N}\}$. Then $B_n(x) \not\subset P_m(x)$ for each $n, m \in \mathbb{N}$. Choose $x_{n,m} \in B_n(x) - P_m(x)$. For $n \geq m$, let $x_{n,m} = y_k$, where $k = m + n(n-1)/2$. Then the sequence $\{y_k : k \in \mathbb{N}\}$ converges to x. Thus , there is $m, i \in \mathbb{N}$ such that $\{y_k : k \geq i\} \bigcup \{x\} \subset P_m(x) \subset U$. Take $j \geq i$ with $y_j = x_{n,m}$ for some $n \geq m$. Then $x_{n,m} \in P_m(x)$. This is a contradiction.

(1) $\Longrightarrow$ (3). Let $\mathcal{P}$ be a locally countable sn-network of X. For each $x \in X$, there is an open neighborhood U of x such that $\mathcal{P}_U = \{P \bigcap U : P \in \mathcal{P}\}$ is countable. It is easy to prove that $\mathcal{P}_U$ is a countable sn-network of subspace U. So U is an sn-second countable space. Thus X is a locally sn-second countable space.

(3) $\Longrightarrow$ (4) $\Longrightarrow$ (5). It is known that sn-second countable $\Longrightarrow \aleph_0 \Longrightarrow$ hereditarily separable. So (3) $\Longrightarrow$ (4) $\Longrightarrow$ (5).

(5) $\Longrightarrow$ (6). It suffices to prove that X is locally hereditarily Lindelöf. Let $x \in X$ and U be a hereditarily separable neighborhood of x. By Lemma 2.11, U is hereditarily Lindelöf. So X is locally hereditarily Lindelöf.

(6) $\Longrightarrow$ (1). Let $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a σ-locally countable sn-network of a Locally Lindelöf space X, where each $\mathcal{P}_n$ is locally countable in X. Let $x \in X$ and let U be a Lindelöf neighborhood of x. Let $n \in \mathbb{N}$. For each $y \in U$, there is an open neighborhood U_y of y such that U_y intersects at most countable many elements of $\mathcal{P}_n$. The open cover $\{U_y : y \in U\}$ of U has countable subcover $\mathcal{V}$. Put $V = \bigcup \mathcal{V}$, then $U \subset V$ and V intersects at most countable many elements of $\mathcal{P}_n$. So U intersects at most countable many elements of $\mathcal{P}_n$. Moreover, U intersects at most countable many elements of $\mathcal{P}$. Thus $\mathcal{P}$ is a locally countable sn-network of X. $\square$

Remark 2.13. In Theorem 2.12, (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (4) $\Longleftrightarrow$ (6) without requiring $MA + \neg CH + TOP$ involved. The reasons are as follows.

(a) $MA + \neg CH + TOP$ are used only in the proof of (5) $\Longrightarrow$ (6).

(b) Because each $\aleph_0$-space is hereditarily $Lindel\ddot{o}f$, (4) $\Longrightarrow$ (6) without requiring $MA + \neg CH + TOP$ involved.

It is natural to ask whether "$MA + \neg CH + TOP$" in Theorem 2.12 can be omitted. The following Theorem 2.16 shows that the answer is "yes" if X is a k-space.

Lemma 2.14. [14, 22]. *The following hold for a space X.*

(1) *If X is a compact space with a point countable k-network, then X is metrizable.*

(2) *If X is a k-space with a point countable k-network, then X is sequential.*

(3) *If X has a point countable cs^*-network and each compact subset of X is metrizable, then X has a point countable k-network.*

Lemma 2.15. *If X is a k-space with a σ-locally countable cs^*-network, then X is sequential.*

Proof. Let $\mathcal{P}$ be a σ-locally countable cs^*-network of X. Whenever K is a compact subset of X, put $\mathcal{P}_K = \{P \bigcap K : P \in \mathcal{P}\}$, then $\mathcal{P}_K$ is a σ-locally countable cs^*-network of K. It is easy to see that $\mathcal{P}_K$ is a countable cs^*-network of K, and so K has a countable k-network by Remark 2.8(4). By Lemma 2.14(1), K is metrizable. So X has a point-countable k-network by Remark 2.14(3), thus X is sequential by Remark 2.14(2). □

Theorem 2.16. *The following are equivalent for a k-space X.*

(1) *X has a locally countable sn-network.*

(2) *X is a topological sum of sn-second countable spaces.*

(3) *X is an sn-metrizable, locally (hereditarily) separable space.*

(4) *X is a locally (hereditarily) separable space with a σ-locally countable sn-network.*

Proof. (1) $\Longrightarrow$ (2). X is a k-space with a locally countable cs-network, so X is a topological sum of $\aleph_0$-spaces([17, Theorem 1]). It is easy to see that sn-first countability is hereditary to subspace. Note that each sn-first countable, $\aleph_0$-space is sn-second countable([13, Theorem 2.1]). So X is a topological sum of sn-second countable spaces.

(2) $\Longrightarrow$ (3). Let $X = \oplus\{X_\alpha : \alpha \in \Lambda\}$, where each X_α is sn-second countable. Note that each X_α is a (hereditarily) separable, open subspace

of X, So X is locally (hereditarily) separable. For each $\alpha \in \Lambda$, let $\{P_{\alpha,n} : n \in \mathbb{N}\}$ be a countable sn-network of X_α. Put $\mathcal{P}_n = \{P_{\alpha,n} : \alpha \in \Lambda\}$ for each $n \in \mathbb{N}$, and put $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$, then $\mathcal{P}$ is a σ-locally finite sn-network of X. So X is an sn-metrizable space.

(3) $\Longrightarrow$ (4). It is clear.

(4) $\Longrightarrow$ (1). By Remark 2.13, it suffices to prove that X is locally Lindelöf. Let $\mathcal{P}$ be a σ-locally countable sn-network of X. X is a sequential space by Lemma 2.15, so $\mathcal{P}$ is a σ-locally countable k-network of X([30, Corollary 1.5]). Recalled a space is meta-Lindelöf if each open cover of it has a point countable open refinement. Thus X is hereditarily meta-Lindelöf([17, Proposition 1]). Each hereditarily meta-Lindelöf, locally separable space is locally Lindelöf([14, Proposition 8.7]), so X is locally Lindelöf. □

C. Liu and M. Dai proved that a space X has a locally countable weak-base if and only if it is a topological sum of g-second countable spaces [24, Theorem 2.1]. Combining Remark 2.8(3), we have the following corollary.

Corollary 2.17. *A space X is a k-space with a locally countable sn-network if and only if X has a locally countable weak-base.*

The following examples to shows that "k" in Theorem 2.16 can not be omitted.

Example 2.18. There is a space with a locally countable sn-network, which is not a topological sum of $\aleph_0$-spaces.

Proof. Let D is a discrete space, where $|D| = 2^\omega$. By [3, Example 4.2], there is an almost disjoint family $\{\mathcal{P}_\alpha : \alpha < 2^\omega\}$ consisting of countable infinite subsets of D such that for each uncountable subset P of D, there is $\alpha < 2^\omega$ such that $P_\alpha \subset P$. Let $\{P_{\alpha,n} : n \in \mathbb{N}\}$ be a mutually disjoint family consisting of infinite subsets of P_α. For each $\alpha < 2^\omega$ and each $n \in \mathbb{N}$, choose $p_{\alpha,n} \in \overline{P_{\alpha,n}} - P_{\alpha,n}$, where $\overline{P_{\alpha,n}}$ is the closure of $P_{\alpha,n}$ in the Stone-Čech compactification βD of D. Put $X = D \bigcup\{p_{\alpha,n} : \alpha < 2^\omega, n \in \mathbb{N}\}$, and X is endowed the subspace topology of βD.

Claim 1. X has a σ-locally countable sn-network.

By [22, Example 5.1.18(1)], X has a σ-locally countable cs-network $\mathcal{P}$. Note that each compact subset of X is finite [22, Example 1.5.5], so each convergent sequence of X is finite. Thus we can assume that $\mathcal{P}$ is closed under finite intersections. It is easy to see that $\mathcal{P}$ is an sn-network of X. So X has a σ-locally countable sn-network.

Claim 2. X is not a topological sum of $\aleph_0$-spaces [22, Example 5.1.18(1)]. $\square$

Example 2.19. There is a space with a locally countable sn-network, which is not an $\aleph$-spaces.

Proof. Let $X = \omega_1 \bigcup(\omega_1 \times \{1/n : n \in \mathbb{N}\})$. Define a neighborhood base $\mathcal{B}_x$ for each $x \in X$ for the desired topology on X as follows.

(1) If $x \in X - \omega_1$, then $\mathcal{B}_x = \{\{x\}\}$.

(2) If $x \in \omega_1$, then $\mathcal{B}_x = \{\{x\} \bigcup(\bigcup\{V(n,x) \times \{1/n\} : n \geq m\}) : m \in \mathbb{N}$ *and* $V(n,x)$ *is a neighborhood of* x *in* ω_1 *with the order topology*$\}$.

By [17, Example 1], X has a locally countable k-network, which is not an $\aleph$-space. It suffices to prove that X is sn-first countable by Remark 2.13.

Let $x \in X$. If $x \in X - \omega_1$, then $\{\{x\}\}$ is a countable sn-network at x in X. If $x \in \omega_1$, put $\mathcal{P}_x = \{P_{x,m} : m \in \mathbb{N}\}$, where $P_{x,m} = \{x\} \bigcup\{(x, 1/n) : n \geq m\}$. Then $\mathcal{P}_x$ is a countable network at x in X. We only need to prove that each $P_{x,m}$ is a sequential neighborhood of x.

Let $\{x_n\}$ be a sequence converging to x. Put $K = \{x\} \bigcup\{x_n : n \in \mathbb{N}\}$, then K is a compact subset of X. By [17, Example 1], we have the following facts.

Fact 1. $K \bigcap \omega_1$ is finite.

Fact 2. $K - \bigcup\{\{y\} \bigcup\{(y, 1/n) : n \in \mathbb{N}\} : y \in K \bigcap \omega_1\}$ is finite.

Case 1. If there is $y \in K \bigcap \omega_1$ such that $y = x_n$ for infinite many $n \in \mathbb{N}$, i.e., there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $y = x_{n_k}$ for each $k \in \mathbb{N}$, then $y = x$, So $\{x_n\}$ is frequently in $P_{x,m}$.

Case 2. If Case 1 does not hold, without loss of the generalization, we may assume $K \bigcap \omega_1 = \{x\}$ by Fact 1. By Fact 2, $K - \{x\} \bigcup\{(x, 1/n) : n \in \mathbb{N}\}$ is finite. If there is $y \in K - \{x\} \bigcup\{(x, 1/n) : n \in \mathbb{N}\}$ such that $y = x_n$ for infinite many $n \in \mathbb{N}$, then $\{x_n\}$ is frequently in $P_{x,m}$ by a similar way in the proof of Case 1. Conversely, there is $k_0 \in \mathbb{N}$ such that $\{x\} \bigcup\{x_n : n \geq k_0\} \subset \{x\} \bigcup\{(x, 1/n) : n \in \mathbb{N}\}$. So $\{x_n\}$ is eventually in $P_{x,m}$.

By the above Case 1 and Case 2, $P_{x,m}$ is a sequential neighborhood of x by Remark 2.2(1). $\square$

Recalled a space X is sequentially separable [6] if X has a countable subset D such that for each $x \in X$, there is a sequence $\{x_n\}$ in D converging to x, where D is a sequentially dense subset of X. It is know that each sequentially separable space is separable.

Proposition 2.20. *Let X have a point countable sn-network $\mathcal{P}$. If X is sequentially separable, then $\mathcal{P}$ is countable. So X is sn-second countable.*

Proof. Let D be a sequentially dense subset of X, and let $\mathcal{P} = \{\mathcal{P}_x : x \in X\}$, where $\mathcal{P}_x$ is an sn-network at x in X for each $x \in X$. For each $x \in D$, $(\mathcal{P})_x$ is countable because $\mathcal{P}$ is point countable, where $(\mathcal{P})_x = \{P \in \mathcal{P} : x \in P\}$. Furthermore, $\bigcup\{(\mathcal{P})_x : x \in D\}$ is countable. For each $x \in X$ and $P \in \mathcal{P}_x$, there is a sequence S in D converging to x. Note that P is a sequential neighborhood of x. S is eventually in P, and so $P \bigcap D \neq \emptyset$. This proves that each element of $\mathcal{P}$ intersects with D, thus $\mathcal{P} = \bigcup\{(\mathcal{P})_x : x \in D\}$. So $\mathcal{P}$ is countable. □

Corollary 2.21. *Let X have a σ-locally countable (or point countable) sn-network $\mathcal{P}$. If X is locally sequentially separable, then $\mathcal{P}$ is locally countable in X. So X has a locally countable sn-network.*

Proof. Since σ-locally countable $\Longrightarrow$ point countable, we only need to prove parenthetic part.

Let X be locally sequentially separable. For each $x \in X$, there is an open neighborhood U of x such that U is sequentially separable. It is clear that $\{P \bigcap U : P \in \mathcal{P}\}$ is a point countable sn-network of U. $\{P \bigcap U : P \in \mathcal{P}\}$ is countable by Proposition 2.20, So $\mathcal{P}$ is locally countable in X. □

The following example shows that “sequentially separable” in Proposition 2.20 can not be relaxed to “separable”, which is due to [16, Example 1].

Example 2.22. There is a separable, sn-metrizable space, which is not an $\aleph_0$-spaces.

Proof. Let $\mathbb{Q} \subset X \subset \mathbb{R}$ and $|X| > \omega$, where $\mathbb{Q}$ and $\mathbb{R}$ are the set of all rational numbers and the set of all real numbers respectively. Let $Y = X \bigcup(\bigcup\{\mathbb{Q} \times \{1/n\} : n \in \mathbb{N}\})$. Define a neighborhood base $\mathcal{B}_y$ for each $y \in Y$ for the desired topology on Y as follows.

(1) If $y \in Y - X$, then $\mathcal{B}_y = \{\{y\}\}$.

(2) If $y \in X$, then $\mathcal{B}_y = \{\{y\} \bigcup(\bigcup\{([a_{y,n}, y) \bigcap \mathbb{Q}) \times \{1/m\} : n \geq m\}) : m \in \mathbb{N} \textit{ and } y > a_{y,n} \in \mathbb{R}\}$.

Then Y is a separable, $\aleph$-space and not an $\aleph_0$-space [16, Example 1]. On the other hand, each compact subset of Y is finite [16, Example 1]. By a similar way as in the proof of Example 2.18(claim 1), we can prove Y has a σ-locally finite sn-network. That is, Y is an sn-metrizable space. □

3. Mappings on Spaces with a Locally Countable *sn*-Network

In this section, we discuss invariance and inverse invariance of spaces with a locally countable *sn*-network under certain mappings

Definition 3.1. Let $f : X \longrightarrow Y$ be a mapping.

(1) f is called a perfect mapping [7] if f is closed and $f^{-1}(y)$ is a compact subset of X for each $y \in Y$.

(2) f is called a *Lindelöf* mapping [31] (respectively strongly *Lindelöf* mapping [31]) if for each $y \in Y$, $f^{-1}(y)$ is a *Lindelöf* subset of X (respectively $f^{-1}(\overline{U})$ is a *Lindelöf* subset of X for some neighborhood U of y in Y).

(3) f is called a 1-sequence-covering mapping [23] if for each $y \in Y$ there is $x \in f^{-1}(y)$, such that whenever $\{y_n\}$ is a sequence converging to y in Y, there is a sequence $\{x_n\}$ converging to x in X with each $x_n \in f^{-1}(y_n)$.

(4) f is called a finite subsequence-covering mapping [25] if for each $y \in Y$ there is a finite subset F of $f^{-1}(y)$, such that for any sequence S in Y converging to y, there is a sequence L in X converging to some $x \in F$ and $f(L)$ is a subsequence of S.

(5) f is a sequentially-quotient mapping [4] if whenever S is a convergent sequence in Y there is a convergent sequence L in X such that $f(L)$ is a subsequence of S.

(6) f is a quotient mapping [7] if whenever $U \subset Y$, $f^{-1}(U)$ is open in X if and only if U is open Y.

We call a space X to be point-G_δ if for each $x \in X$, there is a sequence $\{U_n\}$ of neighborhoods of x in X such that $\{x\} = \bigcap\{U_n : n \in \mathbb{N}\}$. It is known that if a space X has a locally countable *cs*-network, then X is point-G_δ [26, (D)].

Remark 3.2. [19]. (1) 1-sequence-covering mappings (sequentially-quotient and finite-to-one mappings, respectively) $\Longrightarrow$ finite subsequence-covering mappings $\Longrightarrow$ sequentially-quotient mappings.

(2) Closed mappings $\Longrightarrow$ quotient mappings.

(3) If the domain is point-G_δ, then closed mappings $\Longrightarrow$ sequentially-quotient mappings

(4) If the domain is sequential, then quotient mappings $\Longrightarrow$ sequentially-quotient mappings.

(5) Quotient mappings preserve k-spaces and perfect mappings inversely preserve k-spaces.

Definition 3.3. [20]. Let X be a space. Put

$$\sigma = \{P \subset X : P \text{ is sequentially open in } X\}.$$

The (X, σ), the set X with the topology σ, is called the sequential coreflection of X, which is denoted by σX.

Definition 3.4. [2]. Let $T_0 = \{a_n : n \in \mathbb{N}\}$ be a sequence converging to $x_0 \notin T_0$, and let T_n be a sequence converging to $a_n \notin T_n$ for each $n \in \mathbb{N}$. Let T be the topological sum of $\{T_n \bigcup \{a_n\} : n \in \mathbb{N}\}$. S_ω is defined as a quotient space obtained from T by identifying all point $a_n \in T$ to the point x_0.

The following lemma is obtained by combining [20, Theorem 3.6] and [20, Theorem 3.13].

Lemma 3.5. [20]. *Let X be a point-G_δ space and contain no closed subspace having S_ω as its sequential coreflection. If X has a point-countable cs-network, then X is sn-first countable.*

Lemma 3.6. [21]. *Let $f : X \longrightarrow Y$ be a perfect mapping and X have a G_δ-diagonal. If Y has a locally countable k-network, then X has a locally countable k-network.*

Lemma 3.7. [11]. *Let $f : X \longrightarrow Y$ be a closed mapping and X be point-G_δ. If F is sequentially closed in X, then $f(F)$ is sequentially closed in Y.*

Theorem 3.8. *Let $f : X \longrightarrow Y$ be a perfect mapping and X have a G_δ-diagonal. If Y has a locally countable sn-network, then X has a locally countable sn-network.*

Proof. If Y has a locally countable sn-network, then X has a locally countable cs-network by Remark 2.5(1), Lemma 2.9 and Lemma 3.6. We only need to prove that X is sn-first countable by Remark 2.13. Since X has a G_δ-diagonal, X is point-G_δ. By Lemma 3.5, It suffices to prove that X contains no closed subspace having S_ω as its sequential coreflection.

Assume X contains closed subspace S having S_ω as its sequential coreflection. Put $g = f|_{\sigma s} : \sigma S \longrightarrow \sigma f(S)$.

Claim 1. g is closed.

Proof. Let A be a closed subset of σS, then A is sequentially closed in S. It is clear that $f : S \longrightarrow f(S)$ is closed and S is point-G_δ. So $f(A)$ is sequentially closed in $f(S)$ by Lemma 3.7, thus $f(A)$ is closed in $\sigma f(S)$.

Claim 2. $g^{-1}(y)$ is compact in σS for each $y \in \sigma f(S)$.

Proof. Let $y \in \sigma f(S)$. Note that X has a G_δ-diagonal and $f^{-1}(y)$ is compact in X, so $f^{-1}(y)$ is metrizable [5]. Therefore, the topology on the sequential coreflection of $f^{-1}(y)\bigcap S$ is equivalent to the induced topology of subspace S of X. Thus $g^{-1}(y) = f^{-1}(y)\bigcap S$ is compact in σS.

By the above two claims, g is perfect. Since S_ω, which is homeomorphic to σS, is a *Fréchet*, $\aleph$-space and perfect mappings preserve *Fréchet*, $\aleph$-spaces, $\sigma f(S)$ is a *Fréchet*, $\aleph$-space. On the other hand, Y is *sn*-first countable, so $f(S)$, as a subspace of Y, is *sn*-first countable. By [20, Theorem 3.13], $\sigma f(S)$ is g-first countable, so $\sigma f(S)$ is metrizable [11, Theorem 2.4], and so σS is metrizable [5]. This contradicts that S_ω is not metrizable. □

We have the following corollary by Corollary 2.17, Remark 3.2(5) and Theorem 3.8.

Corollary 3.9. *Let $f : X \longrightarrow Y$ be a perfect mapping and X have a G_δ-diagonal. If Y has a locally countable weak-base, then X has a locally countable weak-base.*

Example 3.10. A perfect image of a g-second countable space has not any locally countable *sn*-network.

Proof. Let $X = \{0\}\bigcup\mathbb{N}\bigcup(\mathbb{N}\times\mathbb{N})$, $\mathcal{F} = \{F \subset \mathbb{N} : F\ is\ finite\}$, $\mathbb{N}^{\mathbb{N}} = \{f : f$is a correspondence from $\mathbb{N}$ *to* $\mathbb{N}\}$. For $n, m, k \in \mathbb{N}$, $F \in \mathcal{F}$, and $f \in \mathbb{N}^{\mathbb{N}}$, put $V(n,m) = \{n\}\bigcup\{(n,k) : k \geq m\}$, $H(F,f) = \bigcup\{V(n,f(n)) : n \in \mathbb{N} - F\}$. Define a neighborhood base $\mathcal{B}_x$ for each $x \in X$ for the desired topology on X as follows.

(1) If $x \in \mathbb{N}\times\mathbb{N}$, then $\mathcal{B}_x = \{\{x\}\}$.

(2) If $x \in \mathbb{N}$, then $\mathcal{B}_x = \{V(x,m) : m \in \mathbb{N}\}$.

(3) If $x = 0$, then $\mathcal{B}_x = \{\{x\}\bigcup H(F,f) : F \in \mathcal{F}, f \in \mathbb{N}^{\mathbb{N}}\}$.

Let Y be the quotient space obtained from X by shrinking the set $\{0\}\bigcup\mathbb{N}$ to a point, $f : X \longrightarrow Y$ be a natural mapping. Then

Claim 1. f is perfect and X is g-second countable [18, Example 3.1].

Claim 2. Y is not *sn*-first countable [11, Example 3.2], so Y has not any locally countable *sn*-network. □

Which mappings preserve spaces with a locally countable *sn*-network? We give some answers for this question.

Lemma 3.11. *Let $f : X \longrightarrow Y$ be a finite subsequence-covering mapping. If X is sn-first countable, then Y is sn-first countable.*

Proof. Let $y \in Y$. Then there is a finite subset F of $f^{-1}(y)$, such that for any sequence S in Y converging to y, there is a sequence L in X converging to some $x \in F$ and $f(L)$ is a subsequence of S. X is sn-first countable, for each $x \in F$, let $\mathcal{P}_x = \{P_{x,n} : n \in \mathbb{N}\}$ be a decreasing sn-network at x in X. Put $\mathcal{F}_y = \{\bigcup\{f(P_{x,n}) : x \in F\} : n \in \mathbb{N}\}$. Then $\mathcal{F}_y$ is countable decreasing.

(1) $\mathcal{F}_y$ is a network at y in Y. In fact, let U be an open neighborhood of y, then $F \subset f^{-1}(y) \subset f^{-1}(U)$. For each $x \in F$, there is $n_x \in \mathbb{N}$ such that $x \in P_{x,n_x} \subset f^{-1}(U)$, so $y \in f(P_{x,n_x}) \subset U$. Put $n_0 = max\{n_x : x \in F\}$, then $P_{x,n_0} \subset P_{x,n_x}$ for each $x \in F$. So $y \in \bigcup\{f(P_{x,n_0}) : x \in F\} \subset \bigcup\{f(P_{x,n_x}) : x \in F\} \subset U$.

(2) Let $\bigcup\{f(P_{x,n_1}) : x \in F\}$, $\bigcup\{f(P_{x,n_2}) : x \in F\} \in \mathcal{F}_y$. Put $n_0 = max\{n_1, n_2\}$, then $\bigcup\{f(P_{x,n_0}) : x \in F\} \in \mathcal{F}_y$ and $\bigcup\{f(P_{x,n_0}) : x \in F\} \subset (\bigcup\{f(P_{x,n_1}) : x \in F\}) \bigcap (\bigcup\{f(P_{x,n_2}) : x \in F\})$.

(3) $\bigcup\{f(P_{x,n}) : x \in F\}$ is a sequential neighborhood of y for each $n \in \mathbb{N}$. In fact, let S be a sequence in Y converging to y. Then there is a sequence L in X converging to some $x_0 \in F$ and $f(L)$ is a subsequence of S. For each $n \in \mathbb{N}$. Since $P_{x_0,n}$ is a sequential neighborhood of x, L is eventually in $P_{x_0,n}$. So $f(L)$ is eventually in $f(P_{x_0,n})$, hence S is frequently in $f(P_{x_0,n})$. Moreover, S is frequently in $\bigcup\{f(P_{x,n}) : x \in F\}$. By Remark 2.2(1), $\bigcup\{f(P_{x,n}) : x \in F\}$ is a sequential neighborhood of y. □

Lemma 3.12. *Let $f : X \longrightarrow Y$ be a closed, Lindelöf mapping. If $\mathcal{P}$ is a locally countable family of subsets of X, then $f(\mathcal{P})$ is a locally countable family of subsets of Y.*

Proof. Let $\mathcal{P} = \{P_\alpha : \alpha \in \Lambda\}$ be a locally countable family of subsets of X and let $y \in Y$. For each $x \in f^{-1}(y)$, there is an open neighborhood U_x of x such that $\{\alpha \in \Lambda : U_x \bigcap P_\alpha \neq \emptyset\}$ is countable. $f^{-1}(y) \subset \bigcup\{U_x : x \in f^{-1}(y)\}$ and $f^{-1}(y)$ is Lindelöf, so there is a countable subset B of $f^{-1}(y)$ such that $f^{-1}(y) \subset \bigcup\{U_x : x \in B\}$. Put $U = \bigcup\{U_x : x \in B\}$. It is clear that $\{\alpha \in \Lambda : U \bigcap P_\alpha \neq \emptyset\}$ is countable. Note that f is closed. By [7, Theorem 1.4.13], there is an open neighborhood V of y such that $f^{-1}(V) \subset U$. Thus $\Lambda' = \{\alpha \in \Lambda : f^{-1}(V) \bigcap P_\alpha \neq \emptyset\}$ is countable. It is easy to check that $\{\alpha \in \Lambda : V \bigcap f(P_\alpha) \neq \emptyset\} = \Lambda'$. So $\{\alpha \in \Lambda : V \bigcap f(P_\alpha) \neq \emptyset\}$ is countable. This proves that $f(\mathcal{P})$ is a locally countable family of subsets of Y. □

Theorem 3.13. *Let $f : X \longrightarrow Y$ be a closed, finite-to-one mapping. If X has a locally countable sn-network, then Y has a locally countable sn-network.*

Proof. Let $\mathcal{P}$ be a locally countable sn-network of X. Then f is sequentially quotient by Remark 3.2(3), and so Y is sn-first countable by Remark 3.2(1) and Lemma 3.11. Since sequentially quotient mappings preserve cs^*-networks [19, Proposition 2.7.3], $f(\mathcal{P})$ is a cs^*-network of Y. $f(\mathcal{P})$ is locally countable by Lemma 3.12, so $f(\mathcal{P})$ is a locally countable cs^*-network of Y. Thus Y has a locally countable sn-network by Remark 2.13. □

Question 3.14. Do closed, countable-to-one mappings preserve spaces with a locally countable sn-network?

A clopen mapping means an open and closed mapping.

Theorem 3.15. *Let $f : X \longrightarrow Y$ be a clopen, Lindelöf mapping. If X has a locally countable sn-network, then Y has a locally countable sn-network.*

Proof. Let $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ be a locally countable sn-network of X. Since f is closed, Lindelöf, by a similar way as in the proof of Theorem 3.13, $f(\mathcal{P})$ is a locally countable cs^*-network of Y. It suffices to prove that Y is sn-first countable by Remark 2.13. Let $y \in Y$. Put $\mathcal{F}_y = \{f(P) : P \in \mathcal{P}_x \textit{ and } x \in f^{-1}(y)\}$, then $\mathcal{F}_y \subset f(\mathcal{P})$, so $\mathcal{F}_y$ is locally countable. Note that $y \in \bigcap \mathcal{F}_y$, $\mathcal{F}_y$ is countable. It is clear that $\mathcal{F}_y$ is a network at y in Y. We only need to prove that each element of $\mathcal{F}_y$ is a sequential neighborhood of y. Let $f(P) \in \mathcal{F}_y$ and $\{y_k\}$ be a sequence in Y converging to y. Then there is $x \in f^{-1}(y)$ such that $P \in \mathcal{P}_x$. Since X is point-G_δ, $\{x\} = \bigcap\{U_n : n \in \mathbb{N}\}$, where each U_n is open in X and $\overline{U_{n+1}} \subset U_n$. For each $n \in \mathbb{N}$, $y \in f(U_n)$ and $f(U_n)$ is open as f is open, so there is $m_n \in \mathbb{N}$ such that $y_k \in f(U_n)$ for each $k \geq m_n$. Pick $x_n \in U_n$ such that $f(x_n) = y_{m_n}$. Since f is closed, it is not difficult to prove that the sequence $\{x_n\}$ converges to $x \in P$. P is a sequential neighborhood of x, so $\{x_n\}$ is eventually in P. Consequently, $\{f(x_n)\}$ is eventually in $f(P)$, so $\{y_k\}$ is frequently in $f(P)$. By Remark 2.2(1), $f(P)$ is a sequential neighborhood of y. □

Corollary 3.16. *Let $f : X \longrightarrow Y$ be an open, perfect mapping. If X has a locally countable sn-network, then Y has a locally countable sn-network.*

Clopen mappings preserve spaces with a locally countable weak-base [24, Theorem 4.7]. But the following question is still open.

Question 3.17. Do clopen mappings preserve spaces with a locally countable sn-network (respectively cs-network)?

Lemma 3.18. *Let $f : X \longrightarrow Y$ be a strongly Lindelöf mapping. If $\mathcal{P}$ is a locally countable family of subsets of X, then $f(\mathcal{P})$ is a locally countable family of subsets of Y.*

Proof. Let $\mathcal{P} = \{P_\alpha : \alpha \in \Lambda\}$ be a locally countable family of subsets of X and let $y \in Y$. Then there is a neighborhood W of y in Y such that $f^{-1}(\overline{W})$ is a Lindelöf subset of X. It suffices to prove that $\{\alpha \in \Lambda : W \bigcap f(P_\alpha) \neq \emptyset\}$ is countable. For each $x \in f^{-1}(\overline{W})$, there is an open neighborhood U_x of x such that $\{\alpha \in \Lambda : U_x \bigcap P_\alpha \neq \emptyset\}$ is countable. $f^{-1}(\overline{W}) \subset \bigcup\{U_x : x \in f^{-1}(\overline{W})\}$ and $f^{-1}(\overline{W})$ is Lindelöf, so there is a countable subset B of $f^{-1}(\overline{W})$ such that $f^{-1}(\overline{W}) \subset \bigcup\{U_x : x \in B\}$. It is easy to see that $\{\alpha \in \Lambda : (\bigcup\{U_x : x \in B\}) \bigcap P_\alpha \neq \emptyset\}$ is countable, so $\Lambda' = \{\alpha \in \Lambda : f^{-1}(W) \bigcap P_\alpha \neq \emptyset\}$ is countable. It is easy to check that $\{\alpha \in \Lambda : W \bigcap f(P_\alpha) \neq \emptyset\} = \Lambda'$. This completes the proof. □

Theorem 3.19. *Let X have a locally countable sn-network. If one of the following holds, then Y has a locally countable sn-network.*

(1) f is finite subsequence-covering, strongly Lindelöf.

(2) f is 1-sequence-covering, strongly Lindelöf.

(3) f is sequentially-quotient, finite-to-one, strongly Lindelöf.

Proof. We only need to prove part (1) by Remark 3.2(1). Let $f : X \longrightarrow Y$ be a finite subsequence-covering, strongly Lindelöf mapping and $\mathcal{P}$ be a locally countable sn-network of X. Then Y is sn-first countable by lemma 3.11 and $f(\mathcal{P})$ is a locally countable family of subsets of Y by Lemma 3.18. By a similar way as in the proof of Theorem 3.13, we can prove $f(\mathcal{P})$ is a cs^*-network of Y. So Y has a locally countable sn-network by Remark 2.13. □

The following corollary is obtained by Remark 3.2, Corollary 2.17, Theorem 3.13 and Theorem 3.19.

Corollary 3.20. *Let X have a locally countable weak-base. If one of the following holds, then Y has a locally countable weak-base.*

(1) f is closed, finite-to-one.

(2) f is finite subsequence-covering, quotient, strongly Lindelöf.

(3) f is 1-sequence-covering, quotient, strongly Lindelöf.

(4) f is quotient, finite-to-one, strongly Lindelöf.

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Department of Mathematics, Suzhou University, Suzhou, 215006, P.R.China

E-mail address: zhugexun@163.com

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