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Zhi-Gang Wang and Da-Zhao Chen

ON SUBCLASSES OF CLOSE-TO-CONVEX AND QUASI-CONVEX FUNCTIONS WITH RESPECT TO $2K$-SYMMETRIC CONJUGATE POINTS

(submitted by F. G. Avkhadiev)

Abstract. In the present paper, the authors introduce two new subclasses $\mathcal{S}_{sc}^{(k)}(\lambda, \alpha)$ of close-to-convex functions and $\mathcal{C}_{sc}^{(k)}(\lambda, \alpha)$ of quasi-convex functions with respect to $2k$-symmetric conjugate points. The integral representations and convolution conditions for these classes are provided. Some coefficient inequalities for functions belonging to these classes and their subclasses with negative coefficients are also provided.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbf{C} : |z| < 1\}$. Let $\mathcal{S}$, $\mathcal{S}^*$, $\mathcal{K}$, $\mathcal{C}$ and $\mathcal{C}^*$ denote the familiar subclasses of $\mathcal{A}$ consisting of functions which are univalent, starlike, convex, close-to-convex and quasi-convex in $\mathcal{U}$, respectively (see, for details, [2, 3, 4, 5]).

Al-Amiri, Coman and Mocanu [1] once introduced and investigated a class of functions starlike with respect to $2k$-symmetric conjugate points,

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which satisfy the following inequality

$$\Re\left\{\frac{zf'(z)}{f_{2k}(z)}\right\}>0 \quad (z\in\mathcal{U}),$$

where $k\geq 2$ is a fixed positive integer and $f_{2k}(z)$ is defined by the following equality

$$f_{2k}(z)=\frac{1}{2k}\sum_{\nu=0}^{k-1}\left[\varepsilon^{-\nu}f(\varepsilon^{\nu}z)+\varepsilon^{\nu}\overline{f(\varepsilon^{\nu}\overline{z})}\right] \quad (\varepsilon=\exp(2\pi i/k);\ z\in\mathcal{U}). \tag{1.2}$$

In the present paper, we introduce the following two classes of analytic functions with respect to $2k$-symmetric conjugate points, and obtain some interesting results.

Definition 1. Let $\mathcal{S}_{sc}^{(k)}(\lambda,\alpha)$ denote the class of functions in $\mathcal{A}$ satisfying the following inequality

$$\Re\left\{\frac{zf'(z)+\lambda z^2f''(z)}{(1-\lambda)f_{2k}(z)+\lambda zf'_{2k}(z)}\right\}>\alpha \quad (z\in\mathcal{U}), \tag{1.3}$$

where $0\leq\lambda\leq 1,\ 0\leq\alpha<1$ and $f_{2k}(z)$ is defined by equality (1.2). And a function $f(z)\in\mathcal{A}$ is in the class $\mathcal{C}_{sc}^{(k)}(\lambda,\alpha)$ if and only if $zf'(z)\in\mathcal{S}_{sc}^{(k)}(\lambda,\alpha)$.

In our proposed investigation of the classes $\mathcal{S}_{sc}^{(k)}(\lambda,\alpha)$ and $\mathcal{C}_{sc}^{(k)}(\lambda,\alpha)$, we shall also make use of the following lemmas.

Lemma 1. *Let $\gamma\geq 0$ and $f\in\mathcal{C}$, then*

$$F(z)=\frac{1+\gamma}{z^{\gamma}}\int_0^z f(t)t^{\gamma-1}dt\in\mathcal{C}.$$

This lemma is a special case of Theorem 4 in [6].

Lemma 2 [3]**.** *Let $0<\lambda\leq 1$ and $f\in\mathcal{C}^*$, then*

$$F(z)=\frac{1}{\lambda}z^{1-\frac{1}{\lambda}}\int_0^z f(t)t^{\frac{1}{\lambda}-2}dt\in\mathcal{C}^*\subset\mathcal{C}.$$

Lemma 3. *Let $0\leq\lambda\leq 1$ and $0\leq\alpha<1$, then we have $\mathcal{S}_{sc}^{(k)}(\lambda,\alpha)\subset\mathcal{C}\subset\mathcal{S}$.*

Proof. Let $F(z)=(1-\lambda)f(z)+\lambda zf'(z),\ F_{2k}(z)=(1-\lambda)f_{2k}(z)+\lambda zf'_{2k}(z)$ with $f(z)\in\mathcal{S}_{sc}^{(k)}(\lambda,\alpha)$, substituting z by $\varepsilon^{\mu}z$ $(\mu=0,1,2,\ldots,k-1)$ in (1.1), we get

$$\Re\left\{\frac{\varepsilon^{\mu}zf'(\varepsilon^{\mu}z)+\lambda(\varepsilon^{\mu}z)^2f''(\varepsilon^{\mu}z)}{(1-\lambda)f_{2k}(\varepsilon^{\mu}z)+\lambda\varepsilon^{\mu}zf'_{2k}(\varepsilon^{\mu}z)}\right\}>\alpha. \tag{1.4}$$

From inequality (1.4), we have

$$\Re\left\{\frac{\overline{\varepsilon^{\mu}\overline{z}}\ \overline{f'(\varepsilon^{\mu}\overline{z})}+\lambda\overline{(\varepsilon^{\mu}\overline{z})^{2}}\ \overline{f''(\varepsilon^{\mu}\overline{z})}}{(1-\lambda)\overline{f_{2k}(\varepsilon^{\mu}\overline{z})}+\lambda\overline{\varepsilon^{\mu}\overline{z}}\ \overline{f'_{2k}(\varepsilon^{\mu}\overline{z})}}\right\}>\alpha. \tag{1.5}$$

Note that $f_{2k}(\varepsilon^{\mu}z)=\varepsilon^{\mu}f_{2k}(z)$, $f'_{2k}(\varepsilon^{\mu}z)=f'_{2k}(z)$, $\overline{f_{2k}(\varepsilon^{\mu}\overline{z})}=\varepsilon^{-\mu}f_{2k}(z)$ and $\overline{f'_{2k}(\varepsilon^{\mu}\overline{z})}=f'_{2k}(z)$, thus, inequalities (1.4) and (1.5) can be written as

$$\Re\left\{\frac{zf'(\varepsilon^{\mu}z)+\lambda z^{2}\varepsilon^{\mu}f''(\varepsilon^{\mu}z)}{(1-\lambda)f_{2k}(z)+\lambda zf'_{2k}(z)}\right\}>\alpha, \tag{1.6}$$

and

$$\Re\left\{\frac{z\overline{f'(\varepsilon^{\mu}\overline{z})}+\lambda z^{2}\varepsilon^{-\mu}\overline{f''(\varepsilon^{\mu}\overline{z})}}{(1-\lambda)f_{2k}(z)+\lambda zf'_{2k}(z)}\right\}>\alpha. \tag{1.7}$$

Summing inequalities (1.6) and (1.7), we can obtain

$$\Re\left\{\frac{z\left[f'(\varepsilon^{\mu}z)+\overline{f'(\varepsilon^{\mu}\overline{z})}\right]+\lambda z^{2}\left[\varepsilon^{\mu}f''(\varepsilon^{\mu}z)+\varepsilon^{-\mu}\overline{f''(\varepsilon^{\mu}\overline{z})}\right]}{(1-\lambda)f_{2k}(z)+\lambda zf'_{2k}(z)}\right\}>2\alpha. \tag{1.8}$$

Let $\mu=0,1,2,\ldots,k-1$ in (1.8), respectively, and summing them we can get

$$\Re\left\{\frac{z\frac{1}{2k}\sum_{\mu=0}^{k-1}\left[f'(\varepsilon^{\mu}z)+\overline{f'(\varepsilon^{\mu}\overline{z})}\right]+\lambda z^{2}\frac{1}{2k}\sum_{\mu=0}^{k-1}\left[\varepsilon^{\mu}f''(\varepsilon^{\mu}z)+\varepsilon^{-\mu}\overline{f''(\varepsilon^{\mu}\overline{z})}\right]}{(1-\lambda)f_{2k}(z)+\lambda zf'_{2k}(z)}\right\}>\alpha,$$

or equivalently,

$$\Re\left\{\frac{zf'_{2k}(z)+\lambda z^{2}f''_{2k}(z)}{(1-\lambda)f_{2k}(z)+\lambda zf'_{2k}(z)}\right\}=\Re\left\{\frac{zF'_{2k}(z)}{F_{2k}(z)}\right\}>\alpha,$$

that is $F_{2k}(z)\in\mathcal{S}^{*}(\alpha)$, which is the class of starlike functions of order α in $\mathcal{U}$. Note that $\mathcal{S}^{*}(0)=\mathcal{S}^{*}$, this implies that $F(z)=(1-\lambda)f(z)+\lambda zf'(z)\in\mathcal{C}$. We now split it into two cases to prove.

Case 1. When $\lambda=0$. It is obvious that $f(z)=F(z)\in\mathcal{C}$.

Case 2. When $0<\lambda\le 1$. From $F(z)=(1-\lambda)f(z)+\lambda zf'(z)$ and $0<\lambda\le 1$, we have

$$f(z)=\frac{1}{\lambda}z^{1-\frac{1}{\lambda}}\int_{0}^{z}F(t)t^{\frac{1}{\lambda}-2}dt.$$

Since $\gamma=\frac{1}{\lambda}-1\ge 0$, by Lemma 1, we obtain that $f(z)\in\mathcal{C}$. Hence $\mathcal{S}_{sc}^{(k)}(\lambda,\alpha)\subset\mathcal{C}\subset\mathcal{S}$, and the proof is complete.

By means of Lemma 2, using the similar method as in Lemma 3, we may prove the following Lemma.

Lemma 4. *Let* $0 \leq \lambda \leq 1$ *and* $0 \leq \alpha < 1$, *then we have* $\mathcal{C}_{sc}^{(k)}(\lambda, \alpha) \subset \mathcal{C}^* \subset \mathcal{C}$.

In the present paper, we shall provide the integral representations and convolution conditions for the classes $\mathcal{S}_{sc}^{(k)}(\lambda, \alpha)$ and $\mathcal{C}_{sc}^{(k)}(\lambda, \alpha)$, we shall also provide some coefficient inequalities for functions belonging to these classes and their subclasses with negative coefficients.

2. Integral Representations

We first give the integral representations of functions in the classes $\mathcal{S}_{sc}^{(k)}(\lambda, \alpha)$ and $\mathcal{C}_{sc}^{(k)}(\lambda, \alpha)$.

Theorem 1. *Let* $f(z) \in \mathcal{S}_{sc}^{(k)}(\lambda, \alpha)$ *with* $0 < \lambda \leq 1$, *then we have*

$$f_{2k}(z) = \frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^z \exp\left\{ \frac{1}{2k} \sum_{\mu=0}^{k-1} \int_0^u \frac{2(1-\alpha)}{\zeta} \left[\frac{\omega(\varepsilon^\mu \zeta)}{1-\omega(\varepsilon^\mu \zeta)} + \frac{\overline{\omega(\varepsilon^\mu \overline{\zeta})}}{1-\overline{\omega(\varepsilon^\mu \overline{\zeta})}} \right] d\zeta \right\} u^{\frac{1}{\lambda}-1} du, \quad (2.1)$$

where $f_{2k}(z)$ *is defined by equality* (1.2), $\omega(z)$ *is analytic in* $\mathcal{U}$ *and* $\omega(0) = 0$, $|\omega(z)| < 1$.

Proof. Suppose that $f(z) \in \mathcal{S}_{sc}^{(k)}(\lambda, \alpha)$, we know that the condition (1.3) can be written as

$$\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda) f_{2k}(z) + \lambda z f_{2k}'(z)} \prec \frac{1+(1-2\alpha)z}{1-z},$$

where "$\prec$" stands for the usual subordination, it follows that

$$\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda) f_{2k}(z) + \lambda z f_{2k}'(z)} = \frac{1+(1-2\alpha)\omega(z)}{1-\omega(z)}, \quad (2.2)$$

where $\omega(z)$ is analytic in $\mathcal{U}$ and $\omega(0) = 0$, $|\omega(z)| < 1$. By applying the similar method as in Lemma 3 to equality (2.2), we can obtain

$$\frac{(1-\lambda) z f_{2k}'(z) + \lambda z (z f_{2k}'(z))'}{(1-\lambda) f_{2k}(z) + \lambda z f_{2k}'(z)} = \frac{1}{2k} \sum_{\mu=0}^{k-1} \left[\frac{1+(1-2\alpha)\omega(\varepsilon^\mu z)}{1-\omega(\varepsilon^\mu z)} + \frac{1+(1-2\alpha)\overline{\omega(\varepsilon^\mu \overline{z})}}{1-\overline{\omega(\varepsilon^\mu \overline{z})}} \right]. \quad (2.3)$$

From equality (2.3), we get

$$\frac{(1-\lambda)f_{2k}'(z)+\lambda(zf_{2k}'(z))'}{(1-\lambda)f_{2k}(z)+\lambda zf_{2k}'(z)}-\frac{1}{z}$$
$$=\frac{1}{2k}\sum_{\mu=0}^{k-1}\left[\frac{2(1-\alpha)\omega(\varepsilon^{\mu}z)}{z(1-\omega(\varepsilon^{\mu}z))}+\frac{2(1-\alpha)\overline{\omega(\varepsilon^{\mu}\overline{z})}}{z\left(1-\overline{\omega(\varepsilon^{\mu}\overline{z})}\right)}\right]. \quad (2.4)$$

Integrating equality (2.4), we have

$$\log\left\{\frac{(1-\lambda)f_{2k}(z)+\lambda zf_{2k}'(z)}{z}\right\}$$
$$=\frac{1}{2k}\sum_{\mu=0}^{k-1}\int_0^z\left[\frac{2(1-\alpha)\omega(\varepsilon^{\mu}\zeta)}{\zeta(1-\omega(\varepsilon^{\mu}\zeta))}+\frac{2(1-\alpha)\overline{\omega(\varepsilon^{\mu}\overline{\zeta})}}{\zeta\left(1-\overline{\omega(\varepsilon^{\mu}\overline{\zeta})}\right)}\right]d\zeta,$$

that is,

$$(1-\lambda)f_{2k}(z)+\lambda zf_{2k}'(z)$$
$$=z\cdot\exp\left\{\frac{1}{2k}\sum_{\mu=0}^{k-1}\int_0^z\frac{2(1-\alpha)}{\zeta}\left[\frac{\omega(\varepsilon^{\mu}\zeta)}{1-\omega(\varepsilon^{\mu}\zeta)}+\frac{\overline{\omega(\varepsilon^{\mu}\overline{\zeta})}}{1-\overline{\omega(\varepsilon^{\mu}\overline{\zeta})}}\right]d\zeta\right\}. \quad (2.5)$$

From equality (2.5), we can get equality (2.1) easily. Hence the proof is complete.

Theorem 2. *Let $f(z)\in\mathcal{S}_{sc}^{(k)}(\lambda,\alpha)$ with $0<\lambda\le 1$, then we have*

$$f(z)=\frac{1}{\lambda}z^{1-\frac{1}{\lambda}}\int_0^z\int_0^u$$
$$\exp\left\{\frac{1}{2k}\sum_{\mu=0}^{k-1}\int_0^t\frac{2(1-\alpha)}{\zeta}\left[\frac{\omega(\varepsilon^{\mu}\zeta)}{1-\omega(\varepsilon^{\mu}\zeta)}+\frac{\overline{\omega(\varepsilon^{\mu}\overline{\zeta})}}{1-\overline{\omega(\varepsilon^{\mu}\overline{\zeta})}}\right]d\zeta\right\}$$
$$\cdot\frac{1+(1-2\alpha)\omega(t)}{1-\omega(t)}dtu^{\frac{1}{\lambda}-2}du, \quad (2.6)$$

where $\omega(z)$ is analytic in $\mathcal{U}$ and $\omega(0)=0$, $|\omega(z)|<1$.

Proof. Suppose that $f(z) \in \mathcal{S}_{sc}^{(k)}(\lambda, \alpha)$, from equalities (2.2) and (2.5), we can get

$$\begin{aligned}
&(1-\lambda)f'(z) + \lambda(zf'(z))' \\
&\qquad = \frac{(1-\lambda)f_k(z) + \lambda z f_k'(z)}{z} \cdot \frac{1+(1-2\alpha)\omega(z)}{1-\omega(z)} \\
&\qquad = \exp\left\{\frac{1}{2k}\sum_{\mu=0}^{k-1}\int_0^z \frac{2(1-\alpha)}{\zeta}\left[\frac{\omega(\varepsilon^\mu\zeta)}{1-\omega(\varepsilon^\mu\zeta)} + \frac{\overline{\omega(\varepsilon^\mu\overline{\zeta})}}{1-\overline{\omega(\varepsilon^\mu\overline{\zeta})}}\right]d\zeta\right\} \\
&\qquad\qquad \cdot \frac{1+(1-2\alpha)\omega(z)}{1-\omega(z)}.
\end{aligned}$$

Integrating this equality, we can get equality (2.6) easily. Hence the proof is complete.

Similarly, for the class $\mathcal{C}_{sc}^{(k)}(\lambda, \alpha)$, we have

Corollary 1. *Let $f(z) \in \mathcal{C}_{sc}^{(k)}(\lambda, \alpha)$ with $0 < \lambda \leq 1$, then we have*

$$\begin{aligned}
f_{2k}(z) = \frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^z \int_0^u & \\
\exp\left\{\frac{1}{2k}\sum_{\mu=0}^{k-1}\int_0^\xi \frac{2(1-\alpha)}{\zeta}\left[\frac{\omega(\varepsilon^\mu\zeta)}{1-\omega(\varepsilon^\mu\zeta)} + \frac{\overline{\omega(\varepsilon^\mu\overline{\zeta})}}{1-\overline{\omega(\varepsilon^\mu\overline{\zeta})}}\right]d\zeta\right\}& \\
d\xi u^{\frac{1}{\lambda}-2}du,&
\end{aligned}$$

where $f_{2k}(z)$ is defined by equality (1.2), *$\omega(z)$ is analytic in $\mathcal{U}$ and $\omega(0) = 0$, $|\omega(z)| < 1$.*

Corollary 2. *Let $f(z) \in \mathcal{C}_{sc}^{(k)}(\lambda, \alpha)$ with $0 < \lambda \leq 1$, then we have*

$$\begin{aligned}
f(z) = \frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^z \int_0^u & \\
\frac{1}{\xi}\int_0^\xi \exp\left\{\frac{1}{2k}\sum_{\mu=0}^{k-1}\int_0^t \frac{2(1-\alpha)}{\zeta}\left[\frac{\omega(\varepsilon^\mu\zeta)}{1-\omega(\varepsilon^\mu\zeta)} + \frac{\overline{\omega(\varepsilon^\mu\overline{\zeta})}}{1-\overline{\omega(\varepsilon^\mu\overline{\zeta})}}\right]d\zeta\right\}& \\
\cdot \frac{1+(1-2\alpha)\omega(t)}{1-\omega(t)} dt d\xi u^{\frac{1}{\lambda}-2}du,&
\end{aligned}$$

where $\omega(z)$ is analytic in $\mathcal{U}$ and $\omega(0) = 0$, $|\omega(z)| < 1$.

3. Convolution Conditions

In this section, we give the convolution conditions for the classes $\mathcal{S}_{sc}^{(k)}(\lambda, \alpha)$ and $\mathcal{C}_{sc}^{(k)}(\lambda, \alpha)$. Let $f,\ g \in \mathcal{A}$, where $f(z)$ is given by (1.1) and $g(z)$ is defined by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

then the Hadamard product (or convolution) $f * g$ is defined (as usual) by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z).$$

Theorem 3. *A function $f(z) \in \mathcal{S}_{sc}^{(k)}(\lambda, \alpha)$ if and only if*

$$\begin{aligned} \frac{1}{z}\Bigg\{ f * \Bigg\{ (1-\lambda)\left[\frac{z}{(1-z)^2}(1-e^{i\theta}) - \frac{1+(1-2\alpha)e^{i\theta}}{2}h\right] \\ + \lambda z \left[\frac{z}{(1-z)^2}(1-e^{i\theta}) - \frac{1+(1-2\alpha)e^{i\theta}}{2}h\right]' \Bigg\}(z) \\ - \left[1+(1-2\alpha)e^{i\theta}\right] \cdot \overline{f * \left(\frac{1-\lambda}{2}h + \frac{\lambda}{2}zh'\right)(\overline{z})} \Bigg\} \neq 0 \end{aligned} \tag{3.1}$$

for all $z \in \mathcal{U}$ and $0 \leq \theta < 2\pi$, where $h(z)$ is given by (3.6).

Proof. Suppose that $f(z) \in \mathcal{S}_{sc}^{(k)}(\lambda, \alpha)$, since (1.3) is equivalent to

$$\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f_{2k}(z) + \lambda z f'_{2k}(z)} \neq \frac{1+(1-2\alpha)e^{i\theta}}{1-e^{i\theta}} \tag{3.2}$$

for all $z \in \mathcal{U}$ and $0 \leq \theta < 2\pi$. And the condition (3.2) can be written as

$$\begin{aligned} \frac{1}{z}\big\{ [(1-\lambda)zf'(z) + \lambda z(zf'(z))'\,](1-e^{i\theta}) \\ -[(1-\lambda)f_{2k}(z) + \lambda z f'_{2k}(z)][1+(1-2\alpha)e^{i\theta}]\big\} \neq 0. \end{aligned} \tag{3.3}$$

On the other hand, it is well known that

$$zf'(z) = f(z) * \frac{z}{(1-z)^2}. \tag{3.4}$$

And from the definition of $f_{2k}(z)$, we know

$$f_{2k}(z) = \frac{1}{2}\left[(f*h)(z) + \overline{(f*h)(\overline{z})}\right], \tag{3.5}$$

where

$$h(z) = \frac{1}{k}\sum_{\upsilon=0}^{k-1} \frac{z}{1-\varepsilon^{\upsilon} z}. \tag{3.6}$$

Substituting (3.4) and (3.5) into (3.3), we can get (3.1) easily. This completes the proof of Theorem 3.

Similarly, for the class $\mathcal{C}_{sc}^{(k)}(\lambda,\alpha)$, we have

Corollary 3. *A function* $f(z)\in\mathcal{C}_{sc}^{(k)}(\lambda,\alpha)$ *if and only if*

$$\frac{1}{z}\Bigg\{f*\Bigg\{z\Bigg\{(1-\lambda)\left[\frac{z}{(1-z)^2}(1-e^{i\theta})-\frac{1+(1-2\alpha)e^{i\theta}}{2}h\right] + \lambda z\left[\frac{z}{(1-z)^2}(1-e^{i\theta})-\frac{1+(1-2\alpha)e^{i\theta}}{2}h\right]'\Bigg\}'\Bigg\}(z) - \left[1+(1-2\alpha)e^{i\theta}\right]\cdot\overline{f*\left[z\left(\frac{1-\lambda}{2}h+\frac{\lambda}{2}zh'\right)'\right](\overline{z})}\Bigg\}\neq 0$$

for all $z\in\mathcal{U}$ *and* $0\le\theta<2\pi$, *where* $h(z)$ *is given by (3.6).*

4. Coefficient Inequalities

In this section, we first provide the sufficient conditions for functions belonging to the classes $\mathcal{S}_{sc}^{(k)}(\lambda,\alpha)$ and $\mathcal{C}_{sc}^{(k)}(\lambda,\alpha)$.

Theorem 4. *Let* $0\le\lambda\le 1$ *and* $0\le\alpha<1$. *If*

$$\sum_{n=1}^{\infty}[(1-\lambda)+\lambda(nk+1)][|(nk+1)a_{nk+1}-\Re(a_{nk+1})|+(1-\alpha)\,|\Re(a_{nk+1})|] + \sum_{\substack{n=2\\ n\neq lk+1}}^{\infty} n[(1-\lambda)+\lambda n]\,|a_n|\le 1-\alpha, \quad (4.1)$$

then $f(z)\in\mathcal{S}_{sc}^{(k)}(\lambda,\alpha)$.

Proof. It suffices to show that

$$\left|\frac{zf'(z)+\lambda z^2f''(z)}{(1-\lambda)f_{2k}(z)+\lambda zf'_{2k}(z)}-1\right|<1-\alpha.$$

Note that for $|z|=r<1$, we have

$$\begin{aligned}\left|\frac{zf'(z)+\lambda z^2f''(z)}{(1-\lambda)f_{2k}(z)+\lambda zf'_{2k}(z)}-1\right| &= \left|\frac{\sum_{n=2}^{\infty}[(1-\lambda)+\lambda n](na_n-\Re(a_n)c_n)z^{n-1}}{1+\sum_{n=2}^{\infty}[(1-\lambda)+\lambda n]\Re(a_n)c_nz^{n-1}}\right| \\ &\le \frac{\sum_{n=2}^{\infty}[(1-\lambda)+\lambda n]\,|na_n-\Re(a_n)c_n|\,|z|^{n-1}}{1-\sum_{n=2}^{\infty}[(1-\lambda)+\lambda n]c_n\,|\Re(a_n)|\,|z|^{n-1}} \\ &\le \frac{\sum_{n=2}^{\infty}[(1-\lambda)+\lambda n]\,|na_n-\Re(a_n)c_n|}{1-\sum_{n=2}^{\infty}[(1-\lambda)+\lambda n]c_n\,|\Re(a_n)|},\end{aligned}$$

where

$$c_n = \frac{1}{k}\sum_{\nu=0}^{k-1} \varepsilon^{(n-1)\nu} = \begin{cases} 1, & n = lk+1, \\ 0, & n \neq lk+1. \end{cases} \tag{4.2}$$

This last expression is bounded above by $1-\alpha$ if

$$\sum_{n=2}^{\infty}[(1-\lambda)+\lambda n][|na_n - \Re(a_n)c_n| + c_n(1-\alpha)\,|\Re(a_n)|] \leq 1-\alpha. \tag{4.3}$$

Since inequality (4.3) can be written as inequality (4.1), hence $f(z)$ satisfies the condition (1.3). This completes the proof of Theorem 4.

Similarly, for the class $\mathcal{C}_{sc}^{(k)}(\lambda,\alpha)$, we have

Corollary 4. *Let* $0 \leq \lambda \leq 1$ *and* $0 \leq \alpha < 1$. *If*

$$\sum_{n=1}^{\infty}(nk+1)[(1-\lambda)+\lambda(nk+1)] \cdot [|(nk+1)a_{nk+1} - \Re(a_{nk+1})| + (1-\alpha)\,|\Re(a_{nk+1})|] + \sum_{\substack{n=2\\ n\neq lk+1}}^{\infty} n^2[(1-\lambda)+\lambda n]\,|a_n| \leq 1-\alpha,$$

then $f(z) \in \mathcal{C}_{sc}^{(k)}(\lambda,\alpha)$.

Let $\mathcal{T}$ be the subclass of $\mathcal{A}$ consisting of all functions which are of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \ (a_n \geq 0).$$

For convenience, we write $\mathcal{S}_{sc}^{(k)}(\lambda,\alpha)\cap\mathcal{T}$ as $\mathcal{T}S_{sc}^{(k)}(\lambda,\alpha)$ and $\mathcal{C}_{sc}^{(k)}(\lambda,\alpha)\cap\mathcal{T}$ simple as $\mathcal{T}C_{sc}^{(k)}(\lambda,\alpha)$. We now provide the necessary and sufficient coefficient conditions for functions belonging to the classes $\mathcal{T}S_{sc}^{(k)}(\lambda,\alpha)$ and $\mathcal{T}C_{sc}^{(k)}(\lambda,\alpha)$.

Theorem 5. *Let* $0 \leq \lambda \leq 1$, $0 \leq \alpha < 1$ *and* $f(z) \in \mathcal{T}$, *then* $f(z) \in \mathcal{T}S_{sc}^{(k)}(\lambda,\alpha)$ *if and only if*

$$\sum_{n=1}^{\infty}[(1-\lambda)+\lambda(nk+1)][(nk+1)-\alpha]a_{nk+1} + \sum_{\substack{n=2\\ n\neq lk+1}}^{\infty} n[(1-\lambda)+\lambda n]a_n \leq 1-\alpha. \tag{4.4}$$

Proof. In view of Theorem 4, we need only to prove the necessity. Suppose that $f(z) \in \mathcal{T}S_{sc}^{(k)}(\lambda,\alpha)$, then from (1.3), we can get

$$\Re\left\{\frac{1-\sum_{n=2}^{\infty} n[(1-\lambda)+\lambda n]a_n z^{n-1}}{1-\sum_{n=2}^{\infty}[(1-\lambda)+\lambda n]c_n a_n z^{n-1}}\right\} > \alpha, \tag{4.5}$$

where c_n is given by (4.2). By letting $z \to 1^-$ through real values in (4.5), we can get

$$\frac{1-\sum_{n=2}^{\infty} n[(1-\lambda)+\lambda n]a_n}{1-\sum_{n=2}^{\infty}[(1-\lambda)+\lambda n]c_n a_n} \geq \alpha,$$

or equivalently,

$$\sum_{n=2}^{\infty}[(1-\lambda)+\lambda n](n-\alpha c_n)a_n \leq 1-\alpha. \tag{4.6}$$

Substituting (4.2) into inequality (4.6), we can get inequality (4.4) easily. This completes the proof of Theorem 5.

Similarly, for the class $\mathcal{T}C_{sc}^{(k)}(\lambda,\alpha)$, we have

Corollary 5. *Let* $0 \leq \lambda \leq 1$, $0 \leq \alpha < 1$ *and* $f(z) \in \mathcal{T}$, *then* $f(z) \in \mathcal{T}C_{sc}^{(k)}(\lambda,\alpha)$ *if and only if*

$$\sum_{n=1}^{\infty}(nk+1)[(1-\lambda)+\lambda(nk+1)][(nk+1)-\alpha]a_{nk+1} + \sum_{\substack{n=2 \\ n\neq lk+1}}^{\infty} n^2[(1-\lambda)+\lambda n]a_n \leq 1-\alpha.$$

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Changsha University of Science and Technology, Changsha, 410076
Hunan, People's Republic of China

Department of Mathematics, Shaoyang University, Shaoyang, 422000
Hunan, People's Republic of China
E-mail address: zhigwang@163.com

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