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WEIGHTED TRACE INEQUALITIES OF MONOTONICITY

Abstract. We study the inequality $\mathrm{Tr}(w(A)f(A)) \le \mathrm{Tr}(w(A)f(B))$, where $w\colon \mathbb{R} \to \mathbb{R}^+$ is a "weight function" and A, B are Hermitian matrices with $A \le B$, and find corresponding characterizations of the trace.

Throughout the paper, M_n stands for the algebra of $n \times n$ complex matrices, M_n^{h} and M_n^+ denote the subsets of Hermitian and positive semidefinite matrices respectively. For $A, B \in M_n^{\mathrm{h}}$, the notation $A \le B$ means that $B - A \in M_n^+$. A linear functional φ on M_n is said to be *positive* if $\varphi(A) \ge 0$ for all $A \in M_n^+$. The spectrum of a matrix $A \in M_n$ is denoted by $\sigma(A)$. For a real-valued function f of a real variable and a matrix $A \in M_n^{\mathrm{h}}$, the value $f(A)$ is understood by means of the functional calculus for Hermitian matrices.

Theorem 1. *Let $n \ge 2$ and let a function $f\colon \mathbb{R}^+ \to \mathbb{R}$ be Borel measurable. The inequality*

$$\mathrm{Tr}(Af(A)) \le \mathrm{Tr}(Af(B)) \tag{1}$$

holds for all $A, B \in M_n^+$ with $A \le B$ if and only if the function $g(x) = xf(x)$ is convex on $\mathbb{R}^+$.

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Proof. Let $g(x)$ be convex on $\mathbb{R}^+$. As is well known, if $0 \le A \le B$ then there exists $U \in M_n$ such that $\|U\| \le 1$ and $A^{1/2} = UB^{1/2}$. Then, by Jensen's trace inequality for contractions (see, e.g., [4, Corollary 3] or [2]), we have

$$\begin{aligned}\operatorname{Tr}(Af(B)) &= \operatorname{Tr}(A^{1/2}f(B)A^{1/2}) \\ &= \operatorname{Tr}(UB^{1/2}f(B)B^{1/2}U^*) = \operatorname{Tr}(Ug(B)U^*) \ge \operatorname{Tr}(g(UBU^*)) \\ &= \operatorname{Tr}(g(A)) = \operatorname{Tr}(Af(A)).\end{aligned}$$

Now, let us prove the converse. Take arbitrary elements A, C in M_n^+. Then (1) yields

$$\operatorname{Tr}(Af(A)) \le \operatorname{Tr}(Af(A+C))$$

and

$$\operatorname{Tr}(Cf(C)) \le \operatorname{Tr}(Cf(A+C)).$$

Summing these two inequalities we obtain that the superadditivity inequality

$$\operatorname{Tr}(g(A)) + \operatorname{Tr}(g(C)) \le \operatorname{Tr}(g(A+C))$$

appears to hold for all $A, C \in M_n^+$. By [6, Theorem 2] (see, also, [7, Theorem 1]), the latter implies that g is convex. □

In what follows $w\colon \mathbb{R} \to \mathbb{R}^+$ is a "weight function".

Proposition 2. *Let $A \in M_n^h$ and $A = A_1 - A_2$ for some $A_1, A_2 \in M_n^+$. Then*

$$\operatorname{Tr}(w(A)A^+) \le \operatorname{Tr}(w(A)A_1).$$

Proof. Set $P = \chi_{(0,+\infty)}(A)$, $P' = \chi_{(-\infty,0]}(A)$. Then $A^+ = PA$ and we have:

$$\begin{aligned}\operatorname{Tr}(w(A)A^+) &\le \operatorname{Tr}(w(A)A^+) + \operatorname{Tr}(w(A)PA_2) \\ &= \operatorname{Tr}(w(A)(PA + PA_2)) = \operatorname{Tr}(w(A)PA_1) \\ &\le \operatorname{Tr}(w(A)(PA_1 + P'A_1)) = \operatorname{Tr}(w(A)A_1).\end{aligned}$$

□

Corollary. *Let $A, B \in M_n^h$ be such that $A \le B$. Then*

$$\operatorname{Tr}(w(A)A^+) \le \operatorname{Tr}(w(A)B^+).$$

Proof. This follows from the equality $A = B^+ - (B^- + (B - A))$. □

Theorem 3. *Let $\mathcal{D}$ be a convex subset of $\mathbb{R}$ and let a function $f\colon \mathcal{D} \to \mathbb{R}$ be nondecreasing and convex. Then*

$$\operatorname{Tr}(w(A)f(A)) \le \operatorname{Tr}(w(A)f(B)) \tag{2}$$

provided that $A, B \in M_n^h$, $\sigma(A), \sigma(B) \subset \mathcal{D}$, and $A \le B$.

Proof. Take a real number λ, put $w_1(x) \equiv w(x+\lambda)$ and consider the function $\gamma_\lambda(x) \equiv (x-\lambda)^+$. By Corollary, we have

$$\begin{aligned}\mathrm{Tr}(w(A)\gamma_\lambda(A)) &= \mathrm{Tr}(w_1(A-\lambda I_n)(A-\lambda I_n)^+)\\ &\le \mathrm{Tr}(w_1(A-\lambda I_n)(B-\lambda I_n)^+)\\ &= \mathrm{Tr}(w(A)\gamma_\lambda(B)).\end{aligned}$$

Now, let λ_i $(i = 1, 2, \dots, k)$ be the points of the finite set $\sigma(A) \cup \sigma(B)$ indexed in the increasing order. As is easily seen, we can find the numbers $\mu_0 \in \mathbb{R}$, $\mu_i \in \mathbb{R}^+$ $(i = 1, 2, \dots, k-1)$ such that the function $\widetilde{f}(x) \equiv \mu_0 + \sum_{i=1}^{k-1} \mu_i \gamma_{\lambda_i}(x)$ coincides with $f(x)$ at the points λ_i $(i = 1, 2, \dots k)$. By the above calculation,

$$\begin{aligned}\mathrm{Tr}(w(A)f(A)) &= \mathrm{Tr}(w(A)\widetilde{f}(A))\\ &\le \mathrm{Tr}(w(A)\widetilde{f}(B))\\ &= \mathrm{Tr}(w(A)f(B)).\end{aligned}$$

□

Notice that the proof of Proposition 2 and Theorem 3 is adapted from [5].

Now we turn to the question whether some special cases of inequalities (1), (2) characterize scalar multiples of the trace among all positive linear functionals. The following proposition complements Theorem 1.

Proposition 4. *Let a function* $f\colon \mathbb{R}^+ \to \mathbb{R}$ *be Borel measurable and let a positive linear functional* φ *on* M_n *be such that*

$$\varphi(Af(A)) \le \frac{1}{2}\varphi(Af(B) + f(B)A) \tag{3}$$

whenever $0 \le A \le B$. *Then either* f *is constant on* $(0, +\infty)$ *or* φ *is a scalar multiple of the trace.*

Proof. Let $A, C \in M_n^+$. Then (3) yields

$$\varphi(Af(A)) \le \frac{1}{2}\varphi(Af(A+C) + f(A+C)A)$$

and

$$\varphi(Cf(C)) \le \frac{1}{2}\varphi(Cf(A+C) + f(A+C)C).$$

Summing these two inequalities we get

$$\varphi(g(A)) + \varphi(g(C)) \le \varphi(g(A+C)),$$

where $g(x) = xf(x)$, and it remains to apply [7, Theorem 1]). □

Recently it was proved [1], [3] that if for a positive linear functional φ on M_n the inequality $0 \le A \le B$ entails $\varphi(A^2) \le \varphi(B^2)$, then φ is a scalar multiple of the trace. We will show that similar characterization can be obtained within the framework of weighted inequalities. To do this, we need the following lemma. Its proof uses some constructions from [3].

Lemma 5. *Let p, q be unequal positive numbers. If for a positive linear functional φ on M_n the inequality*

$$\varphi(X^pYX^q + X^qYX^p) \ge 0 \tag{4}$$

holds for all $X, Y \in M_n^+$ then φ is a scalar multiple of the trace.

Proof. Clearly, without loss of generality we can assume that $p > q = 1$. Moreover, it suffices to study the case $n = 2$, $\varphi = \mathrm{Tr}(S\cdot)$, $S = \mathrm{diag}(s, 1)$, $0 \le s \le 1$, and to prove that $s = 1$ (cf., e. g., [1] or [3]).

Take $\alpha \in (0, 1]$, $r \in (0, 1 - \frac{1}{p})$ and consider the matrices

$$A = \begin{pmatrix} \alpha^r & 0 \\ 0 & \alpha \end{pmatrix}, \qquad B = \begin{pmatrix} \alpha^2 & \alpha\sqrt{1-\alpha^2} \\ \alpha\sqrt{1-\alpha^2} & \sqrt{1-\alpha^2} \end{pmatrix}.$$

We have

$$\begin{aligned} &A^pBA + ABA^p \\ =&\alpha^{2+pr} \begin{pmatrix} 2\alpha^r & (1+\alpha^{(p-1)(1-r)})\sqrt{1-\alpha^2} \\ (1+\alpha^{(p-1)(1-r)})\sqrt{1-\alpha^2} & 2\alpha^{p-1-pr}(1-\alpha^2) \end{pmatrix}. \end{aligned}$$

If we consider the unitary matrix

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

and substitute $X = UAU^*$, $Y = UBU^*$ into (4), then we obtain

$$\begin{aligned} &\varphi(X^pYX + XYX^p) \\ =&\varphi(U(A^pBA + ABA^p)U^*) \\ =&\alpha^{2+pr}\varphi\big(U \begin{pmatrix} 2\alpha^r & (1+\alpha^{(p-1)(1-r)})\sqrt{1-\alpha^2} \\ (1+\alpha^{(p-1)(1-r)})\sqrt{1-\alpha^2} & 2\alpha^{p-1-pr}(1-\alpha^2) \end{pmatrix} U^*\big) \\ \ge&0. \end{aligned}$$

Reducing α^{2+pr}, tending α to $+0$ and calculating we get

$$0 \le \varphi\big(U \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} U^*\big) = \mathrm{Tr}\big(S \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\big) = s - 1,$$

i. e., $s \ge 1$, and we conclude that $s = 1$. $\square$

Theorem 6. *Let r be a positive number. Let a positive linear functional φ on M_n be such that*

$$\varphi(A^{2+2r}) \leq \varphi(A^r B^2 A^r) \tag{5}$$

whenever $0 \leq A \leq B$. Then φ is a scalar multiple of the trace.

Proof. Let X, Y be arbitrary matrices in M_n^+ and t be a positive number. Substituting $A = X$, $B = X + tY$ into (5) we obtain

$$t\varphi(X^{r+1}YX^r + X^rYX^{r+1}) + t^2\varphi(X^rY^2X^r) \geq 0,$$

i. e.,

$$\varphi(X^{r+1}YX^r + X^rYX^{r+1}) + t\varphi(X^rY^2X^r) \geq 0,$$

Tending t to $+0$ we get

$$\varphi(X^{r+1}YX^r + X^rYX^{r+1}) \geq 0,$$

which implies, by Lemma 5, that φ is a scalar multiple of the trace. □

References

[1] Bikchentaev A.M. and Tikhonov O.E., *Characterization of the trace by monotonicity inequalities,* Linear Algebra Appl., **422** (2007), 274–278.

[2] Brown L.G. and Kosaki H., *Jensen's inequality in semi-finite von Neumann algebras,* J. Operator Theory, **23** (1990), 3–19.

[3] Sano T. and Yatsu T., *Characterizations of tracial property via inequalities,* J. Inequal. Pure Appl. Math., **7**(1) (2006), article 36.

[4] Tikhonov O.E., *Convex functions and inequalities for a trace,* in *Constructive theory of functions and functional analysis,* No. 6 (Russian), Kazan State University, Kazan, 1987, pp. 77–82.

[5] Tikhonov O.E., *Trace inequalities for spaces in spectral duality,* Studia Mathematica, **104** (1993), 99–110.

[6] Tikhonov O.E., *On matrix-subadditive functions and a relevant trace inequality,* Linear Multilinear Algebra, **44** (1998), 25–28.

[7] Tikhonov O.E., *Subadditivity inequalities in von Neumann algebras and characterization of tracial functional,* Positivity, **9** (2005), 259–264.

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