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POSITIVE PROJECTIONS AS GENERATORS OF J-PROJECTIONS OF TYPE (B)

(submitted by O. E. Tikhonov)

ABSTRACT. Let $\mathcal{A}$ be a von Neumann J-algebra of type (B) acting in an indefinite metric space. The aim of the paper is to study J-projections from $\mathcal{A}$.

In ([1], Chapter XII) the problem of construction of probability theory for quantum mechanics is posed. An analog of boolean algebra of events is quantum logic. An important interpretation of a quantum logic is the set $B(H)^{pr}$ of all orthogonal projections on a Hilbert space H. In construction of measure theory on logics of projections it is important to know the properties and the structure of projections. The problem to construct a quantum field theory sometimes leads to an indefinite metric space ([3]). In indefinite case, the set $\mathcal{P}$ of all J-orthogonal projections is an analog of the logics $B(H)^{pr}$. In the present paper we study J-projections from von Neumann J-algebras of type (B). The main results of this paper were announced in [6].

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1. Introduction

Let H be a complex Hilbert space with an inner product $(.,.)$ and let $B(H)$ be the set of all bounded linear operators in H. Fix a self-adjoint symmetry operator J $(J = J^* = J^{-1},\ J \neq \pm I)$. The form $[x, y] := (Jx, y)$ is said to be an *indefinite metric*, and H with $[.,.]$ is said to be the *Krein space* (J-*space*) (see [2]). Put $P^+ := \frac{1}{2}(I + J)$ and $P^- := I - P^+$. Put also $\Gamma^+ \equiv \{f \in H : [f, f] = 1\}$ and $\Gamma^- \equiv \{f \in H : [f, f] = -1\}$. It is clear that $J\Gamma^\pm = \Gamma^\pm$. The set $\Gamma := \Gamma^+ \cup \Gamma^-$ is an indefinite analog of the unit sphere S of H. Let $A \in B(H)$. The operator $A^\# := JA^*J$ is said to be J-*adjoint* of A. Note that $[Ax, y] = [x, By]$ for all $x, y \in H$ and some $B \in B(H)$ if and only if $B = A^\#$. An operator A is said to be J-*self-adjoint* (J-*positive*, J-*negative*) if $[Ax, y] = [x, Ay]$ ($[Ax, x] \geq 0$, $[Ax, x] \leq 0$) for all $x, y \in H$. Note that A is J-self-adjoint (J-positive, J-negative) if and only if JA is self-adjoint (positive, negative, respectively).

An operator $p \in B(H)$ is said to be a *projection* if $p^2 = p$. Any one-dimensional projection has the form $(., x)y$ where x, $y \in H$ with $(y, x) = 1$. Let $\mathcal{P} := \{p \in B(H) : p^2 = p = p^\#\}$. Thus $\mathcal{P}$ is the set of all J-orthogonal (J-self-adjoint) projections in $B(H)$. Any $p \in \mathcal{P}$ is said to be the J-*projection*. Let $\mathcal{P}^+$ ($\mathcal{P}^-$) be the set of all J-positive (J-negative, respectively) J-projections. It is clear that $\mathcal{P}^+ \cap \mathcal{P}^- = \{0\}$. Any one-dimensional J-projection has the form $p_f := [f, f][., f]f$, $f \in \Gamma$. A vector $f \in \Gamma^+$ ($f \in \Gamma^-$) if and only if p_f, $p_{Jf} \in \mathcal{P}^+$ ($\in \mathcal{P}^-$, respectively).

2. Projections of type (B)

A von Neumann algebra $\mathcal{A}$ in H is said to be a *von Neumann J-algebra* if $A \in \mathcal{A}$ implies $A^\# \in \mathcal{A}$. Following [4], a commutative von Neumann J-algebra $\mathcal{Z}$ is said to be a type (B) algebra if $\mathcal{Z}$ contains a pair P, $Q \in B(H)^{pr}$ such that $P + Q = I$, $Q^\# = P$. A von Neumann J-algebra $\mathcal{A}$ is said to be of type (B) if its center ($=\mathcal{A} \cap \mathcal{A}'$) is of type (B).

Throughout the rest of the paper, $\mathcal{A}$ is a von Neumann J-algebra of type (B), a pair P, Q of orthogonal projections in $\mathcal{A} \cap \mathcal{A}'$ satisfying $P + Q = I$, $Q^\# = P$ is assumed to be fixed. Set $\mathcal{B} := PB(H)P + QB(H)Q$. It is clear that $\mathcal{B}$ is a von Neumann J-algebra of type (B) and $\mathcal{A} \subseteq \mathcal{B}$. Put $\mathcal{P}^{\mathcal{A}} = \mathcal{A} \cap \mathcal{P}$, $\mathcal{P}^{\mathcal{B}} = \mathcal{B} \cap \mathcal{P}$, and $\mathcal{J} := P - Q$. Clearly,

$$\mathcal{J}\Gamma^+ = \Gamma^-, \qquad \mathcal{J}J = -J\mathcal{J}, \quad \text{i.e.,} \quad -\mathcal{J} = \mathcal{J}^\#. \tag{1}$$

Let p, q be projections. Put $p \leq q$ if $pq = qp = p$. Note that if p, $q \in \mathcal{P}$ then $pq = p$ if and only if $qp = p$. With respect to the standard relations, namely, *the ordering* $\leq$, *the orthogonal relation* $p \perp q$ if and only if $pq = qp = 0$, and *the orthocomplementation* $p \mapsto p^{\perp} := I - p$, the set $\mathcal{P}$ is a *quantum* logic and $\mathcal{P}^{\mathcal{A}}$ is a sublogic. Any J-projection from $\mathcal{P}^{\mathcal{B}}$ is said to be a *J-projection of type* (B).

Proposition 1. *The function $\sigma(P) := \mathcal{J}P\mathcal{J}$ is an automorphism of $\mathcal{P}$, for which $\sigma(\mathcal{P}^+) = \mathcal{P}^-$ and $\sigma(P^+) = P^-$.*

Proof. Let $R \in \mathcal{P}$. Then $(\sigma(R))^2 = \sigma(R)$ and $J(\sigma(R))^*J = \mathcal{J}JR^*J\mathcal{J} = \sigma(R)$. Thus $\sigma(R) \in \mathcal{P}$. It is clear that $\sigma(\sigma(R)) = R$. Hence $\sigma(\mathcal{P}) = \mathcal{P}$. Furthermore, $R_1 \leq R_2$ implies $\sigma(R_1) \leq \sigma(R_2)$, $R_1 \perp R_2$ implies $\sigma(R_1) \perp \sigma(R_2)$, and $\sigma(R^{\perp}) = (\sigma(R))^{\perp}$.

Now, let $R \in \mathcal{P}^+$. Then $J(\mathcal{J}R\mathcal{J}) = -\mathcal{J}(JR)\mathcal{J}$ is a negative operator. Hence $\sigma(R)$ is a J-negative J-projection. By (1), $\sigma(P^+) = \frac{1}{2}\mathcal{J}(I+J)\mathcal{J} = \frac{1}{2}(I + \mathcal{J}J\mathcal{J}) = \frac{1}{2}(I - J) = P^-$. □

We will see below (Theorem 1) that points of $\mathcal{P}$ being invariant under σ form the logic $\mathcal{P}^{\mathcal{B}}$.

Proposition 2. *Let $\mathcal{A}^{Js}$ be the set of all J-self-adjoint operators in the von Neumann J-algebra $\mathcal{A}$ of type (B) and let $A \in \mathcal{A}$. Then $A \in \mathcal{A}^{Js}$ if and only if $A = A_P + JA_P^*J$, where $A_P := PAP$.*

Proof. Let $A = A^{\#}$. Then $A = AP + AQ = PAP + (JP)(JAJ)(PJ) = A_P + JA_P^*J$. Conversely, let $A = A_P + JA_P^*J$. Then $A^{\#} = JA^*J = J(A_P^* + JA_PJ)J = JA_P^*J + A_P = A$. □

Corollary 1. *$P\mathcal{A}P \cap \mathcal{A}^{Js} = \{0\}$, $Q\mathcal{A}Q \cap \mathcal{A}^{Js} = \{0\}$.*

Let us denote by $\mathcal{L}_P^{\mathcal{A}}$ the set of all projections from $P\mathcal{A}P$.

Proposition 3. *$\mathcal{P}^{\mathcal{A}} = \{q + Jq^*J : q \in \mathcal{L}_P^{\mathcal{A}}\}$.*

Proof. Let $R \in \mathcal{P}^{\mathcal{A}}$ and $q := PR$. By Proposition 2, $R = PRP + J(PRP)^*J = R_P + JR_P^*J$. Since $(R_P)^2 + J(R_P^*)^2J = R^2 = R = R_P + JR_P^*J$, $(R_P)^2 = R_P$. Hence $R_P \in \mathcal{L}_P^{\mathcal{A}}$.

Conversely, let $q \in \mathcal{L}_P^{\mathcal{A}}$. Then $Jq^*J = q^{\#} \in \mathcal{A}$. Hence $q + Jq^*J \in \mathcal{A}^{Js}$. In addition, $(q+Jq^*J)^2 = q+Jq^*J$. By Proposition 2, $q+Jq^*J \in \mathcal{P}^{\mathcal{A}}$. □

If $\dim H = \infty$ *then the logic $\mathcal{P}^{\mathcal{B}}$ is not a σ-logic* (cf. [5, Proposition 2]).

To prove this fact, we will construct a sequence of mutually orthogonal J-projections $\{R_n\}_1^{\infty} \subset \mathcal{P}^{\mathcal{B}}$ such that the supremum $\sum R_n$ does not exist in $\mathcal{P}^{\mathcal{B}}$. Let $\{\phi_n\}_1^{\infty}$ be an orthonormal family in PH. (Note that $\{J\phi_n\}_1^{\infty}$ is an orthonormal family in QH.) Put $f_k \equiv (k+1)^{\frac{1}{2}}\phi_{2k} + k^{\frac{1}{2}}\phi_{2k-1}$ and

$f_k^- \equiv (k+1)^{\frac{1}{2}}\phi_{2k} - k^{\frac{1}{2}}\phi_{2k-1}$. Then $(f_k^-, f_k) = 1$, $r_k := (., f_k^-)f_k \in \mathcal{L}_P^{\mathcal{B}}$. By the construction, $\{R_k := r_k + Jr_k^* J\}_1^\infty$ is an orthogonal sequence of J-projection (by Proposition 3) from $\mathcal{P}^{\mathcal{B}}$.

Assume now that there exists the supremum $R := \sum\limits_1^\infty R_n \in \mathcal{P}^{\mathcal{B}}$. Put

$$P_m \equiv \sum_1^m R_k + \sum_{2m+1}^\infty ((., J\phi_n)J\phi_n + (., \phi_n)\phi_n).$$

Then $P_m \in \mathcal{P}^{\mathcal{B}}$, $P_m \geq P_{m+1}$, $\forall m$, and $P_m \geq R_n$, $\forall m, n$. Hence $P_m \geq R = \sum\limits_1^\infty R_n \geq \sum\limits_1^m R_k$. Thus $P_m - \sum\limits_1^m R_k \geq R - \sum\limits_1^m R_k$ and $P_m - \sum\limits_1^m R_k \geq (R - \sum\limits_1^m R_k)^*$. Finally,

$$RR^* = (R - \sum_1^m R_k)(R - \sum_1^m R_k)^* + \sum_1^m R_k R_k^*$$

$$= (R - \sum_1^m R_k)(R - \sum_1^m R_k)^* + \sum_1^m (2k+1)((., f_k)f_k + (., Jf_k^-)Jf_k^-), \ \forall m.$$

We get a contradiction, since the norm of the right hand side expression tends to infinity when $m \to \infty$.

Theorem 1. *Let R be a J-projection. The following conditions are equivalent:*

1) *R has type (B);*
2) *$R = \mathcal{J}R\mathcal{J}$;*
3) *$QRP = PRQ = 0$.*

Proof. Let R have type (B), i.e., $R \in \mathcal{P}^{\mathcal{B}}$. Then $R = PRP + QRQ$, hence $R = \mathcal{J}R\mathcal{J}$.

Now, let $R = \mathcal{J}R\mathcal{J}$. Then $\mathcal{J}R = R\mathcal{J}$, hence $PR = \frac{1}{2}(I + \mathcal{J})R = \frac{1}{2}(R + R\mathcal{J}) = RP$, and consequently, $PRQ = RPQ = 0$. Similarly, $QRP = 0$.

Now, let $PRQ = QRP = 0$. Then $R = PRP + QRQ$, hence $R \in \mathcal{P}^{\mathcal{B}}$. $\square$

Corollary 2. *Let $f \in \Gamma$ and $[f, \mathcal{J}f] = 0$. Then $p_f + p_{\mathcal{J}f}$ is a minimal J-projection of type (B).*

It appears interesting to compare Theorem 1 with the following proposition.

Proposition 4. 1) *If either $R \in \mathcal{P}^+$ or $R \in \mathcal{P}^-$ and either $PRQ = 0$ or $QRP = 0$, then $R = 0$.*

2) *Let $\dim H \geq 4$. Then there is $R \in \mathcal{P}$ such that $PRQ = 0$, $QRP \neq 0$.*

3) *If $R \in \mathcal{P}$ and $PRQ = 0$, then PRP and QRQ are projections.*

Proof. 1) Let us consider, for instance, the case $R \in \mathcal{P}^+$ and $PRQ = 0$. We have $0 = JPRQ = Q(JR)Q$. Since JR is a positive operator, the latter implies that $P(JR)P = JR$. Thus $R = JP(JR)P = (JPJ)RP = QRP$. Finally, $R = R^2 = (QRP)(QRP) = 0$.

2) Let $\dim H \geq 4$. Then we can find $\varphi_P \in PH \cap S$ and $\varphi_Q \in QH \cap S$ such that $(\varphi_Q, J\varphi_P) = 0$. Put $f_P := \varphi_P = g_P$ and $f_Q := \frac{1}{2}J\varphi_P + \varphi_Q$, $g_Q := -\frac{1}{2}J\varphi_P + \varphi_Q$. Let us define $f = f_P + f_Q$, $g = g_P + g_Q$. It is easy to verify that $f \in \Gamma^+$, $g \in \Gamma^-$ and $[f, g] = 0$. Thus $p_f + p_g \in \mathcal{P}$. Finally, $P(p_f + p_g)Q = P([., f]f - [., g]g)Q = ((., Jf_P)f_P - (., Jg_P)g_P) = 0$ and $Q(p_f + p_g)P = (., Jf_Q)f_Q - (., Jg_Q)g_Q \neq 0$.

3) Now, let $PRQ = 0$. Then

$$PRP = (PRP)(PRP) + (PRQ)(QRP) = (PRP)(PRP)$$

and

$$QRQ = (PRP)^{\#} = (PRP)^{\#}(PRP)^{\#} = (QRQ)(QRQ).$$

□

By Theorem 1 and Proposition 4.1), we immediately get

Corollary 3. *Let $\mathcal{A}$ be a von Neumann J-algebra of type (B). Then $\mathcal{P}^{\mathcal{A}} \cap \mathcal{P}^+ = \{0\} = \mathcal{P}^{\mathcal{A}} \cap \mathcal{P}^-$.*

Proposition 5. *Let $R \in \mathcal{P}$ be an orthoprojection. Then:*

1) *$P^+R = RP^+$, $P^-R = RP^-$;*

2) *if, in addition, R has type (B) then $\mathcal{J}P^+R\mathcal{J} = P^-R$ and $R = P^+R + \mathcal{J}(P^+R)\mathcal{J}$.*

Proof. 1) Since $R \in \mathcal{P}$ and $R = R^*$, we have $JR = R^*J = RJ$. Hence $P^{\pm}R = RP^{\pm}$.

2) If, in addition, $R = \mathcal{J}R\mathcal{J}$ then $\mathcal{J}P^+R\mathcal{J} = (\mathcal{J}P^+\mathcal{J})\mathcal{J}R\mathcal{J} = P^-R$. This means that $R = (P^+ + P^-)R = P^+R + \mathcal{J}(P^+R)\mathcal{J}$. □

Proposition 6. *Let q, $p \in \mathcal{P}$ and let q be a J-projection of type (B). Then $p \leq q$ if and only if $\mathcal{J}p\mathcal{J} \leq q$.*

Proof. The equality $p = qp$ is equivalent to

$$\mathcal{J}p\mathcal{J} = \mathcal{J}qp\mathcal{J} = \mathcal{J}p\mathcal{J}(\mathcal{J}q\mathcal{J}) = \mathcal{J}p\mathcal{J}q.$$

□

In what follows we will use equations (2) (see below). Let $A \in \mathcal{A}^{Js}$. This means that $JA = A^*J$. Hence

$$P^+AP^+ = P^+(JA)P^* = P^+(A^*J)P^+ = P^+A^*P^+,$$

$$P^-AP^+ = -P^-(A^*J)P^+ = -P^-A^*P^+ = -(P^+AP^-)^*, \tag{2}$$

$$P^-AP^- = -P^-JAP^- = -P^-A^*JP^- = P^-A^*P^-.$$

Consider some properties of projections. Let r be a bounded projection on H. Let us denote by r_{or} the orthogonal projection onto $rH \cap r^*H$. By Proposition 4 [5], r_{or} is the greatest orthogonal projection with the property $r_{or} \leq r$. A projection r is said to be *properly skew projection* if $r_{or} = 0$. Let $x,\, y \in H$, $(x, y) = 1$ and $\|x\| = \|y\|$. Then $(., x)y$ is the properly skew projection if and only if $\|x\| > 1$.

The orthoprojection r_{or} is said to be an *orthogonal component* of r, and $r_s := r - r_{or}$ is said to be a *properly skew component* of r. It is clear that $\mathcal{J} r_{or} \mathcal{J}$ ($\mathcal{J} r_s \mathcal{J}$) is the orthogonal (properly skew, respectively) component of $\mathcal{J} r \mathcal{J}$.

Remark 1. 1) *Let r be a bounded projection. Then $r \neq r^*$ implies that r_s is a properly skew projection.*

2) *$R \in \mathcal{P}^{\mathcal{A}}$ implies $R_{or} \in \mathcal{P}^{\mathcal{A}}$. Let $R = r + Jr^*J$, where $r \in \mathcal{L}_P^{\mathcal{A}}$. Then $R_{or} = r_{or} + Jr_{or}J$.*

3) *$r \in \mathcal{L}_P^{\mathcal{A}}$ is a properly skew projection if and only if $R = r + Jr^*J$ is a properly skew projection.*

Now, let r be a properly skew projection. Let us denote by r_m the orthogonal projection onto rH. Then $r = (r_m + r_m^{\perp})r(r_m + r_m^{\perp}) = r_m + r_m r r_m^{\perp}$. Let $r_m r r_m^{\perp} = v|r_m r r_m^{\perp}|$ be the polar decomposition of $r_m r r_m^{\perp}$, where $|B| := (B^*B)^{1/2}$. Since r is a properly skew projection, $vH = r_m H$. Note that $v^*|r_m^{\perp} r^* r_m|$ is the polar decomposition of $r_m^{\perp} r^* r_m$ and $|r_m^{\perp} r^* r_m| = v|r_m r r_m^{\perp}|v^*$. It is clear that

$$|r - r^*| = |r_m r r_m^{\perp} - r_m^{\perp} r^* r_m| = |r_m r r_m^{\perp}| + |r_m^{\perp} r^* r_m|,$$

the cover projection of $|r_m^{\perp} r^* r_m|$ (of $|r_m r r_m^{\perp}|$) is equal to v^*v (vv^*, respectively) and

$$(v - v^*)|r - r^*| = v|r_m r r_m^{\perp}| - v^*|r_m^{\perp} r^* r_m| = r_m r r_m^{\perp} - r_m^{\perp} r^* r_m = r - r^*. \tag{3}$$

Put $r'_m := v^* r_m v$. By definition, $r'_m r_m = 0$ and $(v - v^*)(r_m + r'_m)(v - v^*)^* = r'_m + r_m$. A straightforward verification shows that

$$(v - v^*)[2(r_m + r'_m) - (r + r^*)](v - v^*)^* = r + r^*. \tag{4}$$

Put $x := r + r^*$ and note that

$$x^2 - 2x = (r - r^*)^*(r - r^*) = |r - r^*|^2 \geq 0. \tag{5}$$

Let $A_i \in B(H)$, $i = 1, .., 4$, be such that $PA_iP = A_i$, for all i. Let us identify $A_1 + A_2J + JA_3 + JA_4J$ with the matrix $\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$. Thus

$$P^+ = \frac{1}{2}(I+J) = \frac{1}{2}(I+JP+JQ) = \frac{1}{2}((P+JPJ)+JP+PJ) = \frac{1}{2}\begin{pmatrix} P & P \\ P & P \end{pmatrix}$$

and $P^- = \frac{1}{2}\begin{pmatrix} P & -P \\ -P & P \end{pmatrix}$. Put $X := \frac{1}{4}\begin{pmatrix} x & x \\ x & x \end{pmatrix}$.

Theorem 2. *Let $\mathcal{A}$ be a von Neumann J-algebra of type (B), $R = r + Jr^*J$, where $r \leq P$ and r is a properly skew projection from $\mathcal{A}$, and let $P^-RP^+ = U|P^-RP^+|$ be the polar decomposition for P^-RP^+. Then $X = P^+RP^+$, $JUJ = -U$ and*

$$R = X + U(X^2 - X)^{1/2} - (X^2 - X)^{1/2}U^* + U(I - X)U^*. \tag{6}$$

Conversely, let $x \in P\mathcal{A}P$ be such that $x = x^*$, $x^2 - 2x \geq 0$ and $w \in P\mathcal{A}P$ be a partial isometry with the initial subspace $\overline{xH}$. Then the formula (6) defines a J-projection. Here $U := \frac{1}{2}\begin{pmatrix} w & w \\ -w & -w \end{pmatrix}$. If, in addition, for x, w the equalities

$$x = w(2P - x)w^* \quad \text{and} \quad w(x^2 - 2x)^{1/2} = -(x^2 - 2x)^{1/2}w^* \tag{7}$$

hold, then R is a J-projection from $\mathcal{A}$.

Proof. A simple matrix computation shows that

$$P^+RP^+ = X \quad \text{and} \quad P^-XP^- = \frac{1}{4}\begin{pmatrix} x & -x \\ -x & x \end{pmatrix} \tag{8}$$

Using (2) with R in place of A and (5) we obtain

$$|P^-RP^+| = [(P^-RP^+)^*(P^-RP^+)]^{1/2} = [(-P^+RP^-)(P^-RP^+)]^{1/2} =$$

$$[(P^+RP^+)(P^+RP^+) - P^+RP^+]^{1/2} = (X^2 - X)^{1/2} = \frac{1}{4}\begin{pmatrix} |r - r^*| & |r - r^*| \\ |r - r^*| & |r - r^*| \end{pmatrix}.$$

Put $V := \frac{1}{2}\begin{pmatrix} (v - v^*) & 0 \\ 0 & -(v - v^*) \end{pmatrix}\begin{pmatrix} P & P \\ P & P \end{pmatrix} = \frac{1}{2}\begin{pmatrix} v - v^* & v - v^* \\ -(v - v^*) & -(v - v^*) \end{pmatrix}$.

Then

$$VV^* = \frac{1}{2}\begin{pmatrix} (vv^* + v^*v) & -(vv^* + v^*v) \\ -(vv^* + v^*v) & (vv^* + v^*v) \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} (r_m + r_m') & -(r_m + r_m') \\ -(r_m + r_m') & (r_m + r_m') \end{pmatrix} \leq P^-,$$

and $V^*V = \frac{1}{2}\begin{pmatrix} (vv^*+v^*v) & (vv^*+v^*v) \\ (vv^*+v^*v) & (vv^*+v^*v) \end{pmatrix} \leq P^+$. This means that V is a partial isometry. Since the cover projection of $|r-r^*|$ is equal to $r_m + r_m^{'}$, the cover projection of $|P^-RP^+|$ is equal to $\frac{1}{2}\begin{pmatrix} (r_m+r_m^{'}) & (r_m+r_m^{'}) \\ (r_m+r_m^{'}) & (r_m+r_m^{'}) \end{pmatrix}$. Hence we have

$$V(X^2-X)^{1/2} = \frac{1}{4}\begin{pmatrix} (v-v^*) & 0 \\ 0 & -(v-v^*) \end{pmatrix}\begin{pmatrix} |r-r^*| & |r-r^*| \\ |r-r^*| & |r-r^*| \end{pmatrix} =$$

$$\text{by (3)} \quad = \frac{1}{4}\begin{pmatrix} r-r^* & r-r^* \\ r-r^* & r-r^* \end{pmatrix} = P^-RP^+.$$

Thus $V = U$, $U(X^2-X)^{1/2} = P^-RP^+$, and, by (2),

$$-(X^2-X)^{1/2}U^* = P^+RP^-. \tag{9}$$

Moreover,

$$U(I-X)U^* = U(P^+-X)U^*$$

$$= \frac{1}{4}\begin{pmatrix} (v-v^*) & 0 \\ 0 & -(v-v^*) \end{pmatrix}\begin{pmatrix} 2P-x & 2P-x \\ 2P-x & 2P-x \end{pmatrix}\begin{pmatrix} v^*-v & 0 \\ 0 & -(v^*-v) \end{pmatrix}$$

$$= \text{ by (4) } = \frac{1}{4}\begin{pmatrix} x & -x \\ -x & x \end{pmatrix} = \text{ by (8) } = P^-RP^-. \tag{10}$$

Summarizing (8), (9), and (10) we have (6). By simple calculations we prove the equality $JUJ = -U$.

Conversely, let $x \in P\mathcal{A}P$ be such that $x = x^*$, $x^2 - 2x \geq 0$ and let $w \in P\mathcal{A}P$ be a partial isometry with the initial subspace $\overline{xH}$. Then one can directly show that (6) defines a J-projection.

Now, let (7) hold true for x, w. The matrix entry of R at the first row and the second column is equal to

$$\frac{1}{4}(x + w(x^2-2x)^{1/2} + (x^2-2x)^{1/2}w^* - w(2P-x)w^*) = 0,$$

and the matrix entry of R at the first row and the first column is equal to

$$\frac{1}{4}(x+w(x^2-2x)^{1/2}-(x^2-2x)^{1/2}w^*+w(2P-x)w^*) = \frac{1}{2}(x+w(x^2-2x)^{1/2}).$$

This means that $R = \begin{pmatrix} r & 0 \\ 0 & r^* \end{pmatrix}$ is a J-projection from $\mathcal{A}$. Here $r := \frac{1}{2}(x + w(x^2-x)^{1/2})$ is a projection from $P\mathcal{A}P$. $\square$

Using Proposition 5.2) we see that (6) is not true if $R_{or} \neq 0$.

Note that for any J-projection p there exists (non unique!) representation $p = p_+ + p_-$, where $p_+ \in \mathcal{P}^+$, $p_- \in \mathcal{P}^-$. Let us show that for any $\mathcal{R} \in \mathcal{P}^{\mathcal{B}}$ there is a unique special representation of such form.

Let us denote by $(pJ)_+$ ($(pJ)_-$) the positive (the negative, respectively) part of self-adjoint operator pJ, $p \in \mathcal{P}$.

Lemma 1. *If $p = p_+ + p_-$, $p \in \mathcal{P}$, $p_+ \in \mathcal{P}^+$, $p_- \in \mathcal{P}^-$ and the subspaces p_+H, p_-H are mutually orthogonal then $p_+ = (pJ)_+J$ and $p_- = -(pJ)_-J$.*

Proof. By the assumption on p_+ and p_-, we have $p_+J = (pJ)_+$ and $p_-J = -(pJ)_-$. Hence $p_+ = (p_+J)J = (pJ)_+J$ and $p_- = (p_-J)J = -(pJ)_+J$. $\square$

Theorem 3. *For any J-projection R of type (B) there exists a unique J-positive J-projection R_+^0 such that $R = R_+^0 + \mathcal{J}R_+^0\mathcal{J}$ and the subspaces R_+^0H, $\mathcal{J}R_+^0\mathcal{J}H$ are mutually orthogonal.*

Proof. Let us prove that the projection R_+^0 exists.

1) First, let R be an orthogonal J-projection of type (B). The J-projection P^+R (P^-R) is J-positive (J-negative, respectively). By Proposition 5.2), $R = P^+R + \mathcal{J}P^+R\mathcal{J}$. By the construction, the subspaces P^+RH and $\mathcal{J}P^+R\mathcal{J}H \subseteq P^-H$ are mutually orthogonal.

2) Now, let R be a properly skew J-projection. By Remark 1.3) and Theorem 2, (6) holds. By Theorem 1 and (10),

$$\mathcal{J}X\mathcal{J} = \mathcal{J}P^+RP^+\mathcal{J} = P^-(\mathcal{J}R\mathcal{J})P^- = U(I-X)U^*. \qquad (11)$$

Hence $\mathcal{J}(X^2 - X)^{1/2}\mathcal{J} = U(X^2 - X)^{1/2}U^*$. Thus

$$(X^2 - X)^{1/2}\mathcal{J}U = \mathcal{J}U(X^2 - X)^{1/2}. \qquad (12)$$

By Theorem 1 and (9),

$$\mathcal{J}(U(X^2-X)^{1/2})\mathcal{J} = \mathcal{J}(P^-RP^+)\mathcal{J} = P^+(\mathcal{J}R\mathcal{J})P^- = -(X^2-X)^{1/2}U^*.$$

Hence

$$-(X^2 - X)^{1/2}U^*\mathcal{J} = \mathcal{J}U(X^2 - X)^{1/2}. \qquad (13)$$

By (12) and (13), $\mathcal{J}U = -U^*\mathcal{J}$, i.e.

$$-U = \mathcal{J}U^*\mathcal{J}. \qquad (14)$$

Let us denote by F_+ (F_-) the cover projection of positive ($= X_+$) (negative ($= X_-$)) part of self-adjoint operator X, respectively. By (11),

$$\mathcal{J}X_+\mathcal{J} + \mathcal{J}(-X_-)\mathcal{J} = \mathcal{J}X\mathcal{J} = U(F_- + X_-)U^* + U(F_+ - X_+)U^*. \qquad (15)$$

Since $[U(F_- + X_-)U^*][U(F_+ - X_+)U^*] = 0$,

$$\mathcal{J}X_+\mathcal{J} = U(F_- + X_-)U^* \quad \text{and} \quad -\mathcal{J}X_-\mathcal{J} = U(F_+ - X_+)U^*. \tag{16}$$

By (14) and (16),

$$\mathcal{J}U(X_+^2 - X_+)^{1/2}\mathcal{J} = -U^*\mathcal{J}(X_+^2 - X_+)^{1/2}\mathcal{J} =$$
$$-U^*U(X_-^2 + X_-)^{1/2}U^* = -(X_-^2 + X_-)^{1/2}U^*. \tag{17}$$

Put

$$R_-^0 := -X_- + U(X_-^2 + X_-)^{1/2} - (X_-^2 + X_-)^{1/2}U^* + U(F_- + X_-)U^*,$$
$$R_+^0 := X_+ + U(X_+^2 - X_+)^{1/2} - (X_+^2 - X_+)^{1/2}U^* + U(F_+ - X_+)U^*.$$

It is easy to verify that $R_\pm^0 \in \mathcal{P}^\pm$, $R_+^0 R_-^0 = 0$, $R = R_+^0 + R_-^0$. Since $R_+^0 J$ is a positive operator, $R_-^0 J$ is a negative operator, and $(R_+^0 J)(R_-^0 J) = 0$, the subspaces $R_+^0 H$ and $R_-^0 H$ are mutually orthogonal. By (16), (17),

$$\mathcal{J}R_+^0\mathcal{J} = \mathcal{J}X_+\mathcal{J} + \mathcal{J}U(X_+^2 - X_+)^{1/2}\mathcal{J}$$
$$-\mathcal{J}(X_+^2 - X_+)^{1/2}U^*\mathcal{J} + \mathcal{J}U(F_+ - X_+)U^*\mathcal{J} =$$
$$U(F_- + X_-)U^* - (X_-^2 + X_-)^{1/2}U^* + U(X_-^2 + X_-)^{1/2} - X_- = R_-^0$$

3) Now, let us consider the general case of R. We have $R = R_{or} + R_s$. We know that R_{or} is an orthogonal J-projection of type (B), and R_s is a properly skew J-projection of type (B). Let $(R_{or})_+ := P^+R_{or}$ and let $(R_s)_+$ be the J-projection from step 2), generated by R_s. Put $R_+^0 := (R_{or})_+ + (R_s)_+$. Thus the projection R_+^0 is that in question.

By Lemma 1, the J-projection R_+^0 is unique. □

A J-projection $R \in \mathcal{P}^+$ is said to be a *generator* (for a J-projection $\mathcal{R}$) if $\mathcal{R} = R + \mathcal{J}R\mathcal{J}$ and the subspaces RH, $\mathcal{J}RH$ are mutually orthogonal. Let F^+ (F^-) be the cover projection of P^+RP^+ (of P^-RP^-, respectively).

Theorem 4. *Let $R \in \mathcal{P}^+$ and let $P^-RP^+ = U|P^-RP^+|$ be the polar decomposition of P^-RP^+. Then R is a generator if and only if the subspaces $\mathcal{J}F^+H$ and UH are mutually orthogonal.*

Proof. 1) Let R be a generator. Put $\mathcal{R} := R + \mathcal{J}R\mathcal{J}$. By Theorem 3, $R = \mathcal{R}_+^0$. Here $\mathcal{R}_+^0$ is the generator for $\mathcal{R}$ from the proof of Theorem 3. From (16) it follows that UH and $\mathcal{J}F^+H$ are mutually orthogonal.

2) Let us prove some properties. Let $R \in \mathcal{P}^+$. Then

$$P^+RP^+ = P^+(JR)P^+ \quad \text{is a positive operator and}$$
$$P^-RP^- = -P^-(JR)P^- \quad \text{is a negative operator.} \tag{18}$$

Furthermore, $P^+R^*RP^+ = P^+(JR^*)RP^+ = P^+RR^*JP^+ = P^+RR^*P^+$ and $P^-R^*RP^- = P^-RR^*P^-$. We have $(P^+RP^+x, x) \geq (F^+x, x)$ for all $x \in H$ (see Proposition 1, [7]) and

$$|RP^+|^2 = P^+R^*RP^+ = P^+RJRP^+ =$$

$$P^+R(2P^+ - I)RP^+ = 2(P^+RP^+)^2 - P^+RP^+.$$

a) Hence follows that *the initial projection of* RP^+ *is equal to the cover projection of* P^+RP^+ (*i.e., is equal to* F^+).

In the same way we can prove the following assertion.

b) *The final projection of* P^-R *is equal to the cover projection of* P^-RP^- (*i.e.* F^-). We have

$$(P^-RP^+)(P^+R^*P^-) = -P^-RP^+JP^+RP^- = -P^-RP^+RP^- =$$

$$-P^-R(I - P^-)RP^- = (P^-RP^-)^2 - P^-RP^-.$$

c) From (18) it follows that *the final projection of* P^-RP^+ (*and hence of* U) *is equal to the cover projection of* P^-RP^- (*i.e., is equal to* F^-)

By b), $P^-R = F^-P^-R$. By a), $RP^+ = RP^+F^+$. Hence

$$RP^+\mathcal{J}R\mathcal{J} = RP^+\mathcal{J}P^-R\mathcal{J} = RP^+(F^+\mathcal{J}F^-)P^-R\mathcal{J}.$$

Now, let $R \in \mathcal{P}^+$ be a J-projection such that UH and $\mathcal{J}F^+H$ are mutually orthogonal. Hence $0 = \mathcal{J}F^+\mathcal{J}(UU^*)$, i.e. by c), $F^+\mathcal{J}F^- = 0$. Finally, $RP^+\mathcal{J}R\mathcal{J} = RP^+(F^+\mathcal{J}F^-)P^-R\mathcal{J} = 0$. Similarly, $RP^-(\mathcal{J}R\mathcal{J}) = 0$. Thus $R(\mathcal{J}R\mathcal{J}) = R(P^+ + P^-)\mathcal{J}R\mathcal{J} = 0$. Hence

$$R + \mathcal{J}R\mathcal{J} \in \mathcal{P}^{\mathcal{B}}, \quad RJ(\mathcal{J}R\mathcal{J}) = R(P^+ - P^-)(\mathcal{J}R\mathcal{J}) = 0.$$

This means that RH and $\mathcal{J}RH$ are mutually orthogonal. Thus R is a generator. □

Let R, $\mathcal{R}$ be J-projections of type (B) and R_+, $\mathcal{R}_+$ be their generators. Then $R\mathcal{R} = 0$ (i.e., the subspaces RH, $\mathcal{R}H$ are J-mutually orthogonal) implies $R + \mathcal{R} = (R_+ + \mathcal{R}_+) + \mathcal{J}(R_+ + \mathcal{R}_+)\mathcal{J}$. But the J-positive J-projection $R_+ + \mathcal{R}_+$ is not a generator for $R + \mathcal{R}$, in general.

3. Two-dimensional (minimal) J-projections of type (B)

In what follows we denote by $\Re$ and $\Im$ the real and the imaginary part of a complex number.

Let $f \in H$. Put $f_P := Pf$, $f_Q := Qf$. In terms of f_P, f_Q we can give another simple description for vectors from Γ.

Proposition 7. 1) $\|f_P\| = \|f_Q\|$ *if and only if* $(f, \mathcal{J}f) = 0$ *if and only if* $\Re(\mathcal{J}P^+f, P^-f) = 0$*;*

2) $f \in \Gamma^+$ $(\in \Gamma^-)$ *if and only if* $2\Re[f_Q, f_P] = 1$ $(= -1)$ *if and only if* $2[f_Q, f_P] - [f, \mathcal{J}f] = 1$ $(= -1)$*;*

3) $[f, \mathcal{J}f] = 2\Im[f_Q, f_P]$ $\forall$ $f \in H$*.*

In connection with Corollary 2 we formulate the following elementary proposition.

Proposition 8. *Let* $f \in \Gamma$*. The following conditions are equivalent:*

1) $p_f p_{\mathcal{J}f} = 0$*;*

2) $[f, \mathcal{J}f] = 0$*;*

3) $\Im[f_Q, f_P] = 0$*;*

4) $(2[f_Q, f_P])^2 = 1$*;*

5) $\Im(\mathcal{J}P^+f, P^-f) = 0$.

Lemma 2. *Let* x*,* $y \in PH$*,* $(x, y) = 1$*. The set* $\mathcal{K} := \{f \in \Gamma^+ : [f, \mathcal{J}f] = 0,\ f_P = \alpha y,\ f_Q = \beta Jx\}$ *is infinite.*

Proof. Let α, β be real numbers such that $2\beta\alpha = 1$. Put $g_P = \alpha y$, $g_Q = \beta Jx$ and $g = g_P + g_Q$. Then $1 = 2\beta\alpha = 2[g_Q, g_P] = 2\Re[g_Q, g_P]$. By Proposition 7 (2), $g \in \Gamma^+$. Since $\Im[g_Q, g_P] = 0$, we have $[g, \mathcal{J}g] = 0$, by Proposition 7 (3). Thus $g \in \mathcal{K}$. Since the set $\{(\alpha, \beta) :\ 2\beta\overline{\alpha} = 1\}$ is infinite, the set $\mathcal{K}$ is infinite too. □

Thus we have

Corollary 4. *For any non-zero* J*-projection* R *of type* (B) *there exists an infinite set of* J*-positive* J*-projections* R^0 *such that*

$$R = R^0 + \mathcal{J}R^0\mathcal{J}.$$

Proof. 1) First, let Q be a two-dimensional J-projection of type (B). Then $Q = (., x)y + (., Jy)Jx$, where x, $y \in PH$ and $(x, y) = 1$. Let $\mathcal{K}$ be from Lemma 2 and let $f \in \mathcal{K}$. Since $[f, \mathcal{J}f] = 0$, we have $p_f p_{\mathcal{J}f} = 0$, by Proposition 8. This means that $p_f + p_{\mathcal{J}f}$ is a two-dimensional J-projection of type (B). Since $\mathcal{J}f, f \in ran(Q)$, we have $Q = p_f + p_{\mathcal{J}f}$. Since $\mathcal{K}$ is infinite and p_f is a J-positive J-projection, Corollary 4 is true for the case of two-dimensional J-projection.

2) Let us consider the general case. For any non-zero J-projection R of type (B) we can find a two-dimensional J-projection Q of type (B) such that $Q \leq R$. Then $R - Q$ is also a J-projection of type (B). Let R^0_+ be the J-positive J-projection such that R^0_+H, $\mathcal{J}R^0_+H$ are mutually orthogonal (Theorem 3) and $R - Q = R^0_+ + \mathcal{J}R^0_+\mathcal{J}$. The set $\{R^0_+ + p_f \colon f \in \mathcal{K}\}$ is

infinite. The operator $R_0 := R^0_+ + p_f$, $f \in \mathcal{K}$ is a J-positive J-projection and $R = R^0 + \mathcal{J} R^0 \mathcal{J}$. $\square$

We shall show that there is a unique special $f_0 \in \mathcal{K}$. In order to prove Proposition 11 we need the following proposition.

Proposition 9. *Let x, $y \in PH$ be such that $(x, y) = 1$, $\|x\| = \|y\|$, and let $\mathcal{N} := \{f \in \Gamma^+ : (f, \mathcal{J} f) = [f, \mathcal{J} f] = 0 \quad \text{and} \quad f_P = \alpha y,\ f_Q = \beta J x\}$. Then $\mathcal{N} = \{\lambda f_0\}$, where $f_0 := \frac{1}{\sqrt{2}}(y + Jx)$ and $|\lambda| = 1$.*

Proof. Let $f \in \mathcal{N}$. By Proposition 7 (2) and 7 (3),

$$1 = 2[f_Q, f_P] = 2\beta\overline{\alpha}(x, y) = 2\beta\overline{\alpha}. \tag{19}$$

We have $0 = (f, \mathcal{J} f) = \|f_P\|^2 - \|f_Q\|^2 = (|\alpha|^2 - |\beta|^2)\|x\|^2$. Hence $|\alpha| = |\beta|$. By (19), $\alpha = \beta$ and $|\alpha| = \frac{1}{\sqrt{2}}$. Finally, $f = \frac{\lambda}{\sqrt{2}}(y + Jx)$, $|\lambda| = 1$. $\square$

Note the following. Let x, $y \in PH$ be such that $(x, y) = 1$ and $\|x\| = \|y\|$ and let f_0 be from Proposition 9. Then $f_0 \in P^+H$ if and only if $\|x\| = 1$ if and only if $(., x)y$ is an orthogonal projection.

A vector $f \in \Gamma^+$ is said to be a *generator* (for a two-dimensional J-projection R of type (B)) if $R = p_f + p_{\mathcal{J} f}$ and $p_f H \perp p_{\mathcal{J} f} H$. It is clear that *a vector $f \in \Gamma^+$ is a generator if and only if the J-projection p_f is a generator*.

Proposition 10. *The following conditions are equivalent:*

1) *a vector f is a generator;*
2) *$f \in \Gamma^+$ and $(f, \mathcal{J} f) = 0 = [f, \mathcal{J} f]$;*
3) *$f \in \Gamma^+$ and $(\mathcal{J} P^+ f, P^- f) = 0$;*
4) *$2[f_Q, f_P] = 1$ and $\|f_P\| = \|f_Q\|$.*

Proof. It is clear that conditions *1)* and *2)* are equivalent.

2)⇒3). Let $(f, \mathcal{J} f) = 0 = [f, \mathcal{J} f]$. By Proposition 7 (1), we have $\Re(\mathcal{J} P^+ f, P^- f) = 0$. By Proposition 8, $\Im(\mathcal{J} P^+ f, P^- f) = 0$. Hence $(\mathcal{J} P^+ f, P^- f) = 0$.

3)⇒4). Since $f \in \Gamma^+$, we have $2\Re[f_Q, f_P] = 1$, by Proposition 7 (2). Since $(\mathcal{J} P^+ f, P^- f) = 0$, we have $\Im[f_Q, f_P] = 0$, by Proposition 8. Hence $2[f_Q, f_P] = 1$. Finally, $(\mathcal{J} P^+ f, P^- f) = 0$ implies (see Proposition 7 (1)) $\|f_P\| = \|f_Q\|$.

4)⇒2). The equality $2[f_Q, f_P] = 1$ implies $f \in \Gamma^+$ (see Proposition 7 (2), and by Proposition 8, $[f, \mathcal{J} f] = 0$. By Proposition 7 (1), $\|f_P\| = \|f_Q\|$ implies $(f, \mathcal{J} f) = 0$. $\square$

The following proposition is a two-dimensional analog of Theorem 3 specifying the structure of J-projection R_+^0.

Proposition 11. *Let x, $y \in PH$ be such that $(x,y)=1$, $\|x\|=\|y\|$, and let $R:=(.,x)y+(.,Jy)Jx$ be a two-dimensional J-projection of type (B). Then $\mathcal{M}:=\{p\in\mathcal{P}^+ : R=p+\mathcal{J}p\mathcal{J}, (JR)_+=Jp, (JR)_-=-J\mathcal{J}p\mathcal{J}\}=\{p_{f_0}\}$, where the vector f_0 is from Proposition 9.*

Proof. Let $p\in\mathcal{M}$. By the definition of R, $p=p_f$, where $f\in\Gamma^+$. It is clear that $p_f(\mathcal{J}p_f\mathcal{J})=p_f p_{\mathcal{J}f}=0$. Hence $[f,\mathcal{J}f]=0$. Since $(JR)_+(JR)_-=0$, $0=(Jp_f)(J\mathcal{J}p_f\mathcal{J})$. This means $(f,\mathcal{J}f)=0$. Since $p_f<R$, $f\in RH$. By the definition of R again, $f_P=\alpha y$, $f_Q=\beta Jx$. By Proposition 9, $f=\lambda f_0$, $|\lambda|=1$. Hence $p_f=p_{f_0}$. □

If $\dim H>2$, then, for any J-projection p_f, $f\in\Gamma$, the set of two-dimensional J-projections q such that $p_f<q$ is infinite. Therefore the following proposition seems to be interesting.

Proposition 12. *For any J-projection p_f, $f\in\Gamma$, there is a unique two-dimensional J-projection R of type (B) such that $p_f<R$.*

Proof. Let us consider, for instance, the case $f\in\Gamma^+$. Fix α such that $\alpha[f_Q,f_P]=1$. Put $y_0:=\overline{\alpha}f_P$, $x_0:=Jf_Q$. Then $(x_0,y_0)=1$. Thus $(.,x_0)y_0$ is a projection. By the construction, $(.,x_0)y_0\le P$. Note that

$$[f,f_p]=[f_P+f_Q,f_P]=[f_Q,f_P],\quad [f,f_Q]=[f_P,f_Q]. \tag{20}$$

Let us define the two-dimensional J-projection

$$R:=(.,x_0)y_0+(.,Jy_0)Jx_0$$

of type (B). From the equality $\alpha[f_Q,f_P]=1$ and (20), it follows that $Rp_f=p_f$. Thus $p_f<R$. By Proposition 6, $p_{\mathcal{J}f}<R$. This means that f, $\mathcal{J}f\in RH$. Hence the lineal generated by the set $\{f,\mathcal{J}f\}$ is equal to the subspace RH. Any J-projection is uniquely determined by its range space. Therefore R is the unique two-dimensional J-projection with the property $p_f<R$. □

Problem: Does there exist $\sup\{p,\mathcal{J}p\mathcal{J}\}$ in $\mathcal{P}^{\mathcal{B}}$ for any $p\in\mathcal{P}$ with respect to the order $\le$? (cf. Proposition 12).

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