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LOWER BOUNDS FOR SUMMABILITY MATRICES ON WEIGHTED SEQUENCE SPACES

(submitted by O. E. Tikhonov)

Abstract. The purpose of this paper is finding a lower bound for summability matrix operators on sequence spaces $l_p(w)$ and Lorentz sequence spaces $d(w,p)$ and also the sequence space $e(w,\infty)$. Also, this study is an extension of some works of Bennett.

1. Introduction

We study the lower bounds of summability matrix operators on $l_p(w)$ and Lorentz sequence spaces $d(w,p)$ and also the Banach sequence space $e(w,\infty)$ considered in [1] on l_p spaces. The problem of finding the upper bound and lower bound of certain matrix operators such as Cesaro, Copson and Hausdorff and Hilbert operators are considered in [3], [4], [5], [6] and [7] on weighted sequence spaces.

Let $1 \leq p < \infty$, l_p be the normed linear space of all sequences (x_n) with finite norm $\|x\|_p$, where

$$\|x\|_p = (\sum_{n=1}^{\infty} |x_n|^p)^{1/p}.$$

If (w_n) is a decreasing non-negative sequence, we define the weighted sequence space $l_p(w)$ as follows:

$$l_p(w) = \{ \ (x_n) \ : \ \sum_{n=1}^{\infty} w_n |x_n|^p < \infty \ \},$$

with norm, $\|.\|_{p,w}$, which is defined in the following way:

$$\|x\|_{p,w} = (\sum_{n=1}^{\infty} w_n |x_n|^p)^{1/p}.$$

Also, if (w_n) is a decreasing non-negative sequence such that $\lim_{n\to\infty} w_n = 0$ and $\sum_{n=1}^{\infty} w_n = \infty$, then the Lorentz sequence space $d(w,p)$ is defined as follows:

$$d(w,p) = \{\ (x_n)\ :\ \sum_{n=1}^{\infty} w_n x_n^{*p} < \infty\ \},$$

where (x_n^*) is the decreasing rearrangement of $(|x_n|)$. Then $d(w,p)$ is the space of null sequences x for which x^* is in $l_p(w)$, with norm $\|x\|_{d(w,p)} = \|x^*\|_{p,w}$.

Let $X_k^* = x_1^* + \cdots + x_k^*$ and $W_k = w_1 + \cdots + w_k$. The conjugate space of $d(w,1)$ is $e(w,\infty)$, where

$$e(w,\infty) = \{\ (x_n)\ :\ \sup_k \frac{X_k^*}{W_k} < \infty\ \},$$

and its norm is defined by

$$\|x\|_{w,\infty} = \sup_k \frac{X_k^*}{W_k}.$$

Let A be a matrix with non-negative entries. We consider lower bounds of the form

$$\|Ax\|_{p,w} \geq L\|x\|_{p,v}, (\|Ax\|_{d(w,p)} \geq L\|x\|_{d(v,p)}),$$

valid for every decreasing non-negative sequence x and L is a constant not depending on x. We seek the largest possible value of L and denote the best lower bound by $L_{p,v,w}$ for matrix operator from $l_p(v)$ into $l_p(w)$ and also it is denoted by $L_{p,w}(A)$ and $L_{d(w,p)}(A)$ on $l_p(w)$ and $d(w,p)$, respectively.

When $0 < p < 1$, we use notation $\|.\|$ without assuming that it is a norm. We denote transpose matrix of A by A^t. Suppose that $A = (a_{n,k})$ is a summability matrix, then A^t is quasi-summability matrix and

$$(A^t x)_n = \sum_{k=n}^{\infty} a_{k,n} x_k.$$

Also, denote by p^* the conjugate exponent of p, so that $p^* = p/(p-1)$.

Throughout this paper, we apply the following lemma and state some statements on weighted sequence space $l_p(w)$ and some results on weighted sequence space $d(w,p)$.

Lemma 1.1. Let $p \geq 0$ and $A = (a_{i,j})$ be a matrix with non-negative entries. The following condition is equivalent to the statement that Ax is decreasing for every decreasing, non-negative sequence x in $d(w, p)$:
(1) $r_{i,n} = \sum_{j=1}^{n} a_{i,j}$ decreases with i for each n.
Proof. Let $x \in d(w, p)$ be a decreasing, non-negative sequence and $y = Ax$. If (1) holds, by Abel summation, we have

$$y_i = \sum_{j=1}^{\infty} a_{i,j} x_j = \sum_{j=1}^{\infty} r_{i,j}(x_j - x_{j+1}),$$

it follows that Ax is decreasing. The converse deduce from the fact that $y_i = r_{i,n}$ when $x = e_1 + \cdots + e_n$.

Above lemma shows that for the matrix A with condition (1), we have

$$L_{p,w}(A) = L_{d(w,p)}(A).$$

2. Summability matrix operator on $d(w, 1)$ and $e(w, \infty)$

In this section, we consider the lower bound problem for summability matrix operators on $d(w, 1)$ and $e(w, \infty)$. These are lower triangular matrices with entries of the form:

(i) $a_{n,k} \geq 0$;
(ii) $a_{n,k} = 0$ if $k > n$;
(iii) $\sum_{k=1}^{n} a_{n,k} = 1$.

We generalize Theorem 1 of [3] for summability matrix operators from $d(v, 1)$ into $d(w, 1)$. We write $\|.\|_w$ instead of $\|.\|_{d(w,1)}$ and denote lower bound by $L_{v,w}(A)$ for matrix operator from $d(v, 1)$ into $d(w, 1)$ and it is denoted by $L_w(A)$ on $d(w, 1)$. Moreover, we denote lower bound of matrix operator A from $e(w, \infty)$ into itself with $L_{w,\infty}(A)$. Throughout this section, we assume A is a summability matrix operator satisfying condition (1) in Lemma 1.1.

Theorem 2.1. Suppose that $A = (a_{i,j})$ is a summability matrix operator from $d(v, 1)$ into $d(w, 1)$ with non-negative entries. We write $S_n = W_n + \sum_{k=n+1}^{\infty} w_k r_{k,n}$ and $V_n = v_1 + \cdots + v_n$. Then

$$L_{v,w}(A) = \inf_n \frac{S_n}{V_n}.$$

Proof. Denote the stated infimum by M. Let x be in $d(v, 1)$ such that $x_1 \geq x_2 \geq \cdots \geq 0$ and $y = Ax$. By Abel summation, we have

$$y_n = \sum_{k=1}^{n-1} r_{n,k}(x_k - x_{k+1}) + x_n.$$

Hence

$$\begin{aligned}\|Ax\|_w &= \sum_{n=1}^{\infty} w_n \left(\sum_{k=1}^{n-1} r_{n,k}(x_k - x_{k+1}) + x_n\right)\\ &= \sum_{n=1}^{\infty} S_n(x_n - x_{n+1})\\ &\geq M\sum_{n=1}^{\infty} V_n(x_n - x_{n+1})\\ &= M\|x\|_v.\end{aligned}$$

Therefore

$$L_{v,w}(A) \geq M.$$

To show that the constant M is the best possible, we take $x_1 = x_2 = \cdots = x_n = 1$ and $x_k = 0$ for all $k \geq n+1$. Then

$$\|x\|_v = V_n, \qquad \|Ax\|_w = S_n.$$

Therefore

$$L_{v,w}(A) = M.$$

The following lemma is needed in the sequel.

Lemma 2.1. Let $\alpha > 0$ and $X_{(n)} = \sum_{k=n}^{\infty} \frac{1}{k^{1+\alpha}}$, then $(n-1)^{\alpha} X_{(n)}$ increases and tends to $\frac{1}{\alpha}$.

Proof. Let $x_n = \frac{1}{n^{1+\alpha}}$ and $y_n = \int_{n-1}^{n} \frac{dt}{t^{1+\alpha}}$, then

$$Y_{(n+1)} = \sum_{k=n+1}^{\infty} y_k = \frac{1}{\alpha n^{\alpha}}.$$

By the usual integral comparison,

$$\frac{1}{\alpha n^{\alpha}} \leq X_{(n)} \leq \frac{1}{\alpha(n-1)^{\alpha}},$$

which implies the stated limit. Write $z_n = n^{1+\alpha} y_n$, then

$$z_{n+1} = (n+1)^{1+\alpha} \int_{n}^{n+1} \frac{dt}{t^{1+\alpha}} = (n+1)^{1+\alpha} \int_{n-1}^{n} \frac{dt}{(t+1)^{1+\alpha}}.$$

For $n-1 \leq t \leq n$, we have $\frac{n+1}{t+1} \leq \frac{n}{t}$, hence $\frac{(n+1)^{1+\alpha}}{(t+1)^{1+\alpha}} \leq \frac{n^{1+\alpha}}{t^{1+\alpha}}$ and $z_{n+1} \leq z_n$. Therefore $(\frac{x_n}{y_n})$ is increasing, then so is $(\frac{X_{(n)}}{Y_{(n)}})$.

Theorem 2.2. Let A be a summability matrix operator from $d(w,1)$ into itself. If for $k > n$

$$r_{k,n} \leq \frac{n}{k},$$

and $w_n \leq \frac{1}{n}$ for all n, then

$$L_w(A) = 1.$$

Proof. We have $S_n = W_n + \sum_{k=n+1}^{\infty} w_k r_{k,n}$, hence $S_n \geq W_n$ for all n, and so

$$L_{1,w}(A) \geq 1.$$

Since $r_{k,n} \leq \frac{n}{k}$ for $k > n$,

$$S_n \leq W_n + n \sum_{k=n+1}^{\infty} \frac{1}{k^2}.$$

Applying Lemma 2.1, $n \sum_{k=n+1}^{\infty} \frac{1}{k^2} \leq 1$. Therefore

$$\frac{S_n}{W_n} \leq 1 + \frac{1}{W_n}.$$

Since $W_n \to \infty$ as $n \to \infty$, hence

$$L_{1,w}(A) \leq 1.$$

This deduces the statement.

We note that the Hausdorff matrix, Nörlund mean matrix, weighted mean matrix, and in particular Cesaro matrix are summability matrix operators.

We denote the Cesaro matrix by C, with entries:

$$c_{i,j} = \begin{cases} \frac{1}{i} & \text{if} \quad i \geq j \\ 0 & \text{if} \quad i < j. \end{cases}$$

The lower bound problem of C is discussed in [3].

Corollary 2.1. If $N \geq 0$ and $w_n = \frac{1}{n+N}$, then

$$L_w(C) = 1.$$

Theorem 2.3. Let A be a summability matrix operator from $e(w, \infty)$ into itself. If $w_1 = 1$ and $w_n \geq a_{n,1}$ for all n, then

$$L_{w,\infty}(A) = 1.$$

Proof. Let x be in $e(w, \infty)$ such that $x_1 \geq x_2 \geq \cdots \geq 0$ and $y = Ax$. Then for all n

$$y_n = \sum_{k=1}^{n} a_{n,k} x_k \geq x_n \sum_{k=1}^{n} a_{n,k} = x_n.$$

Hence

$$\frac{X_n}{W_n} \leq \frac{Y_n}{W_n}, \qquad (\forall n).$$

Therefore $\|x\|_{w,\infty} \leq \|Ax\|_{w,\infty}$ and so $L_{w,\infty}(A) \geq 1$. To show that the constant is the best possible, we take $x_1 = 1$ and $x_n = 0$ for all $n \geq 2$. Then $\|x\|_{w,\infty} = 1$ and $y_n = a_{n,1}$. Thus $y_n \leq w_n$ and

$$Y_n \leq W_n, \qquad (\forall n).$$

Therefore $\|Ax\|_{w,\infty} \leq 1$ and $L_{w,\infty}(A) \leq 1$. This establishes the proof of the theorem.

Theorem 2.4. Suppose that $A = (a_{n,k})$ is a summability matrix operator from $e(w,\infty)$ into itself, then

$$L_{w,\infty}(A^t) = 1.$$

Proof. Let x be in $e(w,\infty)$ such that $x_1 \geq x_2 \geq \cdots \geq 0$ and $y = A^t x$. Then for all n

$$\begin{aligned} Y_n = \sum_{k=1}^{n} y_k &= \sum_{k=1}^{n} (\sum_{i=k}^{\infty} a_{i,k} x_i) \\ &= X_n + \sum_{i=n+1}^{\infty} (\sum_{k=1}^{n} a_{i,k}) x_i. \end{aligned}$$

Hence $X_n \leq Y_n$ for all n and so $\|x\|_{w,\infty} \leq \|A^t x\|_{w,\infty}$. Therefore

$$L_{w,\infty}(A) \geq 1.$$

To show that the constant is the best possible, we take $x_1 = 1$ and $x_n = 0$ for all $n \geq 2$. We have $A^t x = x$, hence $\|A^t x\|_{w,\infty} = \|x\|_{w,\infty}$ and so we have the statement.

3. Summability matrix on $l_p(w)$

In this section, we consider summability matrix operator and its transpose on $l_p(w)$. First, we shall give some lemmas which will be useful in the sequel.

Lemma 3.1. Let (x_n), (y_n) and (w_n) be non-negative sequences. If (w_n) is decreasing and

$$\sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} y_i \qquad (n = 1, 2, \cdots),$$

then

$$\sum_{i=1}^{n} w_i x_i \leq \sum_{i=1}^{n} w_i y_i \qquad (n = 1, 2, \cdots).$$

Proof. Suppose that $X_i = \sum_{k=1}^{i} x_k$ and $Y_i = \sum_{k=1}^{i} y_k$. Fixing n and summing by parts, we have

$$\begin{aligned}\sum_{i=1}^{n} w_i x_i &= \sum_{i=1}^{n-1}(w_i - w_{i+1})X_i + w_n X_n\\ &\le \sum_{i=1}^{n-1}(w_i - w_{i+1})Y_i + w_n Y_n\\ &= \sum_{i=1}^{n} w_i y_i.\end{aligned}$$

Lemma 3.2. Let $p \ge 1$ and (x_n), (y_n) and (w_n) be non-negative sequences. If (x_n) and (w_n) are decreasing and

$$\sum_{i=1}^{n} x_i \le \sum_{i=1}^{n} y_i \qquad (n = 1, 2, \cdots),$$

then

$$\sum_{i=1}^{n} w_i x_i^p \le \sum_{i=1}^{n} w_i y_i^p \qquad (n = 1, 2, \cdots).$$

Proof. Applying Lemma 3.1 and then Hölder's inequality, we have

$$\begin{aligned}\sum_{i=1}^{n} x_i^p &= \sum_{i=1}^{n} x_i x_i^{p-1} \le \sum_{i=1}^{n} y_i x_i^{p-1}\\ &\le (\sum_{i=1}^{n} y_i^p)^{1/p}(\sum_{i=1}^{n} x_i^p)^{1/p^*}.\end{aligned}$$

Therefore $\sum_{i=1}^{n} x_i^p \le \sum_{i=1}^{n} y_i^p$ and again by Lemma 3.1, we deduce

$$\sum_{i=1}^{n} w_i x_i^p \le \sum_{i=1}^{n} w_i y_i^p.$$

In the following, we consider the lower bound problem for quasi-summability matrix operators.

Theorem 3.1. Suppose that $p \ge 1$ and $A = (a_{n,k})$ is a summability matrix operator from $l_p(v)$ into $l_p(w)$. Also, let $v = (v_n)$, $w = (w_n)$ be non-negative decreasing sequences. If $v_1 = w_1$ and

$$\sum_{i=1}^{n} v_i \le \sum_{i=1}^{n} w_i,$$

for all n, then

$$L_{p,v,w}(A^t) = 1.$$

Proof. Let $x = (x_n)$ be a non-negative decreasing sequence and $y = A^t x$. Then

$$\begin{aligned}\sum_{i=1}^{n} y_i &= \sum_{i=1}^{n}\sum_{k=i}^{\infty} a_{k,i}x_k \\ &\geq \sum_{i=1}^{n}\sum_{k=i}^{n} a_{k,i}x_k \\ &= \sum_{i=1}^{n} x_i.\end{aligned}$$

Applying Lemma 3.2, we deduce that $\|y\|_{p,w} \geq \|x\|_{p,w}$. Since (x_i^p) is a non-negative decreasing sequence and $\sum_{i=1}^{n} v_i \leq \sum_{i=1}^{n} w_i$, we have $\|x\|_{p,v} \leq \|x\|_{p,w}$. Therefore $\|y\|_{p,w} \geq \|x\|_{p,v}$ and $L_{p,v,w}(A^t) \geq 1$. Further $a_{1,1} = 1$ and $A^t e_1 = e_1$, hence $\|A^t e_1\|_{p,w} = \|e_1\|_{p,v}$. This completes the proof of the theorem.

We establish a lower bound for summability matrices with increasing rows.

Lemma 3.3. Suppose $p \geq 1$ and $A = (a_{i,j})$, $B = (b_{i,j})$ are summability matrices. If

$$\sum_{j=1}^{n} a_{i,j} \leq \sum_{j=1}^{n} b_{i,j}, \qquad (i, n = 1, 2, \cdots) \qquad (I)$$

then

$$L_{p,w}(A) \leq L_{p,w}(B).$$

Proof. Let x be a decreasing non-negative sequence. Applying Lemma 3.1 for (I), we have

$$\sum_{j=1}^{i} a_{i,j}x_j \leq \sum_{j=1}^{i} b_{i,j}x_j.$$

Hence $\|Ax\|_{p,w} \leq \|Bx\|_{p,w}$, and so

$$L_{p,w}(A) \leq L_{p,w}(B).$$

In the following statement, we compare lower bound of summability matrix with Cesaro matrix.

Theorem 3.2. Suppose $p \geq 1$ and $A = (a_{i,j})$ is a summability matrix. If the rows of A are increasing, then

$$L_{p,w}(A) \leq L_{p,w}(C).$$

Proof. We show that

$$\sum_{j=1}^{n} a_{i,j} \leq \sum_{j=1}^{n} c_{i,j} \qquad (i, n = 1, 2, \cdots).$$

It is clear that for $n \geq i$, we have

$$\sum_{j=1}^{n} a_{i,j} = \sum_{j=1}^{n} c_{i,j} = 1.$$

When $n < i$, since the rows of A are increasing, we have

$$\begin{aligned} i\sum_{j=1}^{n} a_{i,j} &= n\sum_{j=1}^{n} a_{i,j} + (i-n)\sum_{j=1}^{n} a_{i,j} \\ &\leq n\sum_{j=1}^{n} a_{i,j} + n\sum_{j=n+1}^{i} a_{i,j} \\ &= n\sum_{j=1}^{i} a_{i,j} \\ &= n. \end{aligned}$$

Hence

$$\sum_{j=1}^{n} a_{i,j} \leq \frac{n}{i} = \sum_{j=1}^{n} c_{i,j}.$$

We now apply Lemma 3.3 for A and C to establish the theorem.

In the following we state some result of Theorem 3.2 showing the exact value of the lower bound for summability matrix, where $w_n = \frac{1}{n}$.

Corollary 3.1. Suppose $p \geq 1$ and $A = (a_{i,j})$ is a summability matrix with increasing rows. If w is defined by $w_n = \frac{1}{n}$, then

$$L_{p,w}(A) = 1.$$

Proof. Let x be a decreasing, non-negative sequence in $l_p(w)$. We have

$$\begin{aligned} \|Ax\|_{p,w}^p &= \sum_{n=1}^{\infty} w_n (\sum_{k=1}^{n} a_{n,k} x_k)^p \\ &\geq \sum_{n=1}^{\infty} w_n x_n^p (\sum_{k=1}^{n} a_{n,k})^p \\ &= \|x\|_{p,w}^p, \end{aligned}$$

hence $L_{p,w}(A) \geq 1$. Applying Theorem 3.1 and Corollary 2.1, we deduce that

$$L_{p,w}(A) \leq 1.$$

Therefore $L_{p,w}(A) = 1$. Bennett considered summability matrices with increasing or decreasing rows in [1]. For example

$$\Gamma(2) = \begin{bmatrix} 1 & 0 & & & & \\ 1/3 & 2/3 & 0 & & & \\ 1/6 & 2/6 & 3/6 & 0 & & \\ 1/10 & 2/10 & 3/10 & 4/10 & 0 & \\ . & . & . & . & . & . \end{bmatrix}$$

$\Gamma(2)$ is the summability matrix with increasing rows and for $w_n = \frac{1}{n}$, we have

$$L_{p,w}(\Gamma(2)) = 1.$$

4. Quasi-summability matrix

In this section, we establish a lower bound for quasi-summability matrix on $l_p(w)$ for $0 < p \le 1$, where sequences are non-negative. Note that we shall use the norm only as a notation and do not use norm's properties. First, we compare lower bound of quasi-summability matrix with Copson matrix.

Lemma 4.1. Suppose $0 < p \le 1$ and u, v and w are N-tuples with non-negative entries. If u, w are decreasing, and

$$\sum_{i=n}^{N} u_i \ge \sum_{i=n}^{N} v_i \qquad (n = 1, \dots, N),$$

then

$$\sum_{i=n}^{N} w_i u_i^p \ge \sum_{i=n}^{N} w_i v_i^p, \qquad (n = 1, \dots, N).$$

Proof. Set $x_i = u_{N-i+1}$ and $y_i = v_{N-i+1}$ for $1 \le i \le N-n+1$, we have $\sum_{i=1}^{N-n+1} x_i = \sum_{i=n}^{N} u_i$ and $\sum_{i=1}^{N-n+1} y_i = \sum_{i=n}^{N} v_i$. Applying Lemma 3.1 and Hölder's inequality, we deduce the statement.

Lemma 4.2. Suppose $0 < p \le 1$ and $A = (a_{i,j})$, $B = (b_{i,j})$ are summability matrices. If the rows of B are decreasing and

$$\sum_{i=1}^{n} a_{j,i} \ge \sum_{i=1}^{n} b_{j,i}, \qquad (j, n = 1, 2, \dots) \qquad (I)$$

then

$$L_{p,w}(A^t) \le L_{p,w}(B^t).$$

Proof. Let N be fixed and x be a sequence with non-negative entries. We define u and v by

$$u_i = \sum_{j=i}^{N} b_{j,i} x_j, \qquad v_i = \sum_{j=i}^{N} a_{j,i} x_j, \qquad (i = 1, 2, \ldots).$$

It is clear that u_i decreases with i. Definition of summability matrix and (I) follow that

$$\sum_{i=n}^{N} u_i \geq \sum_{i=n}^{N} v_i, \qquad (n = 1, 2, \ldots, N),$$

hence applying Lemma 4.1, we deduce that

$$\sum_{i=n}^{N} w_i u_i^p \geq \sum_{i=n}^{N} w_i v_i^p, \qquad (n = 1, 2, \ldots, N).$$

Therefore $\|B^t x\|_{p,w} \geq \|A^t x\|_{p,w}$, and so

$$L_{p,w}(A^t) \leq L_{p,w}(B^t).$$

The transpose of Cesaro matrix is called the Copson matrix and we denote it with C^t. Applying Theorem 2.1 of [6], we have

$$L_{p,w}(C^t) = p.$$

Theorem 4.1. Suppose $A = (a_{i,j})$ is a summability matrix. If the rows of A are decreasing, then

$$L_{p,w}(A^t) \leq p,$$

where $0 < p \leq 1$.

Proof. It is clear that the rows of C are decreasing. For $n \geq j$, we have

$$\sum_{i=1}^{n} a_{j,i} = \sum_{i=1}^{n} c_{j,i} = 1.$$

Also, when $n < j$, since the j^{th} row of A is decreasing, the average

$$\frac{1}{n} \sum_{i=1}^{n} a_{j,i}$$

decreases with n. Hence

$$\sum_{i=1}^{n} a_{j,i} \geq \frac{n}{j} \qquad (j = 1, 2, \ldots),$$

and so we have (I). We now apply Lemma 4.2 for A^t and C^t to establish the theorem.

In the following, we evaluate lower bound of summability matrix with increasing rows.

Lemma 4.3([1], Lemma 3.13). Let $0 < p \leq 1$, and u be N-tuple with positive entries. Then

$$p\sum_{k=n}^{N} u_k(\sum_{j=k}^{N} u_j)^{p-1} \leq (\sum_{j=n}^{N} u_j)^p, \qquad (n = 1, \ldots, N), \qquad (II)$$

and if $p \geq 1$, the inequality in (II) is reversed. The constant p is best possible in either version of (II) and there is strict inequality unless $p = 1$ or $u = 0$.

Theorem 4.2. Let $0 < p \leq 1$ and $A = (a_{i,j})$ be the summability matrix. If the rows of A are increasing, then

$$L_{p,w}(A^t) \geq p.$$

Proof. Suppose x is a sequence with non-negative terms. Applying Lemma 4.3, we have

$$\begin{aligned}
\|A^t x\|_{p,w}^p = \sum_{n=1}^{\infty} w_n(\sum_{k=n}^{\infty} a_{k,n}x_k)^p \quad &\geq \quad p\sum_{n=1}^{\infty} w_n \sum_{k=n}^{\infty} a_{k,n}x_k(\sum_{j=k}^{\infty} a_{j,n}x_j)^{p-1} \\
&= \quad p\sum_{k=1}^{\infty} x_k \sum_{n=1}^{k} w_n a_{k,n}(\sum_{j=k}^{\infty} a_{j,n}x_j)^{p-1} \\
&\geq \quad p\sum_{k=1}^{\infty} w_k x_k \sum_{n=1}^{k} a_{k,n}(\sum_{j=k}^{\infty} a_{j,n}x_j)^{p-1}.
\end{aligned}$$

Since A has increasing rows,

$$\sum_{j=k}^{\infty} a_{j,n}x_j, \qquad (n = 1, \ldots, k),$$

is increasing with n, hence

$$\sum_{n=1}^{k} a_{k,n}(\sum_{j=k}^{\infty} a_{j,n}x_j)^{p-1} \geq (\sum_{j=k}^{\infty} a_{j,n}x_j)^{p-1}.$$

Thus applying Hölder's inequality, it follows that

$$\begin{aligned}
\|A^t x\|_{p,w}^p &\geq p\sum_{k=1}^{\infty} w_k x_k (\sum_{j=k}^{\infty} a_{j,k} x_j)^{p-1} \\
&\geq p\sum_{k=1}^{\infty} w_k^{1/p} x_k (w_k^{1/p^*(p-1)} \sum_{j=k}^{\infty} a_{j,k} x_j)^{p-1} \\
&\geq p(\sum_{k=1}^{\infty} w_k x_k^p)^{1/p} (\sum_{k=1}^{\infty} (w_k^{1/p} \sum_{j=k}^{\infty} a_{j,k} x_j)^p)^{1/p^*} \\
&= p\|x\|_{p,w} \|A^t x\|_{p,w}^{p-1}.
\end{aligned}$$

Therefore $\|A^t x\|_{p,w} \geq p\|x\|_{p,w}$, and so $L_{p,w}(A^t) \geq p$.

Proposition 4.1([6], Proposition 2.2). Let $0 < p, q < 1$, and A be a matrix with non-negative entries. Then

$$\|Ax\|_{q,w} \geq L\|x\|_{p,w},$$

for all non-negative x if and only if

$$\|A^t y\|_{p^*,w} \geq L\|y\|_{q^*,w}.$$

for all non-negative y.

Theorem 4.3. Let $p < 0$ and $A = (a_{i,j})$ be the summability matrix. If A has increasing rows, then

$$\sum_{j=1}^{\infty} w_j (\sum_{k=1}^{j} a_{j,k} x_k)^p \leq (p^*)^p \sum_{k=1}^{\infty} w_k x_k^p,$$

for any sequence x with positive terms. *Proof.* Since $0 < p^* < 1$, Theorem 4.2 follows that

$$L_{p^*,w}(A^t) \geq p^*.$$

Applying Proposition 4.1, we deduce that

$$L_{p,w}(A) \geq p^*.$$

Hence for any sequence x with positive terms, we have

$$\|Ax\|_{p,w} \geq p^* \|x\|_{p,w}.$$

This completes the proof of theorem.

Corollary 4.1. Let $p > 0$, and x be a positive sequence. Then

$$\sum_{j=1}^{\infty} w_j \max_{1\leq i\leq j} (\frac{j-i+1}{\frac{1}{x_i} + \cdots + \frac{1}{x_j}})^p \leq (\frac{p+1}{p})^p \sum_{k=1}^{\infty} w_k x_k^p.$$

Proof. We apply Theorem 4.3, by replacing p with $-p$ and x_k with $\frac{1}{x_k}$. The left hand side of inequality is $\|Ax\|_{p,w}^p$, where $A = (a_{i,j})$ is a summability matrix operator with

$$a_{j,k} = \begin{cases} \frac{1}{j-i_j+1} & k = i_j, \dots, j \\ 0 & \text{otherwise} \end{cases}$$

where i_j is chosen to be first value of i at which the maximum

$$\max_{1\le i\le j}\left(\frac{1}{j-i+1}\sum_{k=i}^{j} x_k\right)$$

is attained$(j = 1, 2, \dots)$. The rows of A are increasing and its entries depend on x, but this is not damaging, because applying Theorem 4.3, for any positive sequence x, we have

$$\|Ax\|_{p,w} \le p^* \|x\|_{p,w}$$

and this establishes the statement.

The following statement is an extension of famous inequality due to Carlemann([2], Theorem 334).

Corollary 4.2. If (x_k) is a sequence with non-negative terms, then

$$\sum_{i=1}^{\infty} w_j \max_{1\le i\le j}(x_i \cdots x_j)^{\frac{1}{j-i+1}} \le e \sum_{k=1}^{\infty} w_k x_k.$$

Proof. We apply Corollary 4.1, by replacing x_k with $x_k^{1/p}$ and tending $p \longrightarrow 0$, we have the statement.

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