Lobachevskii Journal of Mathematics
http://ljm.ksu.ru
ISSN 1818-9962
Vol. 27, 2007, 41–46

J. Kurek and W.M. Mikulski

RIEMANNIAN STRUCTURES ON HIGHER ORDER FRAME BUNDLES OVER RIEMANNIAN MANIFOLDS

(submitted by M. A. Malakhaltsev)

Abstract. We describe all $\mathcal{M}f_m$-natural operators $A : \mathcal{R}iem \rightsquigarrow \mathcal{R}iemP^r$ transforming Riemannian structures g on m-dimensional manifolds M into Riemannian structures $A(g)$ on the r-th order frame bundle $P^rM = invJ^r_0(\mathbf{R}^m, M)$ over M.

Manifolds and maps are assumed to be of class C^∞. Manifolds assumed to be finite dimensional and without boundaries.

Let $\mathcal{M}f_m$ denote the category of m-dimensional manifolds and their embeddings (i.e. diffeomorphisms onto open subsets) and $\mathcal{FM}$ denote the category of fibred manifolds and their fibred maps.

For any m-manifold M we have the r-th order frame bundle $P^rM = invJ^r_0(\mathbf{R}^m, M)$ of M. This is a principal bundle with the corresponding Lie group $G^r_m = J^r_0(\mathbf{R}^m, \mathbf{R}^m)_0$ acting on the right on P^rM via compositions of jets. Every $\mathcal{M}f_m$-map $\psi : M_1 \to M_2$ induces $P^r\psi : P^rM_1 \to P^rM_2$ by $P^r\psi(j^r_0\varphi) = j^r_0(\psi \circ \varphi)$, where $\varphi : \mathbf{R}^m \to M_1$ is a $\mathcal{M}f_m$-map. The correspondence $P^r : \mathcal{M}f_m \to \mathcal{FM}$ is a bundle functor in the sense of [2].

For any n-manifold N we have the Riemannian bundle $\mathcal{R}iem(N) = \bigcup_{y\in N} Met(T_yN)$ over N, where given a vector space V we denote the set

---

2000 Mathematical Subject Classification. 58A20, 58A32.
Key words and phrases. Riemannian structure, higher order frame bundle, natural operator.

of scalar multiplications $G : V \times V \to \mathbf{R}$ on V by $Met(V)$. (We recall that $G : V \times V \to \mathbf{R}$ is a scalar multiplication if it is symmetric bilinear and positive define.) Clearly, $\mathcal{R}iem(N)$ is an open subbundle in the vector bundle $T^*N \odot T^*N$ of symmetric tensors of type $(0, 2)$ over N. Sections $g : N \to \mathcal{R}iem(N)$ are the so called Riemannian structures on N. Every embedding $\psi : N_1 \to N_2$ induces $\mathcal{R}iem(\psi) : \mathcal{R}iem(N_1) \to \mathcal{R}iem(N_2)$ being the restriction of $T^*\psi \odot T^*\psi : T^*N_1 \odot T^*N_1 \to T^*N_2 \odot T^*N_2$. The correspondence $\mathcal{R}iem : \mathcal{M}f_n \to \mathcal{FM}$ is also a bundle functor in the sense of [2].

In the present short note we study the problem how a Riemannian structure g on an m-dimensional manifold M can induce (canonically) a Riemannian structure $A(g)$ on P^rM. This problem is reflected in the concept of $\mathcal{M}f_m$-natural operators $A : \mathcal{R}iem \rightsquigarrow \mathcal{R}iemP^r$. In the note we describe explicitly all $\mathcal{M}f_m$-natural operators A in question.

A general concept of natural operators can be found in the fundamental monograph [2]. We need only the following partial case of the definition of natural operators.

An $\mathcal{M}f_m$-natural operator $A : \mathcal{R}iem \rightsquigarrow \mathcal{R}iemP^r$ is a family of $\mathcal{M}f_m$-invariant regular operators (functions)

$$A = A_M : \underline{\mathcal{R}iem}(M) \to \underline{\mathcal{R}iem}(P^rM)$$

for any $\mathcal{M}f_m$-object M, where $\underline{\mathcal{R}iem}(N)$ is the set of all Riemannian structures on N (sections of $\mathcal{R}iem(N) \to N$) for any manifold N. The invariance means that if $g_1 \in \underline{\mathcal{R}iem}(M_1)$ and $g_2 \in \underline{\mathcal{R}iem}(M_2)$ are related by an $\mathcal{M}f_m$-map $\psi : M_1 \to M_2$ (i.e. $\mathcal{R}iem(\psi) \circ g_1 = g_2 \circ \psi$) then $A(g_1)$ and $A(g_2)$ are $P^r\psi$-related. The regularity means that A transforms smoothly parametrized families of Riemannian structures into smoothly parametrized ones.

For $r = 1$, P^1M is equivalent with the bundle of linear frames over M. In this case we have the following example basing on a very important classical construction presented in the proof of Theorem 1.5 in [1].

Example 1. Let g be a Riemannian structure on an m-manifold M. Let ∇ be the Levi-Civita connection of g and let $\omega = (\omega^j_k) : TP^1M \to gl(m)$ be its connection form. Let $\theta = (\theta^i) : TP^1M \to \mathbf{R}^m$ be the canonical form on P^1M. We put

$$\tilde{g}(X^*, Y^*) = \sum_i \theta^i(X^*)\theta^i(Y^*) + \sum_{j,k} \omega^j_k(X^*)\omega^j_k(Y^*)\ ,\ X^*, Y^* \in T_uP^1M.$$

Then $\tilde{g}$ is a Riemannian structure on P^1M, see the proof of Theorem 1.5 in [1]. Clearly, the correspondence $A^1 : \mathcal{R}iem \rightsquigarrow \mathcal{R}iemP^1$ given by $A^1(g) = \tilde{g}$ for all $g \in \underline{\mathcal{R}iem}(M)$ is an $\mathcal{M}f_m$-natural operator.

To generalize Example 1 on all r we firstly reformulate it as follows.

Example 2. Let g be a Riemannian structure on M. Let Let (θ^i, ω^j_k) be the basis of 1-forms on P^1M, where $\theta = (\theta^i)$ and $\omega = (\omega^j_k)$ is as in Example 1 for g. Let $X^1(g), \dots, X^{L_1}(g)$, where $L_1 = dim(P^1\mathbf{R}^m) = m + m^2$, be the dual (to (θ^i, ω^j_k)) basis of vector fields on P^1M. Then $\tilde{g} = \sum_{s=1}^{L_1} (X^s(g))^* \odot (X^s(g))^*$ is a Riemannian structure on P^1M (the same as in Example 1).

So, to generalize Example 1 (or Example 2) on all r we need to construct an absolute parallelism (basis of global vector fields) on P^rM canonically dependent on a given Riemannian structure g on M.

From now on let $A_1, \dots, A_{dim(g^r_m)}$ be the standard basis in $g^r_m = \mathcal{L}ie(G^r_m)$ (i.e. the basis $j^r_0(x^\alpha \frac{\partial}{\partial x^i}) \in (J^r_0 T\mathbf{R}^m)_0 = g^r_m$ for $i = 1, \dots, m$, $1 \le |\alpha| \le r$).

Example 3. *Construction of an absolute parallelism on P^rM from a Riemannian structure on M.* Let g be a Riemannian structure on an m-manifold M. Let ∇ be the Levi-Civita connection of g. Let $i = 1, \dots, m$. We have a vector field $Y^i(g)$ on P^rM defined as follows. Let $\sigma = j^r_0\varphi \in (P^rM)_x$, $x \in M$. Let $v^i = T\varphi(\frac{\partial}{\partial x^i}{}_{|0}) \in T_xM$. We extend v^i to the constant vector field $\tilde{v}^i$ on T_xM. Then on some neighborhood of x we have the vector field $V^i(g) = (Exp^\nabla_x)_*\tilde{v}^i$, where $Exp^\nabla_x : T_xM \supset U_{0_x} \to \tilde{U}_x \subset M$ is the exponent of ∇. We define $Y^i(g)_\sigma := \mathcal{P}^r(V^i(g))_\sigma$, where $\mathcal{P}^rV$ is the flow lifting of a vector field V on M to P^rM (if $\{\varphi_t\}$ is the flow of V then $\{P^r\varphi_t\}$ is the flow of $\mathcal{P}^rV$). It is easy to see that $Y^i(g)_\sigma$ projects onto v^i by the bundle projection $P^rM \to M$. So, it is a simple observation that $Y^i(g), A^*_j$ for $i = 1, \dots, m$, $j = 1, \dots, dim(g^r_m)$ is an absolute parallelism on P^rM (canonically depending on g), where given $A \in \mathcal{L}ie(G^r_m)$ we denote the fundamental vector field on the principal bundle P^rM by A^*.

Now, we are in position to extend Example 1 (or 2) on all r.

Example 4. Let g be a Riemannian structure on an m-manifold M. Let $(Y^i(g), A^*_j)$ be the parallelism from Example 3. Let $\omega^s(g)$ for

$s = 1, \dots, dim(P^r\mathbf{R}^m)$ be the dual basis of 1-form on P^rM. We put

$$\tilde{g}^r := \sum_s \omega^s(g) \odot \omega^s(g) \ .$$

Clearly, $\tilde{g}^r$ is a Riemannian structure on P^rM. Clearly, the correspondence $A^{[r]} : \mathcal{R}iem \rightsquigarrow \mathcal{R}iem P^r$ given by $A^{[r]}(g) = \tilde{g}^r$ for all g in question is an $\mathcal{M}f_m$-natural operator. One can observe easily that $A^1 = A^{[1]}$.

To present a general example of $\mathcal{M}f_m$-natural operators $\mathcal{R}iem \rightsquigarrow \mathcal{R}iem P^r$ we need some preparation and notations.

According to the global basis of vector fields $Y^i(g), A_j^*$ on P^rM from Example 3, given $g \in \underline{\mathcal{R}iem}(M)$ we have a canonical (in g) fibred diffeomorphism

$$(*) \qquad I_g : P^rM \times Met(\mathbf{R}^{L_r}) \to \mathcal{R}iem(P^rM)$$

covering id_{P^rM} defined by the condition that the matrix of $I_g(\sigma, G)$ in the basis $(Y^i(g)(\sigma), A_j^*(\sigma))$ is the same as the one of G in the usual canonical basis of $\mathbf{R}^{L_r}$.

Given $g \in \underline{Riem}(M)$ we have the projection

$$Ort(g) : P^1M = LM \to O(M, g)$$

given by the Gramm orthonormalization with respect to g (for $l = (l_i) \in L_xM$, $Ort(g)(l)$ is the orthonormalization of l with respect to g_x).

From now on we denote

$$Q^r = (Ort(g^o) \circ \pi_1^r)^{-1}(l^o) \subset (P^r\mathbf{R}^m)_0 \ ,$$

where g^o is the usual flat Riemannian structure on $\mathbf{R}^m$ and l^o is the usual canonical basis in $\mathbf{R}^m = T_0\mathbf{R}^m$ and $\pi_1^r : P^rM \to P^1M$ is the jet projection. Of course, Q^r is a submanifold in $(P^r\mathbf{R}^m)_0$.

For $s = 0, 1, \dots, \infty$, let $Z^s = J_0^s(\mathcal{R}iem(\mathbf{R}^m))$ be the set of all s-jets j_0^sg of Riemannian structures g on $\mathbf{R}^m$. If s is finite, Z^s is a finite dimensional manifold (as the fibre of the s-jet prolongation $J^s(\mathcal{R}iem(\mathbf{R}^m))$ of the bundle $\mathcal{R}iem(\mathbf{R}^m) \to \mathbf{R}^m$). Z^∞ is a topological space with respect to the inverse limit topology given by the inverse system $\dots \to Z^{s+1} \to Z^s \to \dots \to Z^0$ of jet projections.

Now, we are in position to present the following general construction.

Example 5. *General construction.* Let $\mu : Z^\infty \times Q^r \to Met(\mathbf{R}^{L_r})$, where $L_r = dim(P^r\mathbf{R}^m)$, be a map satisfying the following local finite determination property (a_r):

(a_r) For any $\rho \in Z^\infty$ and $\sigma \in Q^r$ we can find an open neighborhood $U \subset Z^\infty$ of ρ, an open neighborhood $V \subset Q^r$ of σ, a natural number s and a smooth map $f : \pi_s(U) \times V \to Met(\mathbf{R}^{L_r})$ such that $\mu = f \circ (\pi_s \times id_V)$ on $U \times V$, where $\pi_s : Z^\infty \to Z^s$ is the jet projection.

(A simple example of such μ is $\mu = f \circ (\pi_s \times id_{Q^r})$ for smooth $f : Z^s \times Q^r \to Met(\mathbf{R}^{L_r})$ for finite s.) Given a Riemannian structure g on an m-manifold M we define a Riemannian structure $A^{<\mu>}(g)$ on $P^r M$ as follows. Let $\sigma \in (P^r M)_x$, $x \in M$. Choose a g-normal coordinate system ψ on M with center x such that $P^r\psi(\sigma) \in Q^r$. Of course, such ψ exists. Then $germ_x(\psi)$ is uniquely determined. We put

$$(**) \quad A^{<\mu>}(g)_\sigma = \mathcal{R}iem(P^r(\psi^{-1}))(I_{\psi_* g}(P^r\psi(\sigma), \mu(j_0^\infty(\psi_* g), P^r\psi(\sigma)))) \ .$$

Since $germ_x(\psi)$ is uniquely determined the definition (**) is correct. The family $A^{<\mu>} : \mathcal{R}iem \rightsquigarrow \mathcal{R}iem P^r$ is an $\mathcal{M}f_m$-natural operator.

The main result of the present note is the following theorem.

Theorem 1. *Any $\mathcal{M}f_m$-natural operator $A : \mathcal{R}iem \rightsquigarrow \mathcal{R}iem P^r$ is $A^{<\mu>}$ for some $\mu : Z^\infty \times Q^r \to Met(\mathbf{R}^{L_r})$ satisfying the property (a_r).*

Proof. Let $A : \mathcal{R}iem \rightsquigarrow \mathcal{R}iem P^r$ be an $\mathcal{M}f_m$-natural operator. Define $\mu : Z^\infty \times Q^r \to Met(\mathbf{R}^{L_r})$ by

$$(\sigma, \mu(j_0^\infty g, \sigma)) = I_g^{-1}(A(g)(\sigma)) \ .$$

Then by the non-linear Peetre theorem [2], μ satisfies the property (a_r). Then by the definition of μ and $A^{<\mu>}$ we see that $A(g)(\sigma) = A^{<\mu>}(g)(\sigma)$ for any Riemannian structure g on $\mathbf{R}^m$ such that the identity map $id_{\mathbf{R}^m}$ is a g-normal coordinate system with center 0 and any $\sigma \in Q^r$. Then by the invariance of A and $A^{<\mu>}$ with respect to normal coordinates we deduce that $A = A^{<\mu>}$. $\square$

Remark 1. One can observe that $A^{[r]} = A^{<\mu>}$ for constant $\mu : Z^\infty \times Q^r \to Met(\mathbf{R}^{L_r})$ equal to the standard scalar multiplication, where $A^{[r]}$ is as in Example 4.

Remark 2. The map μ from Theorem 1 is not uniquely determined by A. One can observe that $A^{<\mu_1>} = A^{<\mu_2>}$ iff $\mu_1(j_0^\infty g, \sigma) = \mu_2(j_0^\infty g, \sigma)$ for all Riemannian structures g on $\mathbf{R}^m$ such that the identity map $id_{\mathbf{R}^m}$ is a g-normal coordinate system with center 0 and all $\sigma \in Q^r$.

References

[1] Kobayashi S., Nomizu K., Foundations of Differential Geometry, Moskow (Mir) 1981 (in Russian).

[2] Kolář I., Michor P.W., Slovák J., Natural Operations in Differential Geometry, Springer Verlag 1993.

Institute of Mathematics, Maria Curie-Sklodowska University, Pl. Marii Curie Sklodowska 1, Lublin, Poland

E-mail address: kurekgolem.umcs.lublin.pl

Institute of Mathematics, Jagiellonian University, ul. Reymonta 4, Kraków, Poland

E-mail address: mikulskiim.uj.edu.pl

Received August 13, 2007