ASYMPTOTIC BEHAVIOUR OF OSCILLATORY SOLUTIONS OF A FOURTH-ORDER NONLINEAR DIFFERENTIAL EQUATION

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Abstract. Asymptotic behaviour of oscillatory solutions of the fourth-order nonlinear differential equation with quasiderivates $y^{[4]} + r(t)f(y) = 0$ is studied.

Keywords: oscillatory solution, fourth order differential equation

MSC 2000: 34C10

1. Introduction

Consider the differential equation

$$\left(\frac{1}{a_3}\left(\frac{1}{a_2}\left(\frac{y'}{a_1}\right)'\right)'\right)' + r(t)f(y) = 0 \tag{1}$$

where $\mathbb{R}_+ = [0,\infty)$, $\mathbb{R} = (-\infty,\infty)$, $r \in C^0(\mathbb{R}_+)$, $f \in C^0(\mathbb{R})$, $a_i \in C^1(\mathbb{R}_+)$, $a_3/a_1 \in C^2(\mathbb{R}_+)$, a_i are positive on $\mathbb{R}_+$, $i = 1,2,3$ and

$$r(t) > 0 \quad \text{on } \mathbb{R}_+, \qquad f(x)x > 0 \quad \text{for } x \neq 0. \tag{H1}$$

If the quasiderivatives of y are defined as

$$y^{[0]} = y, \quad y^{[i]} = \frac{1}{a_i(t)}(y^{[i-1]})',\ i = 1,2,3, \quad y^{[4]} = (y^{[3]})' \tag{2}$$

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then (1) can be expressed by

$$y^{[4]} + r(t)f(y) = 0.$$

Let a fuction $y\colon I \to \mathbb{R}$ have continuous quasiderivatives up to the order 4 and let (1) hold on I. Then y is called a solution of (1). A solution y is called oscillatory if it is defined on $\mathbb{R}_+$, $\sup\limits_{\tau \leqslant t < \infty} |y(t)| > 0$ for an arbitrary $\tau \in \mathbb{R}_+$ and there exists a sequence of its zeros tending to ∞.

It is proved in [2] that there exist only two types of oscillatory solutions with respect to the distribution of zeros of the quasiderivatives.

Lemma 1 ([2, Th. 3 and Remark 2]). *Let y be oscillatory. Then there exist sequences $\{t_k^i\}$, $i = 0, 1, 2, 3$; $k = 1, 2, \dots$ such that $\lim\limits_{k\to\infty} t_k^0 = \infty$ and either*

$$t_k^0 < t_k^3 < t_k^2 < t_k^1 < t_{k+1}^0, \quad y^{[i]}(t_k^i) = 0, \ i = 0, 1, 2, 3,$$

$$(3) \qquad y^{[j]}(t)y(t) \begin{cases} > 0 \text{ on } (t_k^0, t_k^j), \\ < 0 \text{ on } (t_k^j, t_{k+1}^0), \ j = 1, 2, 3, \ k = 1, 2, \dots \end{cases}$$

or

$$t_k^0 < t_k^1 < t_k^2 < t_k^3 < t_{k+1}^0, \quad y^{[i]}(t_k^i) = 0, \ i = 0, 1, 2, 3,$$

$$(4) \qquad (-1)^{j+1} y^{[j]}(t)y(t) \begin{cases} > 0 \text{ on } (t_k^0, t_k^j), \\ < 0 \text{ on } (t_k^j, t_{k+1}^0), \ j = 1, 2, 3, \ k = 1, 2, \dots \end{cases}$$

Asymptotic properties of an oscillatory solution y fulfilling (3) are studied in [3] and [4]. E.g., it is shown that, under certain assumptions, all local maximas of $|y^{[i]}|$, $i \in \{0, 1, 2\}$ are increasing in a neighbourhood of ∞ and $y^{[j]}$ are unbouned for $j = 0, 1$.

In the present paper the asymptotic behaviour of oscillatory solutions fulfilling (4) will be investigated. Sufficient conditions will be given under which these solutions tend to zero for $t \to \infty$ and the absolute values of all local extremes of the quasiderivatives are decreasing in a neighbourhood of ∞. The problem of "monotonicity" of oscillatory solutions for the second order (the third order) equations has been studied by many authors, see e.g. [1] ([5]).

We do not touch the problem of existence of oscillatory solutions fulfilling (4). In fact this problem is open; it is completely solved only for the case of the usual derivatives in the monographs [1] and [7], i.e. for $a_1 \equiv a_2 \equiv a_3 \equiv 1$.

Denote by $\mathcal{O}$ the set of all oscillatory solutions of (1) that fulfil (4). The following example shows that the above mentioned problems are reasonable.

E x a m p l e 1. The differential equation

$$(((\mathrm{e}^{-\sqrt{3}t}y')')')' + 8\mathrm{e}^{-\sqrt{3}t}y = 0$$

has an oscillatory solution $y = \sin t$ and (4) is valid. Hence, $y \in \mathcal{O}$, y does not tend to zero for $t \to \infty$ and the sequence of the absolute values of all local extremes of y is not decreasing.

2. Decreasing oscillatory solutions

First we state some auxilliary results. The following lemma is a simple consequence of (2) and (H1).

Lemma 2. *Let* (H1) *be valid and let* y *be a solution of* (1) *defined on* $I = [t_1, t_2] \subset \mathbb{R}_+$, $t_1 < t_2$.

(i) *If* $y(t) > 0$ (< 0) *on* I, *then* $y^{[3]}$ *is decreasing (increasing) on* I;

(ii) *if* $i \in \{1, 2, 3\}$ *and* $y^{[i]}(t) > 0$ (< 0) *on* I, *then* $y^{[i-1]}$ *is increasing (decreasing) on* I.

R e m a r k 1. (i) Note that $<$, $>$, increasing and decreasing can be replaced by $\leqslant$, $\geqslant$, nondecreasing and nonincreasing, respectively.

(ii) Let $y \in \mathcal{O}$. It is easy to see that according to (4) and Lemma 2 the sequence $\{|y^{[i]}(t_k^{i+1})|\}_1^\infty$ is the sequence of the absolute values of all local extremes of $y^{[i]}$, $i \in \{0, 1, 2, 3\}$ on $[t_0^0, \infty)$ where $t_k^4 = t_k^0$, $k = 1, 2, \ldots$.

Sometimes, it is useful to transform (1).

Lemma 3. *Let* $a_0 \in C^0(\mathbb{R}_+)$ *be positive. Then the transformation*

$$x(t) = \int_0^t a_0(s)\,\mathrm{d}s,\ Y(x) = y(t),\ t \in \mathbb{R}_+,\ x \in [0, x^*),\ x^* = x(\infty)$$

transforms (1) *into*

$$\Bigl(\frac{1}{A_3}\Bigl(\frac{1}{A_2}\Bigl(\frac{1}{A_1}Y^\bullet\Bigr)^\bullet\Bigr)^\bullet\Bigr)^\bullet + R(x)f(Y) = 0,\ \frac{\mathrm{d}}{\mathrm{d}x} = {}^\bullet, \tag{5}$$

where

$$A_i(x) = \frac{a_i(t(x))}{a_0(t(x))},\ i = 1, 2, 3,\ R(x) = \frac{r(t(x))}{a_0(t(x))},$$

and $t(x)$ is the inverse function to $x(t)$. At the same time

$$Y^{\{i\}}(x) = y^{[i]}(t), \ \ i = 0,1,2,3,4 \tag{6}$$

where

$$Y^{\{0\}} = Y, \ Y^{\{j\}} = \frac{1}{A_j(x)}(Y^{\{j-1\}})^{\bullet}, \ j = 1,2,3, \ Y^{\{4\}} = (Y^{\{3\}})^{\bullet}. \tag{7}$$

P r o o f. Use direct computation or see [4]. □

R e m a r k 2. (6) yields that the transformation from Lemma 3 preserves the relations (3) and (4) for Eq. (5) where $y^{[j]}$ must be substituted by $Y^{\{j\}}$, $j \in \{0,1,2,3\}$.

Theorem 1. *Let* (H1) *be valid. Let $y \in \mathcal{O}$ and let $\{t_k^i\}$, $i = 0,1,2,3$ be given by* (4).

(i) *If $f \in C^2(\mathbb{R})$, $f(-y) = -f(y)$ on $\mathbb{R}$, $f' \geqslant 0$, $(f'/f)' \leqslant 0$ on $(0,\infty)$ and $r \in C^1(\mathbb{R}_+)$, $(r/a_1)' \geqslant 0$ on $\mathbb{R}_+$, then the sequence $\{|y(t_k^1|\}_1^\infty$ is decreasing.*
(ii) *If $(a_2/a_1)' \leqslant 0$ on $\mathbb{R}_+$, then the sequence $\{|y^{[1]}(t_k^2)|\}_1^\infty$ is decreasing.*
(iii) *If $(a_3/a_2)' \leqslant 0$ on $\mathbb{R}_+$, then the sequence $\{|y^{[2]}(t_k^3)|\}_1^\infty$ is decreasing.*
(iv) *If $f \in C^1(\mathbb{R})$, $f' \geqslant 0$ on $\mathbb{R}$; $r \in C^1(\mathbb{R}_+)$ and $(r/a_3)' \leqslant 0$ on $\mathbb{R}_+$, then the sequence $\{|y^{[3]}(t_k^0)|\}_1^\infty$ is decreasing.*

P r o o f. Let $k \in \{1,2,\ldots\}$ and suppose, without loss of generality, that $y(t) < 0$ on (t_k^0, t_{k+1}^0). Then Lemma 1 and Lemma 2 yield

$$\begin{aligned}
&y(t) < 0 \quad \text{is increasing on } (t_k^1, t_{k+1}^0), \ y(t) > 0 \quad \text{on } (t_{k+1}^0, t_{k+1}^2],\\
&y^{[1]}(t) > 0 \ (< 0) \quad \text{on } [t_k^2, t_{k+1}^1) \ (\text{on } (t_{k+1}^1, t_{k+1}^2]),\\
&y^{[1]}(t) \quad \text{is decreasing on } (t_k^2, t_{k+1}^2),\\
&y^{[2]}(t) \quad \text{is increasing on } (t_k^3, t_{k+1}^2),\\
&y^{[3]}(t) \quad \text{is increasing (decreasing) on } (t_k^3, t_{k+1}^0) \ (\text{on } (t_{k+1}^0, t_{k+1}^2)).
\end{aligned} \tag{8}$$

(i) By virtue of (8) there exists t_k^* such that $t_k^* \in (t_k^3, t_{k+1}^0)$ and $y^{[3]}(t_k^*) = y^{[3]}(t_{k+1}^1)$. Let φ and ψ be the inverse functions to $y^{[3]}(t)$:

$$\begin{aligned}
t_k^* \leqslant \varphi(v) \leqslant t_{k+1}^0, &\quad y^{[3]}\left(\varphi(v)\right) = v,\\
t_{k+1}^0 \leqslant \psi(v) \leqslant t_{k+1}^1, &\quad y^{[3]}\left(\psi(v)\right) = v, \quad v \in I = [y^{[3]}(t_k^*), y^{[3]}(t_{k+1}^0)].
\end{aligned}$$

Evidently

$$\varphi(v) < \psi(v), \quad v \in I. \tag{9}$$

We prove by the indirect proof that

$$|y(\varphi(v))| \geqslant y(\psi(v)), \quad v \in I. \tag{10}$$

Define

$$S(v) = -f\left(y(\varphi(v))\right) - f\left(y(\psi(v))\right).$$

Using the assumptions of the theorem, (2), (8) and (9) we have ($' = \mathrm{d}/\mathrm{d}t$)

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}v} S(v) &= -f'\left(y(\varphi(v))\right) \frac{y'(\varphi(v))}{[y^{[3]}(\varphi(v))]'} - f'\left(y(\psi(v))\right) \frac{y'(\psi(v))}{[y^{[3]}(\psi(v))]'} \\ &= \frac{f'\left(y(\varphi(v))\right)}{f\left(y(\varphi(v))\right)} \frac{y^{[1]}(\varphi(v)) a_1(\varphi(v))}{r(\varphi(v))} + \frac{f'\left(y(\psi(v))\right)}{f\left(y(\psi(v))\right)} \frac{y^{[1]}(\psi(v)) a_1(\psi(v))}{r(\psi(v))} \\ &\leqslant \frac{a_1(\psi(v))}{r(\psi(v))} y^{[1]}(\psi(v)) \Big[-\frac{f'\left(|y(\varphi(v))|\right)}{f\left(|y(\varphi(v))|\right)} + \frac{f'\left(y(\psi(v))\right)}{f\left(y(\psi(v))\right)} \Big]. \end{aligned} \tag{11}$$

Suppose, on the contrary, that there exists $\bar{v} \in [y^{[3]}(t_k^*), y^{[3]}(t_{k+1}^0))$ such that $|y(\varphi(\bar{v}))| < y(\psi(\bar{v}))$. As $f' \geqslant 0$ we have $S(\bar{v}) = f\left(|y(\varphi(\bar{v}))|\right) - f\left(y(\psi(\bar{v}))\right) < 0$. Moreover, the assumptions of the theorem and (11) yield

$$|y(\varphi(v))| < y(\psi(v)) \Leftrightarrow S(v) < 0, \quad |y(\varphi(v))| < y(\psi(v)) \Rightarrow S'(v) \leqslant 0$$

for $v \in [y^{[3]}(t_k^*), y^{[3]}(t_{k+1}^0))$. Thus we can conclude $S\left(y^{[3]}(t_{k+1}^0)\right) < 0$ which contradicts $S\left(y^{[3]}(t_{k+1}^0)\right) = 0$, which holds by the definition of S. Thus (10) holds and (8) and (10) for $v = y^{[3]}(t_k^*)$ yield $|y(t_k^1)| > |y(t_k^*)| > y(t_{k+1}^1)$. The statement is proved.

(ii) (8) yields that there exists t_{k+1}^* such that $t_{k+1}^* \in (t_{k+1}^0, t_{k+1}^1)$ and $y(t_{k+1}^*) = y(t_{k+1}^2)$. Let functions φ and ψ be the inverse functions to y:

$$\begin{aligned} &t_{k+1}^* \leqslant \varphi(v) \leqslant t_{k+1}^1, \quad y(\varphi(v)) = v, \\ &t_{k+1}^1 \leqslant \psi(v) \leqslant t_{k+1}^2, \quad y(\psi(v)) = v, \quad v \in I = [y(t_{k+1}^*), y(t_{k+1}^1)]. \end{aligned}$$

Define $S(v) = y^{[1]}(\varphi(v)) - |y^{[1]}(\psi(v))|$. We prove by the indirect proof that

$$y^{[1]}(\varphi(v)) \geqslant |y^{[1]}(\psi(v))| \quad \text{for } v \in I. \tag{12}$$

Then, using (2), (8) and $(\frac{a_2}{a_1})' \leqslant 0$ we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}v} S(v) &= \frac{y^{[2]}(\varphi(v)) a_2(\varphi(v))}{y'(\varphi(v))} + \frac{y^{[2]}(\psi(v)) a_2(\psi(v))}{y'(\psi(v))} \\ &= \frac{y^{[2]}(\varphi(v))}{y^{[1]}(\varphi(v))} \frac{a_2(\varphi(v))}{a_1(\varphi(v))} + \frac{y^{[2]}(\psi(v))}{y^{[1]}(\psi(v))} \frac{a_2(\psi(v))}{a_1(\psi(v))} \\ &\leqslant y^{[2]}(\psi(v)) \frac{a_2(\psi(v))}{a_1(\psi(v))} \Big[\frac{1}{y^{[1]}(\varphi(v))} - \frac{1}{|y^{[1]}(\psi(v))|} \Big]. \end{aligned}$$

Hence, for $v \in [y(t^*_{k+1}), y(t^1_{k+1}))$,

$$S(v) < 0 \Rightarrow S'(v) < 0. \tag{13}$$

On the contrary, let there exists $\bar{v} \in [y(t^*_{k+1}), y(t^1_{k+1}))$ such that $S(\bar{v}) < 0$. Then (13) yields $S(y(t^1_{k+1})) < 0$ that contradicts $S(y(t^1_{k+1})) = 0$, which holds by the definition. Then (8), (12) (for $v = y(t^*_{k+1})$)and (13) yield

$$y^{[1]}(t^2_k) > y^{[1]}(t^*_{k+1}) \geqslant |y^{[1]}(t^2_{k+1})|$$

and thus the conclusion follows for $k = 0, 1, 2, \ldots$.

(iii), (iv) The proof is analogous to case (ii). □

R e m a r k. (i) The assumptions of Theorem 1 (i) posed on f are fulfilled e.g. for $f(x) = |x|^\lambda \operatorname{sgn} x$, $\lambda > 0$.

(ii) Note that the assumptions of (i) (either (ii) or (iii) or (iv)) are not (are) fulfilled in case of the differential equation in Example 1.

3. Oscillatory solutions vanishing at infinity

Theorem 1 gives sufficient conditions for the sequence of the absolute values of all local extremes of $y \in \mathcal{O}$ to be decreasing. Thus a question arises when $\lim_{t\to\infty} y(t) = 0$. This property is natural in the case of the usual derivatives, i.e. if $a_1 \equiv a_2 \equiv a_3 \equiv 1$ is valid.

Theorem A ([1, Th. 3.13]). *Let $a_i \equiv 1$, $i = 1, 2, 3$, (H1) be valid and let there exist a constant $M > 0$ such that $r(t) \geqslant \frac{M}{1+t}$ on $\mathbb{R}_+$. Then $\lim_{t\to\infty} y^{(j)}(t) = 0$ for $y \in \mathcal{O}$, $j = 0, 1$.*

Further, we will investigate this problem for Eq. (1). We start with some lemmas. The next one brings up a result concerning solutions fulfilling (3).

Lemma 4. *Let (H1) be valid and let $\{t^i_k\}$, $i = 0, 1, 2, 3$; $k = 1, 2$ and t^0_3 be numbers and y a solution of (1) such that (3) holds for $t \in [t^0_1, t^0_3]$. Let*

$$\Big(\frac{a_2}{a_1}\Big)' \geqslant 0, \ \Big(\frac{a_3}{a_1}\Big)' \geqslant 0 \quad \text{on } [t^0_1, t^0_3]. \tag{14}$$

Then

$$\sqrt{2}|y^{[1]}(t^2_1)| < |y^{[1]}(t^2_2)|.$$

P r o o f. According to Remark 2 it is sufficient to prove the result for Eq. (5), $a_0 \equiv a_1$ only. As, according to (14) $A_1 \equiv 1$, $A_2^{\bullet}(x) \geqslant 0$, $A_3^{\bullet} \geqslant 0$, the assertion is proved for an oscillatory solution Y of (5) fulfilling (3) (applied to Y) in [1], Lemmas 2 and 4 (with $\beta = 2$, $i = 1$). However, as follows from the proof only the information on $[t_1^0, t_3^0]$ was used. Thus the statement is valid for our solution, too. $\square$

Lemma 5. *Let $\{t_k^i\}$, $i = 0,1,2,3$; $k = 1,2$ and t_3^0 be numbers and let y be a solution of* (1) *such that* (4) *hold for $t \in [t_1^0, t_3^0]$. Let*

$$\left(\frac{a_2}{a_1}\right)' \leqslant 0 \quad \textit{and} \quad \left(\frac{a_3}{a_1}\right)' \leqslant 0 \quad \textit{on } [t_1^0, t_3^0]. \tag{15}$$

Then

$$|y^{[1]}(t_k^2)| \leqslant 2^{\frac{1-k}{2}} |y^{[1]}(t_1^2)|, \quad k = 1,2,\ldots \tag{16}$$

P r o o f. Put $t_1 = t_1^2$ and $t_2 = t_2^2$ for simplicity. Let us transform Eq. (1) into (5) according to Lemma 3 with $a_0 \equiv a_1$. Then x_i, $x_i = x(t_i)$, $i = 1,2$ are consecutive zeros of $Y^{\{2\}}$, $x_1 < x_2$ and $x(t_1^0) < x_1$.

Another transformation

$$\sigma = x_2 - x,\ Y(x) = Z(\sigma),\ x \in [x_1, x_2],\ \sigma \in [0, x_2 - x_1] \tag{17}$$

transforms Eq. (5) into

$$\left(\frac{1}{b_3(\sigma)}\left(\frac{1}{b_2(\sigma)}\left(\frac{1}{b_1(\sigma)}Z^{\mathbb{D}}\right)^{\mathbb{D}}\right)^{\mathbb{D}}\right)^{\mathbb{D}} + \bar{r}(\sigma)f(Z) = 0, \quad {}^{\mathbb{D}} = \frac{\mathrm{d}}{\mathrm{d}\sigma} \tag{18}$$

where

$$b_1 \equiv 1, \quad b_j(\sigma) = \frac{a_j(t(x(\sigma)))}{a_1(t(x(\sigma)))}, \quad j = 2,3, \quad \bar{r}(\sigma) = \frac{r(t(x(\sigma)))}{a_1(t(x(\sigma)))}.$$

This implies (15) and as $t(x)$ is increasing (see Lemma 3) we have

$$\left(\frac{b_j(\sigma)}{b_1(\sigma)}\right)^{\mathbb{D}} \geqslant 0 \quad \text{on } [0, x_3 - x_1] \text{ for } j = 2,3.$$

Moreover, it is easy to see that $\sigma_1 = 0$, $\sigma_2 = x_2 - x_1$ are two consecutive zeros of $(Z^{\mathbb{D}})^{\mathbb{D}}$ and (17) transforms (4) into (3). Hence, the assumptions of Lemma 4 are fulfilled for Eq. (18) and thus

$$\sqrt{2}|Z^{\mathbb{D}}(\sigma_1)| < |Z^{\mathbb{D}}(\sigma_2)|.$$

Using (17) and (6) we have $Z^{\mathbb{D}}(\sigma 1) = Y^{\{1\}}(x_2) = y^{[1]}(t_2)$ and $Z^{\mathbb{D}}(\sigma_2) = Y^{\{1\}}(x_1) = y^{[1]}(t_1)$. Hence,

$$\sqrt{2}|y^{[1]}(t_2)| < |y^{[1]}(t_1)|.$$

Consequently, the inequality (16) holds and $\lim\limits_{t\to\infty} y^{[1]}(t) = 0$. $\square$

Theorem 2. *Let* (H1) *and* $(\frac{a_2}{a_1})' \leqslant 0$, $(\frac{a_3}{a_1})' \leqslant 0$ *on* $\mathbb{R}_+$ *be valid and let* $y \in \mathcal{O}$. *Then* $\lim_{t\to\infty} y^{[i]}(t) = 0$, $i = 0, 1$, *if one of the following assumptions holds*:

(i) $(\frac{r}{a_3})' \leqslant 0$, $\frac{r}{a_1} \geqslant M > 0$ *on* $\mathbb{R}_+$ *and* $f' \geqslant 0$ *on* $\mathbb{R}$;

(ii) $\int_0^\infty a_1(t)\,\mathrm{d}t < \infty$.

Proof. Let $y \in \mathcal{O}$. According to Lemma 3 with $a_0 \equiv a_1$ it is sufficient to prove the result for Eq. (5) only. Denote by $\{x_k^i\}$, $i = 0, 1, 2, 3$; $k = 1, 2, \ldots$ the sequences given by Lemma 2 for Eq. (5) (i.e. $x_k^i = t_k^i$) and put $\Delta_k = [x_k^0, x_k^1]$. Then according to (4)

(19) $Y^{\{1\}}(x)Y(x) \geqslant 0$, $Y^{\{3\}}(x)Y(x) \geqslant 0$, $|Y^{\{1\}}|$ and $|Y^{\{3\}}|$ are decreasing on Δ_k.

Further,

$$\Big(\frac{R(x)}{A_3(x)}\Big)^{\bullet} = \Big(\frac{r(t(x))}{a_3(t(x))}\Big)^{\bullet} = \Big(\frac{r(t)}{a_3(t)}\Big)' t^{\bullet}(x) \leqslant 0 \quad \text{on } I = [0, x^*),$$

$$R(x) = \frac{r(t(x))}{a_1(t(x))} \geqslant M > 0 \quad \text{on } I,$$

$$\Big(\frac{A_i(x)}{A_1(x)}\Big)^{\bullet} = \Big(\frac{a_i(t(x))}{a_1(t(x))}\Big)^{\bullet} = \Big(\frac{a_i(t(x))}{a_1(t(x))}\Big)' t^{\bullet}(x) \leqslant 0 \quad \text{on } I,\ i = 1, 2$$

and the assumptions of Lemma 5 applied to (5) are fullfiled. Thus $\lim_{x\to x^*} Y^{\{1\}}(x) = 0$ and

(20) $$|Y^{\{1\}}(x_k^0)| \leqslant |Y^{\{1\}}(x_{k-1}^2)| \leqslant 2^{\frac{2-k}{2}} |Y^{\{1\}}(x_1^2)|, \quad k \geqslant 2;$$

note that the first inequality follows from (19).

We prove indirectly that

(21) $$\lim_{x\to x^*} Y(x) = 0.$$

Thus, suppose without loss of generality that

(22) $$|Y(x_k^1)| \geqslant M_1 > 0, \quad k = 1, 2, \ldots.$$

Then according to (19) there exists a sequence $\{\bar{x}_k\}_1^\infty$ such that $x_k^0 < \bar{x}_k < x_k^1$ and

(23) $$|Y(\bar{x}_k)| = \frac{M_1}{2}, \quad \frac{M_1}{2} \leqslant |Y(x)| \leqslant |Y(x_k^1)| \quad \text{on } \overline{\Delta}_k = [\bar{x}_k, x_k^1].$$

Denote $\delta_k = x_k^1 - \bar{x}_k$. Then using (7), (19), (20), (22) and (23) we obtain

$$\begin{aligned} \frac{M_1}{2} \leqslant |Y(x_k^1) - Y(\bar{x}_k)| = \int_{\overline{\Delta}_k} |Y^{\{1\}}(x)|\,\mathrm{d}x \leqslant |Y^{\{1\}}(x_k^0)|\delta_k \\ \leqslant 2^{\frac{2-k}{2}} |Y^{\{1\}}(x_1^2)|\delta_k,\ k \geqslant 2 \end{aligned}$$

and hence,

$$\lim_{k\to\infty} \delta_k = \infty. \tag{24}$$

(i) By virtue of (19), (22) and (23) we have

$$\begin{aligned}|Y^{\{3\}}(x_k^0)| &\geqslant [Y^{\{3\}}(\bar{x}_k) - Y^{\{3\}}(x_k^1)]\operatorname{sgn} Y(x_k^1) = \int_{\overline{\Delta}_k} |Y^{\{4\}}(x)|\,\mathrm{d}x \\ &= \int_{\overline{\Delta}_k} R(x)|f(Y(x))|\,\mathrm{d}x \geqslant M\delta_k \min_{\frac{M_1}{2}\leqslant |s|\leqslant M_1} |f(s)| > 0.\end{aligned}$$

Hence, (24) yields $\lim\limits_{k\to\infty} |Y^{\{3\}}(x_k^0)| = \infty$, which contradicts Th.1 (iv) applied to (5), the assumptions of which are fulfilled.

(ii) In this case $x^* < \infty$, I is bounded, which contradicts (24). □

R e m a r k . Note that the differential equation in Example 1 fulfils all assumptions of Theorem 2 (i) except of $\frac{r}{a_1} \geqslant M > 0$ on $\mathbb{R}_+$.

A powerful tool for investigations of the asymptotic behaviour of solutions of (1) consists in applying energy functions, see [6].

Let y be a solution of (1) defined on $\mathbb{R}_+$. Put

$$\begin{aligned}Z(t) &= -y(t)y^{[2]}(t) + \frac{a_1(t)}{a_2(t)}(y^{[1]}(t))^2 \\ &\quad - \int_0^t \Big[\Big(\frac{a_1(s)}{a_2(s)}\Big)' + \frac{a_3(s)}{2}W(s)\Big](y^{[1]}(s))^2\,\mathrm{d}s, \\ W(t) &= \frac{1}{a_2(t)}\Big(\frac{a_1(t)}{a_3(t)}\Big)', \quad t \in \mathbb{R}_+\end{aligned} \tag{25}$$

and

$$F(t) = -y(t)y^{[3]}(t) + \frac{a_1(t)}{a_3(t)}y^{[1]}(t)y^{[2]}(t) - \frac{W(t)}{2}(y^{[1]}(t))^2. \tag{26}$$

Then direct computation and (1) yield

$$Z'(t) = a_3(t)F(t), \tag{27}$$

$$F'(t) = r(t)y(t)f(y(t)) - \frac{1}{2}W'(t)(y^{[1]}(t))^2 + \frac{a_1(t)a_2(t)}{a_3(t)}(y^{[2]}(t))^2. \tag{28}$$

Lemma 6. *Let* (H1) *be valid,*

$$W'(t) \leqslant 0 \quad \textit{on } \mathbb{R}_+ \tag{H3}$$

and let $y \in \mathcal{O}$. Then

$$F(t) < 0,\ F \textit{ is nondecreasing on } \mathbb{R}_+. \tag{29}$$

P r o o f. As $y \in \mathcal{O}$, (4) is valid. Consequently, $y^{[1]}(t_k^1) = 0$, $y(t_k^1)y^{[3]}(t_k^1) > 0$ and thus $F(t_k^1) < 0$. Moreover, (H1), (H3) and (28) yield $F'(t) \geqslant 0$. The conclusion follows for $k \to \infty$. □

Theorem 3. *Let* (H1), (H3) *and*

$$\frac{r^2(t)}{a_1^2(t)} W(t) \geqslant M > 0 \quad \textit{for } t \in \mathbb{R}_+. \tag{30}$$

Then $\lim_{t\to\infty} y(t) = 0$ *holds for* $y \in \mathcal{O}$.

P r o o f. As $y \in \mathcal{O}$, (4) holds and, first, we prove that

$$W(t)(y^{[1]}(t))^2 \quad \text{is bounded on } I_k = [t_k^0, t_k^1],\ k = 1, 2, \ldots. \tag{31}$$

As the assumptions of Lemma 6 are fulfilled, (29) yields that the function F is bounded,

$$-\infty < F(0) \leqslant F(t) \leqslant 0, \quad t \in \mathbb{R}_+. \tag{32}$$

Moreover, (30) and (H3) yield $W(t) > 0$, W is nonincreasing. Further, we obtain from (4) that $|y^{[1]}(t)| \leqslant |y^{[1]}(t_k^0)|$ on I_k and $y^{[1]}(t_k^0)y^{[3]}(t_k^0) > 0$, $k = 1, 2, \ldots$. This together with (26) and (32) yields

$$\begin{aligned} 0 \leqslant W(t)(y^{[1]}(t))^2 &\leqslant W(t_k^0)(y^{[1]}(t_k^0))^2 \\ &= -2F(t_k^0) + 2\frac{a_1(t_k^0)}{a_3(t_k^0)} y^{[1]}(t_k^0)y^{[2]}(t_k^0) \leqslant -2F(0) \quad \text{on } I_k. \end{aligned} \tag{33}$$

Hence, (31) is valid.

We prove indirectly that $\lim_{t\to\infty} y(t) = 0$. Suppose on the contrary that there exist a constant $C > 0$ and a subsequence of natural numbers $\mathbb{N}_1$ such that

$$|y(t_k^1)| \geqslant C,\ k \in \mathbb{N}_1. \tag{34}$$

It will be clear that we can put $\mathbb{N}_1 = \{1, 2, \ldots\}$ without loss of generality. According to (4) and (34) there exist numbers τ_k and σ_k such that

$$\begin{aligned} &t_k^0 < \tau_k < \sigma_k \leqslant t_k^1, \quad |y(\tau_k)| = \frac{C}{2}, \quad |y(\sigma_k)| = C, \\ &\frac{C}{2} \leqslant |y(t)| \leqslant C \quad \text{for } t \in J_k = [\tau_k, \sigma_k],\ k = 1, 2, \ldots \end{aligned} \tag{35}$$

As (4) yields $y(t)y^{[1]}(t) > 0$ on J_k, using (28), (29), (30), (32), (33), (35) and (H3) we have

$$\begin{aligned}\infty > F(\infty) - F(0) &= \int_0^\infty F'(s)\,\mathrm{d}s \geqslant \int_0^\infty r(s)y(s)f(y(s))\,\mathrm{d}s\\ &\geqslant \sum_{k=1}^\infty \int_{J_k} \frac{r(s)y(s)f\left(y(s)\right)y'(s)}{a_1(s)y^{[1]}(s)}\,\mathrm{d}s\\ &= \sum_{k=1}^\infty \int_{J_k} \frac{r(s)\sqrt{W(s)}y(s)f\left(y(s)\right)|y'(s)|}{a_1(s)\sqrt{W(s)}|y^{[1]}(s)|}\,\mathrm{d}s\\ &\geqslant \Big(\frac{M}{2|F(0)|}\Big)^{1/2} \sum_{k=1}^\infty \left(\operatorname{sgn} y(t_k^1)\right) \int_{J_k} y(s)f\left(y(s)\right)y'(s)\,\mathrm{d}s\\ &= \Big(\frac{M}{2|F(0)|}\Big)^{1/2} \sum_{k=1}^\infty \int_{C\nu_k/2}^{C\nu_k} tf(t)\,\mathrm{d}t = \infty,\ \nu_k = \operatorname{sgn} y(t_k^1).\end{aligned}$$

The contradiction proves the conclusion. $\square$

It is evident that (30) is not valid for the differential equation without quasiderivatives, i.e. for $a_i \equiv 1$, $i = 1,2,3$. The following theorem removes this drawback.

Theorem 4. *Let* (H1), (H3) *be satisfied and let*

$$\frac{r^2(t)}{a_1(t)a_2(t)} \geqslant M > 0,\ \Big(\frac{a_1(t)}{a_2(t)}\Big)' + \frac{a_3(t)}{2}W(t) \leqslant 0 \quad \text{on } \mathbb{R}_+. \tag{36}$$

Then $\lim\limits_{t\to\infty} y(t) = 0$ *holds for* $y \in \mathcal{O}$.

P r o o f. As $y \in \mathcal{O}$, (4) holds and Lemma 6 yields $F < 0$. From this and from (27) we conclude that the function Z given by (25) is decreasing. Moreover, it is positive and bounded,

$$0 < Z(t) \leqslant M_1,\ t \in \mathbb{R}_+, \tag{37}$$

as according to (25) and (36) $Z(t_k^0) > 0$, $k = 1,2,\ldots$. As $y(t)y^{[2]}(t) \leqslant 0$ on $[t_k^0, t_k^1]$, (25), (36) and (37) yield

$$\frac{a_1(t)}{a_2(t)}(y^{[1]}(t))^2 \leqslant M_1,\ t \in [t_k^0, t_k^1],\ k = 1,2,\ldots. \tag{38}$$

We prove indirectly that $\lim\limits_{t\to\infty} y(t) = 0$. Suppose, on the contrary, that there exist $C > 0$ and a subsequence of natural numbers $\mathbb{N}_1$ such that

$$|y(t_k^1)| \geqslant C,\ k \in \mathbb{N}_1; \tag{39}$$

we can put without loss of generality $\mathbb{N}_1 = \{1,2,\dots\}$. Then according to (4) there exist numbers τ_k and σ_k such that (35) holds and we have similarly to the proof of Theorem 3

$$\begin{aligned}
\infty > F(\infty) - F(0) &\geqslant \sum_{k=1}^{\infty} \int_{J_k} \frac{r(s)y(s)f\left(y(s)\right)y'(s)}{a_1(s)y^{[1]}(s)}\,\mathrm{d}s \\
&= \sum_{k=1}^{\infty} \int_{J_k} \frac{r(s)y(s)f\left(y(s)\right)|y'(s)|}{\sqrt{a_1(s)a_2(s)}\sqrt{\frac{a_1(s)}{a_2(s)}}|y^{[1]}(s)|}\,\mathrm{d}s \\
&\geqslant \left(\frac{M}{M_1}\right)^{1/2} \sum_{k=1}^{\infty} \int_{C\nu_k/2}^{C\nu_k} tf(t)\,\mathrm{d}t = \infty
\end{aligned}$$

where $\nu_k = \operatorname{sgn} y(t_k^1)$. □

Remark. (i) Let $a_i \equiv 1$ for $i = 1,2,3$ and $r(t) \geqslant M > 0$ on $\mathbb{R}_+$. Then $\lim\limits_{t\to\infty} y(t) = 0$ for $y \in \mathcal{O}$.

(ii) Note that the conclusions of Theorem 3 and Theorem 4 hold without further assumptions on the nonlinearity f.

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