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A RIEMANN-TYPE DEFINITION FOR THE DOUBLE DENJOY INTEGRAL OF CHELIDZE AND DJVARSHEISHVILI

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Abstract. We give a Riemann-type definition of the double Denjoy integral of Chelidze and Djvarsheishvili using the new concept of CD filtering.

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1. Introduction

Chelidze and Djvarsheishvili [1] defined the double Denjoy integral in the plane and discussed its basic properties.

A Riemann-type definition for the Denjoy integral on the real line has been given by Bambang and Lee [2]. A crucial step in the Riemann-type definition is to prove the existence of a partition of a given interval. However, it is not known whether a similar result holds true in the plane.

Recently, Chew and Lee [3] presented an approach to nonabsolute integrals using Vitali covers so that a Riemann-type definition is still possible without having to prove the existence of a partition. In this note, we modify the integral introduced by Chew and Lee and prove that the resulting integral, which is of Riemann type, is equivalent to the double Denjoy integral of Chelidze and Djvarsheishvili.

A new concept of CD filtering is introduced in Section 4 below, which makes the main result of this paper possible.

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2. The B-integral

First, we define the B-integral of Chew and Lee [3].

Let I_0 be a nondegenerate closed bounded interval in the plane $\mathbb{R}^2$. Let Ψ be the class of all nondegenerate closed subintervals of I_0. An element $(I, x) \in \Psi \times I_0$ is called an *interval-point pair.* The point x is called the *associated point* of the interval I.

Let β be a collection of interval-point pairs. Then β is said to be a *Vitali cover* of a set $E \subset \mathbb{R}^2$ if for each $\delta > 0$ and any x in E there is an interval-point pair $(I, x) \in \beta$ such that $x \in I$ and $|I| < \delta$, where $|I|$ denotes the area of the interval I.

Let B be a collection of Vitali covers β of I_0. Then B is said to be *filtering* if for any $\beta_1, \beta_2 \in B$ there is $\beta_3 \in B$ such that $\beta_3 \subset \beta_1 \cap \beta_2$. Suppose $\beta \in B$ and $D \subset \beta$. For brevity, we write $D = \{(I, x)\}$ where (I, x) denotes a typical interval-point pair in D.

D is said to be a *partial β-partition* of the interval I_0 if $\{I; (I, x) \in D\}$ is a collection of nonoverlapping subintervals of I_0.

A partial β-partition $D = \{(I, x)\}$ of I_0 is a *β-partition* of I_0 if $\bigcup\limits_{(I,x)\in D} I = I_0$.

An interval function F defined on I_0 is a mapping from Ψ into the real line $\mathbb{R}$. An interval function F on I_0 is *additive* if $F(I \cup J) = F(I) + F(J)$ for any pair of nonoverlapping subintervals I and J of I_0 for which $I \cup J$ is an interval. A function $h\colon \Psi \times I_0 \to \mathbb{R}$ is called an *interval-point function.* We may view an interval function $F\colon \Psi \to \mathbb{R}$ as a special case of the interval-point function by putting $F(I, x) = F(I)$.

Let $h : \Psi \times I_0 \to \mathbb{R}$ be an interval-point function and $\beta \in B$. We write

$$V(h, \beta) = \sup\{(D)\Sigma|h(I, x)|; D = \{(I, x)\} \text{ a partial } \beta\text{-partition of } I_0\},$$

and refer to $V(h, \beta)$ as the variation of h over β. The variation of h over B is defined to be

$$V(h, B) = \inf\{V(h, \beta); \beta \in B\}.$$

Assume that $D_1 = \{(I, x)\}$ and $D_2 = \{(J, y)\}$ are two partial β-partitions. Then D_2 is said to be finer than D_1, denoted by $D_2 \geqslant D_1$, if for each (J, y) in D_2, there exists (I, x) in D_1 such that $J \subset I$.

An additive interval function F is said to be $AC^{**}_B(X)$, where $X \subseteq I_0$, if for every $\varepsilon > 0$ there exist $\beta \in B$ and $\eta > 0$ such that for any two partial β-partitions D_1 and D_2 with associated points in X and $D_2 \geqslant D_1$ satisfying $(D_1 \setminus D_2)\Sigma|I| < \eta$, we have

$$|(D_1 \setminus D_2)\Sigma F(I)| < \varepsilon,$$

where $(D_1 \setminus D_2)\Sigma$ denotes the difference $(D_1)\Sigma - (D_2)\Sigma$. Here D_2 may be void.

In the above definition, if we consider only one partial β-partition D_1, with D_2 being void, then F is said to be $AC^*_B(X)$.

An additive interval function F is said to be ACG^{**}_B if $I_0 = \bigcup_{i=1}^{\infty} X_i$ so that F is $AC^{**}_B(X_i)$ for each i.

Also, ACG^*_B can be similarly defined.

A collection B of Vitali covers β of I_0 is said to be *complete* if for any ACG^{**}_B additive interval function F on I_0 with $V(F, B) = 0$, we have $F \equiv 0$.

In the following definition, we assume that B is filtering and complete.

Definition 1. A measurable function f defined on I_0 is said to be *B-integrable* if there exists an ACG^{**}_B additive interval function F on I_0 such that

$$V(F - h, B) = 0,$$

where $h(I, x) = f(x)|I|$, i.e., for every $\varepsilon > 0$ there exists $\beta \in B$ such that

$$(D)\Sigma\big|F(I) - f(x)|I|\big| < \varepsilon$$

whenever $D = \{(I, x)\}$ is a partial β-partition of I_0.

If f is B-integrable on I_0, then so is f on a subinterval $I \subset I_0$. We write $F(I) = \int_I f$ and call F the primitive of f. In view of the completeness of B, the primitive F of a B-integrable function is uniquely determined. In view of the filtering of B, many properties of the B-integral can be proved, for example, if f and g are B-integrable on I_0, so is $f + g$.

We remark (see [3]) that the ACG^{**}_B condition in Definition 1 can be replaced by the following δ-fine property on B.

A collection B of Vitali covers β is said to have the *δ-fine property* if for every $\delta(x) > 0$ on I_0 there exists $\beta \in B$ which is δ-fine.

As usual, a Vitali cover β is said to be δ-fine [5] if $x \in I \subset N(x, \delta(x))$ whenever $(I, x) \in \beta$, where $N(x, \delta(x))$ denotes the open ball with centre x and radius $\delta(x)$.

3. The double Denjoy integral

Next, we define the double Denjoy integral of Chelidze-Djvarsheishvili [1], [4].

An interval function F is said to be *absolutely continuous* in the sense of Chelidze-Djvarsheishvili, or AC-CD, on a bounded set $E \subset \mathbb{R}^2$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $\{I_1, I_2, \dots, I_n\}$ is a finite collection of nonoverlapping intervals, each of which has at least one pair of opposite vertices in E, such that if $\sum_{i=1}^{n} |I_i| < \delta$, then $\sum_{i=1}^{n} |F(I_i)| < \varepsilon$.

An interval function F is *generalized absolutely continuous* in the sense of Chelidze-Djvarsheishvili, or ACG-CD, on I_0 if I_0 is expressible as the union of a sequence of sets on each of which F is AC-CD.

An interval function F is said to be *Saks-continuous* if $|I| \to 0$ implies $|F(I)| \to 0$.

The approximate derivative of F at x is defined by

$$D_{\mathrm{ap}}F(x) = \text{ap-}\lim_{I \to x} \frac{|F(I)|}{|I|},$$

in which x is a vertex of I and its opposite vertex runs over a set of density 1 at x. A set E is of density 1 at x if

$$\lim_{I \to x} \frac{|E \cap I|}{|I|} = 1,$$

where I is a square centered at x.

Definition 2. A function f is *double Denjoy integrable in the sense of Chelidze-Djvarsheishvili*, or CD-integrable, on I_0 if there exists an additive interval function F which is Saks-continuous and ACG-CD on I_0 such that $D_{\mathrm{ap}}F(x) = f(x)$ almost everywhere on I_0.

Furthermore, there is a monotonicity theorem in [1] which shows that the CD-integral is uniquely determined. The properties of the CD-integral can be found in [1]. See also [4].

4. A Riemann-type definition of the CD-integral

We will modify the B-integral as follows. Let $\beta \in B$ and $E \subset I_0$. We write

$$\beta[E] = \{(I, x) \in \beta; x \in E\}.$$

Two Vitali covers β_1 and β_2 are said to be *equivalent*, or $\beta_1 \sim \beta_2$, if $\beta_1[X] = \beta_2[X]$ where $|I_0 - X| = 0$.

A collection B of Vitali covers β is said to be *CD filtering* if for any $\beta_1, \beta_2 \in B$ there is $\beta_3 \in B$ such that $\beta_3 \subset \beta_1^* \cap \beta_2^*$, where $\beta_1^* \sim \beta_1$ and $\beta_2^* \sim \beta_2$.

It is important to consider CD filtering here and not filtering. As will become clear later, the Vitali covers we consider normally consist of two kinds of (I, x), one coming from the differentiability of F at x and the other from elsewhere, for example from ACG-CD and Saks-continuity.

Given two such Vitali covers β_1 and β_2, it is not always possible to find $\beta_3 \subset \beta_1 \cap \beta_2$. However, it is possible to find β_3 possessing the property given by the definition of CD filtering.

Let $h: \Psi \times I_0 \to R$ be an interval-point function and $\beta \in B$. We define $V(h, \beta, X)$ as in $V(h, \beta)$ above with an additional condition that $D = \{(I, x)\}$ is a partial β-partition of I_0 with $x \in X$. Next, we define

$$V(h, B, X) = \inf\{V(h, \beta, X); \beta \in B\}.$$

Further, B is said to be *CD complete* if for any Saks-continuous and ACG-CD additive interval function F on I_0 there exists a subset X_F of I_0 with $|I_0 - X_F| = 0$ such that if $V(F, B, X_F) = 0$ then $F \equiv 0$. Let B_{CD} denote a collection of Vitali covers β which is CD filtering and CD complete. We call $\beta_{CD} \in B_{CD}$ a *CD cover.*

Definition 3. A measurable function f defined on I_0 is said to be *approximately continuous integrable in the sense of CD*, or AP_{CD}-integrable, on I_0 if there exist an additive Saks-continuous interval function F which satisfies ACG-CD on I_0 and a subset X of I_0 which satisfies $|I_0 - X| = 0$ such that $V(F - h, B_{CD}, X) = 0$, where $h(I, x) = f(x)|I|$, i.e., there exists $X \subset I_0$ such that $|I_0 - X| = 0$ and for every $\varepsilon > 0$ there exists $\beta_{CD} \in B_{CD}$ such that

$$(D)\Sigma|F(I) - f(x)|I|\,| < \varepsilon$$

whenever $D = \{(I, x)\}$ is a partial β_{CD}-partition of I_0 with $x \in X$.

We shall prove in what follows that the above provides a Riemann-type definition of the CD-integral.

Theorem 4. *If f is CD-integrable on I_0, then there exists a collection B_{CD} of CD covers such that f is AP_{CD}-integrable on I_0.*

P r o o f. Since f is CD-integrable on I_0, there exists an additive Saks-continuous interval function F and F is ACG-CD on I_0 such that $D_{\mathrm{ap}}F(x) = f(x)$ almost everywhere on I_0.

Let E be the set of points $x \in I_0$ at which either $D_{\mathrm{ap}}(x)$ does not exist or if it does then $D_{\mathrm{ap}}(x) \neq f(x)$. Obviously, $|E| = 0$. For convenience, we may assume that $f(x) = 0$ if $x \in E$.

Now we construct β_{CD} as follows. Since F is approximately differentiable at each $x \in I_0 - E$, there is a Vitali cover β_x such that whenever $(I, x) \in \beta_x$ with x being a vertex of I we have

$$\bigl|F(I) - f(x)|I|\bigr| < \varepsilon|I|.$$

In view of Definition 3, the Vitali cover β_x can be defined arbitrarily for $x \in E$. Finally, put $\beta_{CD} = \{\beta_x; x \in I_0\}$. Write $X = I_0 - E$. Then for any β_{CD}-partition

$D = \{(I, x)\}$ of I_0 with $x \in X$ we have

$$(D) \sum_{x \in X} |f(x)|I| - F(I)| < \varepsilon |I_0|.$$

Here $(D) \sum\limits_P$ means that the sum is taken over $(I, x) \in D$ such that P is satisfied.

Let B_{CD} be the collection of all β_{CD} as defined above. It is easy to see that B_{CD} is CD filtering, though not filtering. Since the monotonicity theorem holds true for the CD-integral, hence B_{CD} is CD complete as shown in [3; Theorem 10]. Therefore f is AP_{CD}-integrable on I_0. □

In the following theorem, let B_{CD} be a collection of CD covers as defined in the proof of Theorem 4.

Theorem 5. *If f is AP_{CD}-integrable on I_0, then f is CD-integrable on I_0.*

Proof. Since f is AP_{CD}-integrable on I_0, there exists an additive Saks-continuous and ACG-CD interval function F on I_0 and there is $X \subset I_0$ which satisfies $|I_0 - X| = 0$ such that $V(F - h, B_{CD}, X) = 0$ where $h(I, x) = f(x)|I|$. It follows from [1] that $D_{\text{ap}}F(x)$ exists for almost all $x \in I_0$. It remains to show that $D_{\text{ap}}F(x) = f(x)$ almost everywhere on I_0.

Note that for every $\varepsilon > 0$ there is a CD cover $\beta_1 \in B_{CD}$ such that for any partial β_1-partition D of I_0 with $x \in X$ we have

$$(D)\Sigma \big| F(I) - f(x)|I| \big| < \varepsilon.$$

Let E be the set of $x \in I_0$ such that $D_{\text{ap}}F(x)$ exists but $D_{\text{ap}}F(x) \neq f(x)$. Since the approximate derivative $D_{\text{ap}}(x)$ exists, so does the regular derivative $DF(x)$ relative to a set of density 1 at x. Thus for every $x \in E$ there are $\eta(x) > 0$ and a sequence of interval-point pairs (Q_k, x) for $k = 1, 2, \ldots$ where Q_k are regular[1] intervals, x is a vertex of Q_k, and $|Q_k| \to 0$ as $k \to \infty$, such that for $k = 1, 2, \ldots$ we have

$$\big| F(Q_k) - f(x)|Q_k| \big| \geqslant \eta(x)|Q_k|.$$

Hence we form a Vitali cover β_2 of E. Note that Q_k above are all regular. Since B_{CD} is CD filtering, we can choose β_2 such that $\beta_2[E] \subset \beta_1[E]$. Put $E_n = \{x \in E; \eta(x) \geqslant 1/n\}$. It follows from the Vitali covering theorem that for every $\varepsilon > 0$ there is a partial division $D = \{(I, x)\} \subset \beta_2$ with $x \in E_n$ such that

$$|E_n| < \varepsilon + (D)\Sigma|I| \leqslant \varepsilon + (D)\Sigma \big| F(I) - f(x)|I| \big| / \eta(x) \leqslant \varepsilon + n\varepsilon.$$

[1] An interval $I = [a_1, b_1] \times [a_2, b_2] \subset I_0$ is called regular if there is a constant $\lambda \geqslant 1$ such that $1/\lambda \leqslant (b_1 - a_1)/(b_2 - a_2) \leqslant \lambda$; the regularity constant λ corresponding to Q_k can depend on k in our case.

Consequently, $|E_n| = 0$ for all n and $|E| = 0$. Hence $D_{\text{ap}}F(x) = f(x)$ almost everywhere in I_0. Therefore f is CD-integrable on I_0. □

Combining the above two theorems, we see that the AP_{CD}-integral is uniquely determined as the CD-integral is, and we have

Theorem 6. *There is a collection B_{CD} of CD covers such that a function f is AP_{CD}-integrable on I_0 if and only if f is CD-integrable on I_0.*

Since the CD covers are CD filtering, we can prove that if f and g are AP_{CD}-integrable on I_0 then so is $f + g$. Indeed, we can prove further properties.

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