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SEMISIMPLICITY AND GLOBAL DIMENSION OF A FINITE VON NEUMANN ALGEBRA

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Abstract. We prove that a finite von Neumann algebra $\mathscr{A}$ is semisimple if the algebra of affiliated operators $\mathscr{U}$ of $\mathscr{A}$ is semisimple. When $\mathscr{A}$ is not semisimple, we give the upper and lower bounds for the global dimensions of $\mathscr{A}$ and $\mathscr{U}$. This last result requires the use of the Continuum Hypothesis.

Keywords: finite von Neumann algebra, algebra of affiliated operators, semisimple ring, global dimension

MSC 2000: 16W99, 46L10, 46L99, 16K99

1. Introduction

A finite von Neumann algebra $\mathscr{A}$ comes equipped with a faithful and normal trace that enables us to define the dimension not just of a finitely generated projective module over $\mathscr{A}$ but also of any $\mathscr{A}$-module. This property makes $\mathscr{A}$ an interesting algebra and creates the possibilities for various applications. For applications of group von Neumann algebras in geometry and algebra, see [10].

The algebra $\mathscr{A}$ has some nice ring theoretic properties. It mimics the ring $\mathbb{Z}$ in such a way that every finitely generated module is a direct sum of a torsion and torsion-free part. The dimension faithfully measures the torsion-free part and vanishes on the torsion part. Although not without zero-divisors and, as we are going to see, rarely Noetherian, a finite von Neumann algebra is $\mathbb{Z}$-like: it is semihereditary (i.e., every finitely generated submodule of a projective module is projective) and has the classical quotient ring, constructed in the same way as $\mathbb{Q}$ is constructed from $\mathbb{Z}$. The classical ring of quotients $\mathscr{U}$ of $\mathscr{A}$ can be defined solely within the operator theory as the algebra of affiliated operators. Although $\mathscr{U}$ has many nice properties as a ring, it is not necessarily semisimple (Artinian and with trivial Jacobson radical) like $\mathbb{Q}$ is.

Let us consider the conditions.

(1) $\mathscr{U}$ is semisimple.

(2) $\mathscr{A}$ is $*$-isomorphic to the finite sum of algebras of $m_i \times m_i$ matrices over $L^\infty(n_i)$, $m_i > 0$, $n_i \geqslant 0$, $i = 1, \dots, k$ for some $k > 0$.

(3) $\mathscr{A}$ is isomorphic to the finite sum of rings of $m_i \times m_i$ matrices over $\mathbb{C}^{n_i}$, $m_i > 0$, $n_i \geqslant 0$, $i = 1, \dots, k$ for some $k > 0$.

(4) $\mathscr{A}$ is semisimple.

(5) $\mathscr{A}$ has finite $\mathbb{C}$-dimension.

It is well known that the conditions (2)–(5) are equivalent. Also, it is not hard to see that conditions (2)–(5) imply (1). Here, we shall prove that (1) implies the rest of the conditions (Theorem 4).

If a ring is not semisimple, its global dimension measures how close it is to being semisimple. The bounds for global dimension of $\mathscr{U}$ and $\mathscr{A}$ will be given in the infinite dimensional case. This result uses the Continuum Hypothesis (CH). Namely, in Theorem 8, we shall show:

(1) (CH) If $\dim_{\mathbb{C}} \mathscr{A} = \aleph_1$ then $\operatorname{gl.dim} \mathscr{U} = 2$ and $2 \leqslant \operatorname{gl.dim} \mathscr{A} \leqslant 3$.

(2) (CH) If $\dim_{\mathbb{C}} \mathscr{A} = \aleph_n$, $n > 0$, then $2 \leqslant \operatorname{gl.dim} \mathscr{U} \leqslant n+1$ and $2 \leqslant \operatorname{gl.dim} \mathscr{A} \leqslant n+2$.

The paper is organized as follows. In Sections 2 and 3, we list some results on a finite von Neumann algebra and its algebra of affiliated operators. In Sections 4 and 5, we list the preliminary facts and results that we need. In Section 6, we prove the result on the semisimplicity. In Section 7, we give the upper and lower bounds for the global dimension of non-semisimple $\mathscr{A}$ and $\mathscr{U}$.

The paper is written to be accessible both to an algebraist and an operator theorist, so sometimes even well known results from the fields are referenced for the sake of readability.

2. Finite von Neumann algebras

Let H be a Hilbert space and $\mathscr{B}(H)$ be the algebra of bounded operators on H. The space $\mathscr{B}(H)$ is equipped with five different topologies: norm, strong, ultrastrong, weak and ultraweak. The statements that a $*$-closed unital subalgebra $\mathscr{A}$ of $\mathscr{B}(H)$ is closed in weak, strong, ultraweak and ultrastrong topologies are equivalent. For details see [4] or [7] (Theorem 5.3.1).

A *von Neumann algebra* $\mathscr{A}$ is a $*$-closed unital subalgebra of $\mathscr{B}(H)$ which is closed with respect to weak (equivalently strong, ultraweak, ultrastrong) operator topology.

A $*$-closed unital subalgebra $\mathscr{A}$ of $\mathscr{B}(H)$ is a von Neumann algebra if and only if $\mathscr{A} = \mathscr{A}''$ where $\mathscr{A}'$ is the commutant of $\mathscr{A}$. The proof can be found in [7] (Theorem 5.3.1).

Let $Z(\mathscr{A})$ denotes the center of $\mathscr{A}$. A von Neumann algebra $\mathscr{A}$ is *finite* if there is a linear function $\operatorname{tr}_{\mathscr{A}}\colon \mathscr{A} \to Z(\mathscr{A})$ called *center-valued* (or *universal*) *trace* uniquely determined by the properties that

(1) $\operatorname{tr}_{\mathscr{A}}(ab) = \operatorname{tr}_{\mathscr{A}}(ba)$.
(2) $\operatorname{tr}_{\mathscr{A}}(a^*a) \geqslant 0$.
(3) $\operatorname{tr}_{\mathscr{A}}$ is *normal*: it is continuous with respect to ultraweak topology.
(4) $\operatorname{tr}_{\mathscr{A}}$ is *faithful*: $\operatorname{tr}_{\mathscr{A}}(a) = 0$ for some $a \geqslant 0$ (i.e. $a = bb^*$ for some $b \in \mathscr{A}$) implies $a = 0$.

The trace function extends to matrices over $\mathscr{A}$ in a natural way: the trace of a matrix is the sum of the traces of the elements on the main diagonal. This provides us with a way of defining a convenient notion of, not necessarily integer valued, dimension of any module.

If P is a finitely generated projective $\mathscr{A}$-module, there exist n and $f\colon \mathscr{A}^n \to \mathscr{A}^n$ such that $f = f^2 = f^*$ and the image of f is P. Then, the *dimension* of P is

$$\dim_{\mathscr{A}}(P) = \operatorname{tr}_{\mathscr{A}}(f).$$

For details see [13]. The center-valued dimension was also studied in [10].

If M is any $\mathscr{A}$-module, the *dimension* $\dim_{\mathscr{A}}(M)$ is defined as

$$\dim_{\mathscr{A}}(M) = \sup\{\dim_{\mathscr{A}}(P);\ P \text{ fin. gen. projective submodule of } M\}$$

where the supremum on the right side is an element of $Z(\mathscr{A})$ if it exists and is a new symbol ∞ otherwise. We define $a + \infty = \infty + a = \infty = \infty + \infty$ and $a \leqslant \infty$ for every $a \in Z(\mathscr{A})$.

The dimension satisfies the following properties.

(1) If $0 \to M_0 \to M_1 \to M_2 \to 0$ is a short exact sequence of $\mathscr{A}$-modules, then $\dim_{\mathscr{A}}(M_1) = \dim_{\mathscr{A}}(M_0) + \dim_{\mathscr{A}}(M_2)$.
(2) If M is a finitely generated projective module, then $\dim_{\mathscr{A}}(M) = 0$ if and only if $M = 0$.

The proof can be found in [13]. Part (1) follows from the first part of Proposition 13 in [13]. Part (2) follows from Theorem 17 in [13] (Additivity). For more properties of dimension, see [10] or [13].

As a ring, a finite von Neumann algebra $\mathscr{A}$ is *semihereditary* (i.e., every finitely generated submodule of a projective module is projective or, equivalently, every finitely generated ideal is projective). This follows from two facts. First, every von Neumann algebra is an AW^*-algebra and, hence, a Rickart C^*-algebra (see Chapter 1.4 in [3]). Second, a C^*-algebra is semihereditary as a ring if and only if it is Rickart (see Corollary 3.7 in [1]). The fact that $\mathscr{A}$ is Rickart also gives us that $\mathscr{A}$ is *nonsingular* (see 7.6 (8) and 7.48 in [9]).

Note also that every statement about left ideals over $\mathscr{A}$ can be converted to an analogous statement about right ideals. This is the case because $\mathscr{A}$ is a ring with involution (which gives a bijection between the lattices of left and right ideals and which maps a left ideal generated by a projection to a right ideal generated by the same projection).

3. Algebras of affiliated operators

A finite von Neumann algebra $\mathscr{A}$ is a pre-Hilbert space. Let $l^2(\mathscr{A})$ denote the Hilbert space completion of $\mathscr{A}$. $\mathscr{A}$ can be identified with the set of $\mathscr{A}$-equivariant bounded operators on $l^2(\mathscr{A})$, $\mathscr{B}(l^2(\mathscr{A}))^{\mathscr{A}}$, using the right regular representations (see section 9.1.4 in [10] for details).

Let a be a linear map $a\colon \operatorname{dom} a \to l^2(\mathscr{A})$, $\operatorname{dom} a \subseteq l^2(\mathscr{A})$. We say that a is *affiliated to $\mathscr{A}$* if

- i) a is densely defined (the domain $\operatorname{dom} a$ is a dense subset of $l^2(\mathscr{A})$);
- ii) a is closed (the graph of a is closed in $l^2(\mathscr{A}) \oplus l^2(\mathscr{A})$);
- iii) $ba = ab$ for every b in the commutant of $\mathscr{A}$.

Let $\mathscr{U} = \mathscr{U}(\mathscr{A})$ denote the *algebra of operators affiliated to $\mathscr{A}$*.

Proposition 1. *Let $\mathscr{A}$ be a finite von Neumann algebra and $\mathscr{U} = \mathscr{U}(\mathscr{A})$ its algebra of affiliated operators.*

1. *$\mathscr{A}$ is an Ore ring and $\mathscr{U}$ is the classical ring of quotients $Q_{\mathrm{cl}}(\mathscr{A})$ of $\mathscr{A}$.*
2. *$\mathscr{U}$ is a von Neumann regular (fin. gen. submodule of fin. gen. projective module is a direct summand), left and right self-injective ring equal to the maximal ring of quotients $Q_{\max}(\mathscr{A})$.*
3. *$\mathscr{U}$ is the injective envelope $E(\mathscr{A})$ of $\mathscr{A}$.*
4. *The set of projections (idempotents) in $\mathscr{U}$ is the same as the set of projections (idempotents) in $\mathscr{A}$.*

The proof of (1) can be found in [10] (Theorem 8.22). The proof of (2) is in [2] (Lemma 1, Theorem 2, Theorem 3). (3) follows from Theorem 13.36 in [9] (note that $\mathscr{A}$ is nonsingular). (4) follows from Theorem 1 in Section 48 and Corollary 1 in Section 49 from [3]. For a review of the ring theoretic notions of Ore ring, classical and maximal ring of quotients, self-injective ring and injective envelope see [9].

From this proposition it follows that the algebra $\mathscr{U}$ can be defined using purely algebraic terms (ring of quotient, injective envelope) on one hand and using just the language of operator theory (affiliated operators) on the other.

$K_0(\mathscr{A})$ and $K_0(\mathscr{U})$ are isomorphic. The isomorphism $\mu\colon K_0(\mathscr{A}) \cong K_0(\mathscr{U})$ is induced by the map $\mathrm{Proj}(\mathscr{A}) \to \mathrm{Proj}(\mathscr{U})$ given by $[P] \mapsto [\mathscr{U} \otimes_{\mathscr{A}} P]$ for any finitely generated projective module P (Theorem 8.22 in [10]). In [12] (Theorem 5.2) the explicit description of the map $\mathrm{Proj}(\mathscr{U}) \to \mathrm{Proj}(\mathscr{A})$ that induces the inverse of the isomorphism μ is obtained. Namely, the following holds.

Theorem 2. *There is an one-to-one correspondence between direct summands of $\mathscr{A}$ and direct summands of $\mathscr{U}$ given by $I \mapsto \mathscr{U} \otimes_{\mathscr{A}} I = E(I)$. The inverse map is given by $L \mapsto L \cap \mathscr{A}$. This correspondence induces an isomorphism of monoids $\mu\colon \mathrm{Proj}(\mathscr{A}) \to \mathrm{Proj}(\mathscr{U})$ and an isomorphism $\mu\colon K_0(\mathscr{A}) \to K_0(\mathscr{U})$ given by $[P] \mapsto [\mathscr{U} \otimes_{\mathscr{A}} P]$ with the inverse $[Q] \mapsto [Q \cap \mathscr{A}^n]$ if Q is a direct summand of $\mathscr{U}^n$.*

For proof, see Theorem 5.2 in [12].

4. Algebraic preliminaries

Let R be a ring with unit. Let $M_n(R)$ denotes the ring of $n \times n$ matrices over R.

4.1. Wedderburn-Artin Theorem asserts that a ring R is semisimple (Artinian with trivial Jacobson radical) if and only if there are positive integers m and n_i, $i = 1, \ldots, m$ and division rings D_i, $i = 1, \ldots, m$ such that R is isomorphic to the product of matrix rings $M_{n_i}(D_i)$, $i = 1, \ldots, m$. This result can be found in most of the algebra textbooks (e.g. Theorem 3.3 in [5]).

4.2. Morita Invariance Theorem asserts that $K_0(R) \cong K_0(M_n(R))$ for every ring R and every positive integer n (see, for example, [11], Theorem 1.2.4).

4.3. If R is a semisimple ring, $K_0(R)$ is a finitely generated abelian group. To show this, let R be a semisimple ring. By Wedderburn-Artin Theorem, R is isomorphic to $\prod_{i=1}^{m} M_{n_i}(D_i)$ for some $m > 0$, $n_i > 0$, $i = 1, 2, \ldots, m$ and division rings D_i. Then

$$K_0(R) \cong K_0\left(\prod_{i=1}^{m} M_{n_i}(D_i)\right) \cong \bigoplus_{i=1}^{m} K_0(M_{n_i}(D_i)) \cong \bigoplus_{i=1}^{m} K_0(D_i) \cong \mathbb{Z}^m.$$

For details see 1.2.8 and 1.1.6 in [11].

4.4. A ring R with unit is semisimple if and only if R is a direct sum of minimal left ideals of the form Re_i, $i = 1, \ldots, m$ where $e_1, e_2, \ldots e_m$ are orthogonal idempotents with $e_1 + e_2 + \ldots + e_m = 1$. This result can be found in various algebra textbooks (e.g. Theorem 3.7 in [5]).

Note that if $\mathscr{A}$ is a finite von Neumann algebra $\mathscr{A}$, p a projection of $\mathscr{A}$ and the left ideal $\mathscr{A}p$ is minimal, then p is a minimal projection. This is because for the projections $p, q \in \mathscr{A}$, $p = q$ if and only if $\mathscr{A}p = \mathscr{A}q$ (as in any ring with involution). Since $\mathscr{U}$ does not have any new projections (part (4) of Proposition 1), the same holds for $\mathscr{U}$: if the left ideal $\mathscr{U}p$ is minimal, then p is a minimal projection.

Also note that if e is an idempotent in $\mathscr{A}$, then there is a projection p such that $\mathscr{A}e = \mathscr{A}p$. This is true for every Rickart $*$-ring (see, for example, section 3 in [3]) so it holds for $\mathscr{A}$. Since the algebra of affiliated operators $\mathscr{U}$ is also Rickart, the same holds for $\mathscr{U}$ and an idempotent $e \in \mathscr{U}$. In that case, note that p is in $\mathscr{A}$ (part (4) of Proposition 1).

4.5. If R is a ring with unit, then R is semisimple if and only if the ring of $n \times n$ R-matrices $M_n(R)$ is semisimple. This follows from the following two well known facts. First, R is Artinian if and only if $M_n(R)$ is Artinian (Exercise 5 in 8.1 of [5]). Second, $J(M_n(R)) = M_n(J(R))$, where $J(R)$ denotes the Jacobson radical of the ring R (Exercise 13 in 9.2 of [5]).

Also, a product of rings $\prod_{i=1}^{n} R_i$ is semisimple if and only if each R_i, $i = 1, \ldots, n$ is semisimple (see Theorem 2.17 in 9.2 and Corollary 1.6 in 8.1 in [5]).

4.6. If $E(R)$ is an injective envelope of a ring R, it is easy to see that $M_n(E(R))$ is an injective $M_n(R)$-module which is an essential extension of $M_n(R)$. Thus, $M_n(E(R)) = E(M_n(R))$.

This gives us that $\mathscr{U}(M_n(\mathscr{A})) = M_n(\mathscr{U}(\mathscr{A}))$ for a finite von Neumann algebra $\mathscr{A}$ and its algebra of affiliated operators $\mathscr{U} = E(\mathscr{A})$.

5. Operator theory preliminaries

5.1. If $\mathscr{A}$ is a finite von Neumann algebra, $\mathscr{A}$ has a natural and unique decomposition as a direct sum of von Neumann algebras of types I_n for some positive integer n, and an algebra of type II_1. For the definition of types of von Neumann algebras and more details, see Section 6.5 in [8]. By definition of the type I_1, $\mathscr{A}$ is abelian if and only if $\mathscr{A}$ is of type I_1.

5.2. If $\mathscr{A}$ is of type I_n, $\mathscr{A}$ is $*$-isomorphic to the algebra $M_n(Z(\mathscr{A}))$ where $Z(\mathscr{A})$ is the center of $\mathscr{A}$ (Theorem 6.6.5. in [8]).

5.3. If a finite von Neumann algebra is of type II_1, the group $K_0(\mathscr{A})$ is isomorphic to the group $Z(\mathscr{A})^{\mathbb{Z}_2} = \{a \in Z(A);\ a = a^*\}$, the subgroup of $\mathbb{Z}_2$-invariants of $Z(\mathscr{A})$ with the action of $\mathbb{Z}_2$ by involution. This follows from the proof of Theorem 8.4.4. in [8]. Also, see Theorem 9.13. in [10].

As a consequence of this, $K_0(\mathscr{A})$ of a nontrivial von Neumann algebra of type II_1 is not finitely generated. Indeed, if $\mathscr{A}$ is nontrivial, the group $Z(\mathscr{A})^{\mathbb{Z}_2}$ is nontrivial as well. If nontrivial, the group $Z(\mathscr{A})^{\mathbb{Z}_2}$ contains a copy of $\mathbb{R}$ because it contains all the projections of the form $r1$ where $r \in \mathbb{R}$ and 1 is the unit of $\mathscr{A}$.

5.4. If S is a compact space and μ a finite measure on the Borel algebra of S, we use the standard notation $L_2(S, \mu)$ and $L^\infty(S, \mu)$ in their usual sense. If S is $\{1, 2, \ldots, n\}$, $\{1, 2, \ldots\} = \aleph_0$ or $[0, 1]$, the corresponding algebras are denoted by $L^\infty(n)$, $L^\infty(\aleph_0)$ and $L^\infty([0, 1])$ respectfully. Note that the first one is semisimple and finitely generated while the other two are not.

5.5. We use the term maximal abelian von Neumann algebra in its usual sense (e.g. see [7]). Recall that every abelian von Neumann algebra is $*$-isomorphic to a maximal abelian algebra. For the proof of this see section 9.4. in [8] or section I 7 in [4].

Every two $*$-isomorphic maximal abelian von Neumann algebras are unitarily equivalent. Moreover, the Hilbert spaces on which these two algebras act are isomorphic (Theorem 9.3.1. [8]).

5.6. Theorem 9.4.1. from [8], asserts that every maximal abelian von Neumann algebra that acts on a separable Hilbert space is unitarily equivalent to exactly one of the algebras $L^\infty(n)$, $L^\infty(\aleph_0)$, $L^\infty([0, 1])$, $L^\infty(n) \oplus L^\infty([0, 1])$ or $L^\infty(\aleph_0) \oplus L^\infty([0, 1])$.

6. Semisimplicity

For the main result, we need the following lemma.

Lemma 3. *If $\mathscr{A}$ is an abelian von Neumann algebra such that the algebra of affiliated operators $\mathscr{U}$ is semisimple, then $\mathscr{A}$ is finite dimensional.*

Proof. Let $\mathscr{A}$ be an abelian von Neumann algebra with semisimple $\mathscr{U}$. There are orthogonal idempotents e_i, $i = 1, 2, \ldots, n$, such that the left ideals $\mathscr{U} e_i$ are minimal $i = 1, 2, \ldots, n$, $\sum_{i=1}^{n} e_i = 1$ and $\mathscr{U} = \bigoplus_{i=1}^{n} \mathscr{U} e_i$ by 4.4. There are projections $p_i \in \mathscr{A}$, such that $\mathscr{U} e_i = \mathscr{U} p_i$, $i = 1, 2, \ldots, n$ (4.4). The projections p_i, $i = 1, 2, \ldots, n$, are minimal projections since the ideals $\mathscr{U} p_i$, $i = 1, 2, \ldots, n$, are minimal (4.4).

$\mathscr{A} p_i$ is a direct summand of $\mathscr{A}$ and so $\mathscr{A} p_i = \mathscr{A} \cap (\mathscr{U} \otimes_{\mathscr{A}} \mathscr{A} p_i)$ (by Theorem 2). But $\mathscr{U} \otimes_{\mathscr{A}} \mathscr{A} p_i = \mathscr{U} p_i$ and so $\mathscr{A} p_i = A \cap \mathscr{U} p_i$.

If $j \neq i$, then $\mathscr{A} p_i \cap \mathscr{A} p_j = \mathscr{A} \cap (\mathscr{U} p_i \cap \mathscr{U} p_j) = \mathscr{A} \cap (\mathscr{U} e_i \cap \mathscr{U} e_j) = 0$ and so $\bigoplus_{i=1}^{n} \mathscr{A} p_i \subseteq \mathscr{A}$. Moreover,

$$\mathscr{A} = \mathscr{A} \left(\sum_{i=1}^{n} e_i \right) \subseteq \sum_{i=1}^{n} \mathscr{A} e_i \subseteq \sum_{i=1}^{n} \mathscr{A} \cap \mathscr{U} e_i = \sum_{i=1}^{n} \mathscr{A} \cap \mathscr{U} p_i = \sum_{i=1}^{n} \mathscr{A} p_i = \bigoplus_{i=1}^{n} \mathscr{A} p_i.$$

The spaces $\mathscr{A} p_i$, $i = 1, 2, \ldots, n$, are one-dimensional so $\mathscr{A}$ is finite dimensional.

□

Theorem 4. *Let $\mathscr{A}$ be a finite von Neumann algebra with the algebra of affiliated operators $\mathscr{U}$. The following are equivalent:*

(1) *$\mathscr{U}$ is semisimple.*

(2) *$\mathscr{A}$ is $*$-isomorphic to the finite sum of algebras of $m_i \times m_i$ matrices over $L^\infty(n_i)$, $m_i > 0$, $n_i \geqslant 0$, $i = 1, \ldots, k$ for some $k > 0$.*

(3) *$\mathscr{A}$ is isomorphic to the finite sum of rings of $m_i \times m_i$ matrices over $\mathbb{C}^{n_i}$, $m_i > 0$, $n_i \geqslant 0$, $i = 1, \ldots, k$ for some $k > 0$.*

(4) *$\mathscr{A}$ is semisimple.*

(5) *$\mathscr{A}$ has finite $\mathbb{C}$-dimension.*

It is well known that the conditions (2)–(5) are equivalent. Also, it is not hard to see that conditions (2)–(5) imply (1). The main result here is that (1) implies the rest of the conditions.

P r o o f. (1) $\Rightarrow$ (2). Let $\mathscr{U}$ be semisimple. Then $K_0(\mathscr{U})$ is finitely generated by 4.3. Since $K_0(\mathscr{U}) \cong K_0(\mathscr{A})$ (Theorem 2), $K_0(\mathscr{A})$ is finitely generated as well. Hence, $\mathscr{A}$ has no summand of type II_1 and has just finitely many summands I_{m_i}, $m_i > 0$, $i = 1, \ldots, k$ by 5.1 and 5.3. Let us denote these summands by $\mathscr{A}_{m_i}$. For each i, $\mathscr{A}_{m_i}$ is $*$-isomorphic to the algebra $M_{m_i}(Z(\mathscr{A}_{m_i}))$ by 5.2. Note that $K_0(Z(\mathscr{A}_{m_i})) = K_0(\mathscr{A}_{m_i})$ by 4.2 and $\mathscr{U}(\mathscr{A}_{m_i}) = \mathscr{U}(M_m(Z(\mathscr{A}_{m_i}))) = M_m(\mathscr{U}(Z(\mathscr{A}_{m_i})))$ by 4.6. By 4.5, $\mathscr{U}(Z(\mathscr{A}_{m_i}))$ is semisimple. Thus, $Z(\mathscr{A}_{m_i})$ is finite dimensional by Lemma 3.

$Z(\mathscr{A}_{m_i})$ is $*$-isomorphic to a maximal abelian algebra by 5.5. As $Z(\mathscr{A}_{m_i})$ is finite dimensional, it is $*$-isomorphic to $L^\infty(n_i)$ for some nonnegative integer n_i by 5.6. Thus, $\mathscr{A}_{m_i}$ is $*$-isomorphic to the algebra of $m_i \times m_i$ matrices over $L^\infty(n_i)$.

(2) $\Rightarrow$ (3). $L^\infty(n)$ is isomorphic to $\mathbb{C}^n$ as rings.

(3) $\Rightarrow$ (4). If condition (3) is satisfied, then $\mathscr{A}$ is semisimple by 4.1 and 4.5.

(4) $\Rightarrow$ (1). If $\mathscr{A}$ is semisimple, then $\mathscr{A}$ is self-injective and, hence, equal to its injective envelope $\mathscr{U}$. Thus, $\mathscr{U} = \mathscr{A}$ is also semisimple.

(5) $\Rightarrow$ (4). If $\mathscr{A}$ has finite $\mathbb{C}$-dimension, then $\mathscr{A}$ is $*$-isomorphic to the direct sum of finitely many algebras of the form $M_{n_i}(\mathbb{C})$, $i = 1, \ldots, k$. This follows from Proposition 6.6.6, Theorem 6.6.1 and comments preceding Proposition 6.6.6 from [8] Thus, $\mathscr{A}$ is semisimple by 4.1.

(2) $\Rightarrow$ (5) is clear. □

Clearly, $\mathscr{A}$ is abelian if and only if $m = 1$ in (2) and (3).

Corollary 5. *Let $\mathscr{A}$ be a finite von Neumann algebra with the algebra of affiliated operators $\mathscr{U}$. The following conditions are equivalent to conditions* (1)–(5) *from Theorem 4:*

(6) *$\mathscr{A}$ is Noetherian (every ideal is finitely generated).*

(7) *$\mathscr{U}$ is Noetherian.*

(8) *$\mathscr{A}$ is hereditary (every ideal is projective).*
(9) *$\mathscr{U}$ is hereditary.*
(10) *$\mathscr{A}$ has finite universal dimension (every direct sum of nonzero submodules contained in $\mathscr{A}$ is finite).*
(11) *$\mathscr{U}$ has finite universal dimension.*

Note that the only nontrivial part is that (8) implies any of the other conditions.

Proof. Clearly, (1) implies (7), (9) and (11), and (4) implies (6), (8), and (10). Since (1) and (4) are equivalent, to show the equivalence of all of the conditions, it is sufficient to prove that each of the conditions (6)–(11) implies (1). We shall show (6) $\Rightarrow$ (7) $\Rightarrow$ (1), (8) $\Rightarrow$ (9) $\Rightarrow$ (1) and (11) $\Leftrightarrow$ (10) $\Rightarrow$ (1).

(6) $\Rightarrow$ (7) $\Rightarrow$ (1). The classical ring of quotients of an Ore and Noetherian ring is Noetherian (Proposition 10.32 (6) in [9]). Since $\mathscr{U} = Q_{\mathrm{cl}}(\mathscr{A})$ and $\mathscr{A}$ is Ore, (6) $\Rightarrow$ (7). A von Neumann regular and Noetherian ring is semisimple (Corollary 5.60 and Example 5.62a in [9]). Thus, (7) $\Rightarrow$ (1).

(8) $\Rightarrow$ (9) $\Rightarrow$ (1). A self-injective and hereditary ring is semisimple (Theorem 7.52 in [9]). Since $\mathscr{U}$ is self-injective, (9) $\Rightarrow$ (1). We shall show (8) $\Rightarrow$ (9), in two steps. First, we show that for every ideal J of $\mathscr{U}$, $J = \mathscr{U} \otimes_{\mathscr{A}} (J \cap \mathscr{A})$. Second, we show that if $\mathscr{A}$ is hereditary, $\mathscr{U} \otimes_{\mathscr{A}} (J \cap \mathscr{A})$ is projective. This will give us that every ideal J of $\mathscr{U}$ is projective.

Let J be an ideal of $\mathscr{U}$. Consider first the case when J is finitely generated. Since $\mathscr{U}$ is semihereditary, J is (finitely generated and) projective. Thus, there is a positive integer n such that J is a direct summand of $\mathscr{U}^n$. $\mathscr{U}$ is self-injective, so J is a direct summand of an injective module and therefore J is injective. So, the inclusion $J \hookrightarrow \mathscr{U}$ splits, so J is a direct summand of $\mathscr{U}$. Then, Theorem 2 gives us that $J = \mathscr{U} \otimes_{\mathscr{A}} (J \cap \mathscr{A})$.

If J is any ideal of $\mathscr{U}$, J is the directed union of its finitely generated submodules J_i (directed with respect to the inclusion maps). Then,

$$\begin{aligned}\mathscr{U} \otimes_{\mathscr{A}} (J \cap \mathscr{A}) &= \mathscr{U} \otimes_{\mathscr{A}} ((\varinjlim J_i) \cap \mathscr{A}) = \mathscr{U} \otimes_{\mathscr{A}} \varinjlim (J_i \cap \mathscr{A}) \\ &= \varinjlim \mathscr{U} \otimes_{\mathscr{A}} (J_i \cap \mathscr{A}) = \varinjlim J_i = J.\end{aligned}$$

Now, let us show that $\mathscr{U} \otimes_{\mathscr{A}} (J \cap \mathscr{A}) = J$ is projective for $\mathscr{A}$ hereditary. If $\mathscr{A}$ is hereditary, the module $J \cap \mathscr{A}$ is projective and so a direct summand of some free module $\bigoplus \mathscr{A}$. But then, $\mathscr{U} \otimes_{\mathscr{A}} (J \cap \mathscr{A})$ is a direct summand of $\bigoplus \mathscr{U}$ and so, projective. This gives us that every ideal of $\mathscr{U}$ is projective, so $\mathscr{U}$ is hereditary.

(11) $\Leftrightarrow$ (10) $\Rightarrow$ (1). If a ring R is Ore, the uniform dimension of R is equal to the uniform dimension of $Q_{\mathrm{cl}}(R)$ (Corollary 10.35 in [9]). Thus, (10) $\Leftrightarrow$ (11). If a nonsingular ring R has finite universal dimension, then $Q_{\max}(R)$ is semisimple

(Theorem 13.40 in [9]). Since $\mathscr{A}$ is nonsingular and $\mathscr{U} = Q_{\max}(\mathscr{A})$, (10) $\Rightarrow$ (1) follows. $\square$

7. Global dimensions of $\mathscr{U}$ and $\mathscr{A}$

In this section, we shall examine the global dimension of rings $\mathscr{U}$ and $\mathscr{A}$. The global dimension of a ring measures how close the modules over that ring are to being projective, therefore how close the ring is to being semisimple. The bounds for global dimension of $\mathscr{U}$ and $\mathscr{A}$ will be given.

The global dimension of a ring R is defined via the *projective dimension* of a left R-module M

$$\mathrm{pd}_R(M) = \min\{n;\ M \text{ has a projective resolution of length } n\}.$$

If this minimum does not exist, we define $\mathrm{pd}_R(M)$ to be ∞. Clearly, a left module M is projective if and only if $\mathrm{pd}(M) = 0$.

The *left global dimension* of a ring R is

$$\mathrm{l.gl.\,dim}\, R = \sup\{\mathrm{pd}_R(M);\ M \text{ is a left } R\text{-module}\}.$$

The left global dimension can be computed using ideals solely:

$$\mathrm{l.gl.dim} R = \sup\{\mathrm{pd}(R/I);\ I \text{ is a left } R\text{-ideal}\}.$$

See Corollary 5.51 in [9] for details.

The right global dimension is defined similarly. If left and right global dimensions of a ring are equal, we write just $\mathrm{gl.dim}\, R$ for $\mathrm{l.gl.dim}\, R = \mathrm{r.gl.dim}\, R$. This is the case for $\mathscr{A}$ and $\mathscr{U}$ since they are rings with involution so every statement about left ideals can be converted to an analogous statement about right ideals.

Clearly, a ring R is semisimple iff $\mathrm{r.gl.dim}\, R = 0$ iff $\mathrm{l.gl.dim}\, R = 0$. Also, R is left hereditary (every submodule of a projective left module is projective) if and only if $\mathrm{l.gl.dim}\, R \leqslant 1$.

We have seen that $\mathrm{gl.dim}\, \mathscr{U} = 0$ just if $\mathscr{A}$ is finite dimensional. Suppose that $\mathrm{gl.dim}\, \mathscr{U} = 1$. Then, $\mathscr{U}$ is hereditary. But every self-injective and hereditary ring is semisimple (we already used this in (9) $\Rightarrow$ (1) of Corollary 5), so $\mathrm{gl.dim}\, \mathscr{U} = 0$. So, if $\mathscr{A}$ is infinite dimensional, the global dimension of $\mathscr{U}$ is at least 2.

Tor functor defines another dimension of a ring. For review of: Let R be a ring and M a left R-module. The *weak dimension* of M is

$$\mathrm{wd}(M) = \sup\{n;\ \mathrm{Tor}_n^R(\_\_, M) \neq 0\}.$$

Clearly, M is a flat left module if and only if wd $(M) = 0$. If M is a right module, we can define its weak dimension as the supremum of dimensions n of nonvanishing $\mathrm{Tor}_n^R(M, \_\,)$. It can be shown that the supremum of weak dimensions of left modules is the same as the supremum of weak dimensions of right modules and that is the same as $\sup\{n;\ \mathrm{Tor}_n^R(\_,\_\,) \neq 0\}$ so, we can define the *weak global dimension* of R as

$$\mathrm{wd}R = \sup\{n;\ \mathrm{Tor}_n^R(\_,\_\,) \neq 0\}.$$

Since this definition is left-right symmetric, we do not have to distinguish left and right weak global dimension. For more details, see section 5D in [9].

A ring R is von Neumann regular if and only if all modules are flat (Theorem 4.21, [9]). Thus, R is von Neumann regular if and only if wd $R = 0$. So,

$$\mathrm{wd}\,\mathscr{U} = 0.$$

For any ring R, wd $R \leqslant 1$ if and only if a submodule of a flat module is flat. Since all semihereditary rings have this property (see Theorem 4.67 in [9]),

$$\mathrm{wd}\,\mathscr{A} \leqslant 1.$$

There are von Neumann algebras with weak global dimension 1 (Example 2.9 in [10]).

The following theorem of Jensen (Theorem 5.2 in [6]) connects the global dimension of a ring with its cardinality and its weak global dimension. Recall that $\aleph_0$ denotes the first infinite cardinal, the cardinality of the set of integers. Then, $\aleph_{n+1}$ is defined as the successor cardinal of $\aleph_n$, the least cardinal strictly larger than $\aleph_n$.

Theorem 6. *If R is a ring of cardinality $\aleph_n$, $n \geqslant 0$, then*

$$\text{l.gl.dim}R \leqslant \mathrm{wd}\,R + n + 1$$

and

$$\text{r.gl.dim } R \leqslant \mathrm{wd}\,R + n + 1.$$

If $\mathscr{A}$ is a finite von Neumann algebra, the cardinality of $\mathscr{A}$ is at least the continuum c (the cardinality of $\mathbb{C}$) since $\mathscr{A}$ contains a copy of the set of complex numbers $\{z1_{\mathscr{A}};\ z \in \mathbb{C}\}$ where $1_{\mathscr{A}}$ is the identity operator in $\mathscr{A}$. Also, since $\mathscr{U} = Q_{\mathrm{cl}}(\mathscr{A})$, $\mathscr{A} \subseteq \mathscr{U} \subseteq \mathscr{A} \times \mathscr{A}$. The cardinality of $\mathscr{A}$ is the same as the cardinality of $\mathscr{A} \times \mathscr{A}$ since both are infinite, so the cardinality of $\mathscr{A}$ and $\mathscr{U}$ are the same.

Since wd $\mathscr{U} = 0$ and wd $\mathscr{A} \leqslant 1$, the Theorem 6 gives us the following.

Corollary 7. *If* $\mathscr{A}$ *is a finite von Neumann algebra of cardinality* $\aleph_n$, $n > 0$, *with the algebra of affiliated operators* $\mathscr{U}$, *then*

$$\mathrm{gl.dim}\,\mathscr{U} \leqslant n + 1$$

and

$$\mathrm{gl.dim}\,\mathscr{A} \leqslant n + 2.$$

If we were to use this result, we would like to identify the cardinality of $\mathscr{A}$ as one of $\aleph$'s. Note that the case of $\aleph_\lambda$ when lambda is not finite in Jensen's theorem is trivial.

If V is an infinite dimensional complex space, its cardinality and its dimension over $\mathbb{C}$ are closely connected. Namely, if $\dim_{\mathbb{C}} V = \lambda$, where λ is infinite, then the cardinality of V is equal to the cardinality of finite subsets of the set $\mathbb{C} \times \lambda$. This cardinality is the same as cardinality of $\mathbb{C} \times \lambda$. Since λ is infinite, this is the maximum of the continuum c and λ.

Thus, using Corollary 7 for infinite dimensional $\mathscr{A}$ implies identifying the maximum of the dimension of $\mathscr{A}$ and the continuum c as one of the $\aleph$'s. That requires the use of the *Continuum Hypothesis* (CH). Recall that CH states that

$$c = \aleph_1.$$

In the sequel, we shall emphasize the use of CH.

Theorem 8. *Let* $\mathscr{A}$ *be a finite von Neumann algebra with the algebra of affiliated operators* $\mathscr{U}$, *then*

(1) $\dim_{\mathbb{C}} \mathscr{A} < \aleph_0$ *if and only if* $\mathrm{gl.dim}\,\mathscr{U} = \mathrm{gl.dim}\,\mathscr{A} = 0$.
(2) (CH) *If* $\dim_{\mathbb{C}} \mathscr{A} = \aleph_1$ *then* $\mathrm{gl.dim}\,\mathscr{U} = 2$ *and* $2 \leqslant \mathrm{gl.dim}\,\mathscr{A} \leqslant 3$.
(3) (CH) *If* $\dim_{\mathbb{C}} \mathscr{A} = \aleph_n$, $n > 0$, *then* $2 \leqslant \mathrm{gl.dim}\,\mathscr{U} \leqslant n + 1$ *and* $2 \leqslant \mathrm{gl.dim}\,\mathscr{A} \leqslant n + 2$.

Proof. (1) is proven in Theorem 4.

If $\dim_{\mathbb{C}} \mathscr{A} = \aleph_1$ (note that $\dim_{\mathbb{C}} \mathscr{A}$ cannot be $\aleph_0$ as $\mathscr{A}$ is a Banach space) then the cardinalities of $\mathscr{A}$ and $\mathscr{U}$ are both $\aleph_1$. So, $\mathrm{gl.dim}\,\mathscr{U} \leqslant 2$ by Corollary 7. If $\mathrm{gl.dim}\,\mathscr{U} \leqslant 1$, then $\mathscr{U}$ is hereditary and therefore semisimple (Corollary 5). But then $\dim_{\mathbb{C}} \mathscr{A} < \aleph_0$. Thus, $\mathrm{gl.dim}\,\mathscr{U} = 2$.

Similarly, $\mathrm{gl.dim}\,\mathscr{A} \leqslant 3$ by Corollary 7. But $\mathrm{gl.dim}\,\mathscr{A} \leqslant 1$ ($\mathscr{A}$ hereditary) is equivalent with the conditions from Theorem 4 and Corollary 5 and implies $\dim_{\mathbb{C}} \mathscr{A} < \aleph_0$. So, $\mathrm{gl.dim}\,\mathscr{A} \geqslant 2$.

(3) is proven analogously. □

Let D denote the statement: if $\dim_{\mathbb{C}} \mathscr{A} = \aleph_1$, then $\mathrm{gl.dim}\, \mathscr{U} = 2$. We have seen that CH implies D. The following questions are open:

(1) Does D hold without assuming CH? If the answer is yes, the proof would probably be very enlightening. If the answer is no, the next question will be of great interest.

(3) Does D imply CH? In other words, is D equivalent with CH? If so, we have another equivalent of CH in our hands.

(3) Can the bounds for global dimension of $\mathscr{A}$ and $\mathscr{U}$ be narrowed i.e. can Theorem 8 be improved?

(4) What can we say about global dimensions of $\mathscr{A}$ and $\mathscr{U}$ if the $\mathbb{C}$-dimension of $\mathscr{A}$ is $\aleph_\lambda$ with λ an infinite ordinal?

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