EXISTENCE OF POSITIVE SOLUTION OF A SINGULAR PARTIAL DIFFERENTIAL EQUATION

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Abstract. Motivated by Vityuk and Golushkov (2004), using the Schauder Fixed Point Theorem and the Contraction Principle, we consider existence and uniqueness of positive solution of a singular partial fractional differential equation in a Banach space concerning with fractional derivative.

Keywords: mixed Riemann-Liouville fractional derivative, function space concerning fractional derivative, existence and uniqueness, positive solution, fixed point theorem

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1. Introduction

Let $r = (\alpha, \beta)$, $0 < \alpha$, $\beta \leqslant 1$ and $0 < a < +\infty$, $0 < b < +\infty$. For $f \in L((0,a) \times (0,b))$, the expression

$$(I_0^r f)(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^x \int_0^y (x-s)^{\alpha-1}(y-t)^{\beta-1} f(s,t)\,\mathrm{d}s\,\mathrm{d}t$$

where $\Gamma(\cdot)$ is the Euler gamma function, is called [1] the left-sided mixed Riemann-Liouville integral of order r. In particular,

$$(I_0^r f)(x,y) = \int_0^x \int_0^y f(s,t)\,\mathrm{d}s\,\mathrm{d}t, \quad (I_0^0 f)(x,y) = f(x,y)$$

for almost all $(x,y) \in L((0,a) \times (0,b))$.

The mixed fractional Riemann-Liouville derivative of order r is defined [1] by the expression

$$(D_0^r f)(x,y) = D_{xy} f_{1-r}(x,y)$$

where $f_{1-r}(x,y) = (I_0^{1-r} f)(x,y)$ and $D_{xy} = \partial^2/\partial x \partial y$.

E x a m p l e 1.1.

$$(I_0^r)x^\lambda y^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+\alpha)\Gamma(1+\omega+\beta)}x^{\lambda+\alpha}y^{\omega+\beta}, \quad \lambda > -1,\ \omega > -1$$
$$(D_0^r)x^\lambda y^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda-\alpha)\Gamma(1+\omega-\beta)}x^{\lambda-\alpha}y^{\omega-\beta}, \quad \lambda > -1,\ \omega > -1$$

Proposition 1.1 [1]. *Let $q = (\gamma, \delta)$, $\gamma, \delta > 0$, the following relation is true*

$$(I_0^q I_0^r f)(x,y) = (I_0^{q+r} f)(x,y)$$

for all $f \in L((0,a)\times(0,b))$.

From the definition of the mixed Riemann-Liouville fractional derivative and integral, we have the following results.

Proposition 1.2. *The relation*

$$(D_0^r)(I_0^r)f(x,y) = f(x,y)$$

for all $f \in L((0,a)\times(0,b))$ holds.

Proposition 1.3. *Let f be a continuous function defined on $[0,a]\times[0,b]$. Assume that $(D_0^r f)(x,y)$ exists, $r = (\alpha,\beta)$. Then for $0 < r < 1$, the following relation*

$$(I_0^r D_0^r f)(x,y) = f(x,y)$$

holds.

Recently, there appeared many papers where the existence of solutions of initial value problem for partial differential equation of fractional order is considered, see [2]–[4]. In particular, A. N. Vityuk and A. V. Golushkov [5] consider the existence of solutions of systems of partial differential equations of fractional order in spaces of integrable functions

$$(D_0^{r_i} u_i)(x,y) = f_i[x,y,u(x,y)] \equiv f_i[x,y,u_1(x,y),\dots,u_n(x,y)]$$

with the initial value conditions

$$u_{i,1-r_i}(x,0) = \varphi_i(x), \quad 0 \leqslant x \leqslant a,$$
$$u_{i,1-r_i}(0,y) = \psi_i(y), \quad 0 \leqslant y \leqslant b, \quad \varphi_i(0) = \psi_i(0)$$

where $r_i = (\alpha_i, \beta_i)$, $0 < \alpha_i, \beta_i \leqslant 1$, $u_{i,1-r_i}(x,y) = (I_0^{1-r_i} u_i)(x,y)$, $\varphi_i(x) \in AC([0,a])$ and $\psi_i(y) \in AC([0,b])$, $i = 1, \ldots, n$. Motivated by [5], by means of the Schauder Fixed Point Theorem and Contraction Principle, we consider the existence and uniqueness of positive solution of the following singular partial differential equation of fractional order, in the function spaces concerning the mixed Riemann-Liouville fractional derivative

$$(1) \qquad \begin{cases} (D_0^r u)(x,y) = f(x,y,u(x,y),(D_0^{\varrho_1} u)(x,y),\ldots,(D_0^{\varrho_n} u)(x,y)), (x,y) \in p, \\ u(x,0) = u(0,y) = 0, \end{cases}$$

where $p = (0,a] \times (0,b]$, and $\varrho_i = (\gamma_i, \delta_i)$, $0 \leqslant \gamma_i, \ \delta_i < r$, $i = 1, \ldots, n$, that is $0 \leqslant \gamma_i < \alpha$, $0 \leqslant \delta_i < \beta$.

Definition. In this paper, the positive solution of problem (1) means that $u(0,y) = u(x,0) = 0$ and $u(x,y) > 0$ for $(x,y) \in (0,a] \times (0,b]$.

2. Function spaces concerning the mixed Riemann-Liouville fractional derivative

Let $P = [0,a] \times [0,b]$. Motivated by [6], we define function spaces as following

$$X = \{u \in C(P) \quad \text{having the mixed Riemann-Liouville fractional derivative of order } \varrho_i = (\gamma_i, \delta_i), \text{ and } (D_0^{\varrho_i} u) \in C(P), \ i = 1, \ldots, n\}$$

where $C(P)$ is the usual space of continuous functions on P, which is a Banach space endowed with the norm

$$\|u\|_0 = \max_{(x,y) \in P} |u(x,y)|$$

Theorem 2.1. *The space X endowed with the norm*

$$\|u\| = \|u\|_0 + \sum_{i=1}^{n} \|(D_0^{\varrho_i} u)\|_0$$

is a Banach space.

In order to prove this theorem, we first prove the following two results.

Lemma 2.2. *Let $\varrho_i = (\gamma_i, \delta_i)$, $i = 1, \dots, n$. If sequence of functions $w_n(x, y) \in C(P)$ converges uniformly to a function $w(x, y) \in C(P)$, then the sequence $(I_0^{\varrho_i} w_n)(x, y)$ converges uniformly to a function $(I_0^{\varrho_i} w)(x, y)$ in $C(P)$, $i = 1, \dots, n$.*

P r o o f. By the definition of operator $(I_0^{\varrho_i})$, $i = 1, \dots, n$, there has

$$\begin{aligned}
&|(I_0^{\varrho_i} w_n)(x, y) - (I_0^{\varrho_i} w)(x, y)| \\
&\quad = \left| \frac{1}{\Gamma(\gamma_i)\Gamma(\delta_i)} \int_0^x \int_0^y (x-s)^{\gamma_i - 1} (y-t)^{\delta_i - 1} (w_n(s,t) - w(s,t)) \, \mathrm{d}s \, \mathrm{d}t \right| \\
&\quad \leqslant \frac{1}{\Gamma(\gamma_i)\Gamma(\delta_i)} \int_0^x \int_0^y (x-s)^{\gamma_i - 1} (y-t)^{\delta_i - 1} \|w_n - w\|_0 \, \mathrm{d}s \, \mathrm{d}t \\
&\quad \leqslant \frac{1}{\Gamma(1+\gamma_i)\Gamma(1+\beta_i)} a^{\gamma_i} b^{\delta_i} \|w_n - w\|_0
\end{aligned}$$

which completes the proof. □

Lemma 2.3. *Let $\varrho_i = (\gamma_i, \delta_i)$, $0 \leqslant \gamma_i, \delta_i < 1$, $i = 1, 2, \dots, n$, and let $u_n(x, y) \in C(P)$ be a sequence, having the continuous mixed Riemann-Liouville fractional derivative of order $\varrho_i = (\gamma_i, \delta_i)$, $i = 1, 2, \dots, n$. Assume that the sequence $u_n(x, y)$ converges to the function $u(x, y)$ in $C(P)$-norm and that the sequence $(D_0^{\varrho_i} u_n)(x, y)$ converges to the function $v_i(x, y)$, $i = 1, 2, \dots, n$, in $C(P)$-norm, then, $(D_0^{\varrho_i} u)(x, y) = v_i(x, y)$, $i = 1, 2, \dots, n$.*

P r o o f. Setting $w_n(x, y) = (D_0^{\varrho_i} u_n)(x, y)$, $i = 1, 2, \dots, n$, then by Proposition 1.3 and Lemma 2.2,

$$(I_0^{\varrho_i} w_n)(x, y) = u_n(x, y), \ i = 1, 2, \dots, n$$

converge to the function $(I_0^{\varrho_i} v_i)(x, y)$, $i = 1, 2, \dots, n$ in C_0-norm. This means

$$u(x, y) = (I_0^{\varrho_i} v_i)(x, y), \ i = 1, 2, \dots, n$$

Hence, by Proposition 1.2, $(D_0^{\varrho_i} u)(x, y) = v_i(x, y)$, $i = 1, 2, \dots, n$. □

P r o o f o f T h e o r e m 2 . 1. Let $(u_n(x, y))_{n \in \mathbb{N}}$ be a Cauchy sequence in X. That is, for each $\varepsilon > 0$ there exists an index n_* such that for all $n, m > n_*$

$$\|u_n - u_m\| < \varepsilon$$

From the definition of X-norm, it follows that sequences $u_n(x, y)$, $(D_0^{\varrho_i} u_n)(x, y)$, $i = 1, 2, \dots, n$, are two Cauchy sequences in $C(P)$, which are complete. So, denoting by $u(x, y)$ the limit of sequence $u_n(x, y)$ and $v_i(x, y)$ the limit of sequence $(D_0^{\varrho_i} u_n)(x, y)$, $i = 1, 2, \dots, n$, Lemma 2.3 implies that $(D_0^{\varrho_i} u)(x, y) = v_i(x, y)$, $i = 1, 2, \dots, n$. This proves that X is a Banach space.

We assume that

(H1) $x^\mu y^\nu f\colon [0,a]\times[0,b]\times\mathbb{R}^{n+1}\to[0,+\infty)$, is continuous, where $0\leqslant\mu\leqslant\alpha-\gamma_i$, $0\leqslant\nu\leqslant\beta-\delta_i$, $i=1,2,\dots,n$;

Lemma 3.1. *Assume that* (H1) *holds, then a function* $u(x,y)\in X$ *is a solution of problem* (1) *if and only if* $u(x,y)$ *satisfies the following integral equation*

$$u(x,y)=(I_0^r f)(x,y,u(x,y),(D_0^{\varrho_1}u)(x,y),\dots,(D_0^{\varrho_n}u)(x,y)) \tag{2}$$

P r o o f. Let us first prove the necessity. If $u\in X$ is a solution of problem (1), then applying operator I_0^r to both sides of equation of (1), by the assumption (H1) and Proposition 1.3, we have

$$u(x,y)=(I_0^r f)(x,y,u(x,y),(D_0^{\varrho_1}u)(x,y),\dots,(D_0^{\varrho_n}u)(x,y))$$

for all $(x,y)\in P:=[0,a]\times[0,b]$. If we denote the right-hand side of this relation by $Tu(x,y)$, then we can check that it is in X. That is, that T maps X into itself. Indeed, for $u\in X$, by the definition of space X, for each $\varepsilon>0$, there exist $\eta_i>0$, $i=0,1,2,\dots,n$ such that, for each $(x_0,y_0)\in P$, when $|(x,y)-(x_0,y_0)|<\eta_i$, $i=0,1,2,\dots,n$, $(x,y)\in P$, we have

$$\|u(x,y)-u(x_0,y_0)\|_0<\varepsilon$$
$$\|(D_0^{\varrho_i}u)(x,y)-(D_0^{\varrho_i}u)((x_0,y_0)\|_0<\varepsilon,\ i=1,2,\dots,n.$$

Then, taking into account the assumption (H1), for any $(x_0,y_0)\in P$ and $(x,y)\in P$ such that $|(x,y)-(x_0,y_0)|<\delta_i$, $i=0,1,2,\dots,n$ we have

$$|x^\mu y^\nu f(x,y,u,(D_0^{\varrho_1}u),\dots,(D_0^{\varrho_n}u))-x_0^\mu y_0^\nu f(x_0,y_0,u,(D_0^{\varrho_1}u),\dots,(D_0^{\varrho_n}u))|<\varepsilon$$

Thus, for $u\in X$, combining with these facts and the definition of T, for each $\varepsilon>0$, $(x_0,y_0)\in P$, let

$$\theta=\min\Big\{\eta_i,\Big(\frac{\Gamma(1-\mu+\alpha)\Gamma(1-\nu+\beta)}{2ka^{1+\alpha-\mu}b^{1+\beta-\nu}\Gamma(1-\mu)\Gamma(1-\nu)}\Big)^{\frac{1}{\alpha-\mu}};$$
$$\Big(\frac{\Gamma(1-\mu+\alpha)\Gamma(1-\nu+\beta)}{2ka^{1+\alpha-\mu}b^{1+\beta-\nu}\Gamma(1-\mu)\Gamma(1-\nu)}\Big)^{\frac{1}{\beta-\nu}},\ i=0,1,2,\dots,n\Big\},$$

where k is maximum number of $x^\mu y^\nu |f(x,y,u,(D_0^{\varrho_1}u,\ldots,(D_0^{\varrho_n}u)|+1$ on $P\times[-\|u\|,\|u\|]^{n+1}$, when, for $|(x,y)-(x_0,y_0)|<\theta$, $(x,y)\in P$, we have

$$
\begin{aligned}
&|Tu(x,y)-Tu(x_0,y_0)|\\
&=\Big|\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1\int_0^1(1-s)^{\alpha-1}(1-t)^{\beta-1}(x^\alpha y^\beta f(xs,yt,u,(D_0^{\varrho_1}u),\ldots,(D_0^{\varrho_n}u))\\
&\quad -x_0^\alpha y_0^\beta f(x_0s,y_0t,u,(D_0^{\varrho_1}u),\ldots,(D_0^{\varrho_n}u)))\,\mathrm{d}s\,\mathrm{d}t\Big|\\
&\leqslant \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1\int_0^1(1-s)^{\alpha-1}(1-t)^{\beta-1}|x^\alpha y^\beta f(xs,yt,u,(D_0^{\varrho_1}u),\ldots,(D_0^{\varrho_n}u))\\
&\quad -x_0^\alpha y_0^\beta f(x_0s,y_0t,u,(D_0^{\varrho_1}u),\ldots,(D_0^{\varrho_n}u)))|\,\mathrm{d}s\,\mathrm{d}t\\
&\leqslant \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1\int_0^1(1-s)^{\alpha-1}(1-t)^{\beta-1}\\
&\qquad\qquad \times(x^{\alpha-\mu}y^{\beta-\nu}|(x)^\mu(y)^\nu f(xs,yt,u,(D_0^{\varrho_1}u),\ldots,(D_0^{\varrho_n}u))\\
&\quad -(x_0)^\mu(y_0)^\nu f(x_0s,y_0t,u,(D_0^{\varrho_1}u),\ldots,(D_0^{\varrho_n}u))|\\
&\quad +|x^{\alpha-\mu}y^{\beta-\nu}-x_0^{\alpha-\mu}y_0^{\beta-\nu}||(x_0)^\mu(y_0)^\nu f(x_0s,y_0t,u,(D_0^{\varrho_1}u),\ldots,(D_0^{\varrho_n}u))|)\,\mathrm{d}s\,\mathrm{d}t\\
&\leqslant \frac{\varepsilon}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1\int_0^1(1-s)^{\alpha-1}(1-t)^{\beta-1}x^{\alpha-\mu}y^{\beta-\nu}s^{-\mu}t^{-\nu}\,\mathrm{d}s\,\mathrm{d}t\\
&\quad +\frac{k|x^{\alpha-\mu}y^{\beta-\nu}-x_0^{\alpha-\mu}y_0^{\beta-\nu}|}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1\int_0^1(1-s)^{\alpha-1}(1-t)^{\beta-1}s^{-\mu}t^{-\nu}\,\mathrm{d}s\,\mathrm{d}t\\
&=\frac{\varepsilon}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x\int_0^y(x-s)^{\alpha-1}(y-t)^{\beta-1}xys^{-\mu}t^{-\nu}\,\mathrm{d}s\,\mathrm{d}t\\
&\quad +\frac{k|x^{\alpha-\mu}y^{\beta-\nu}-x_0^{\alpha-\mu}y_0^{\beta-\nu}|}{\Gamma(\alpha)\Gamma(\beta)}\\
&\qquad\qquad \times\int_0^x\int_0^y(x-s)^{\alpha-1}(y-t)^{\beta-1}x^{1-\alpha+\mu}y^{1-\beta+\nu}s^{-\mu}t^{-\nu}\,\mathrm{d}s\,\mathrm{d}t\\
&\leqslant\frac{\varepsilon a^{1+\alpha-\mu}b^{1+\beta-\nu}\Gamma(1-\mu)\Gamma(1-\nu)}{\Gamma(1-\mu+\alpha)\Gamma(1-\nu+\beta)}\\
&\quad +\frac{kab\Gamma(1-\mu)\Gamma(1-\nu)}{\Gamma(1-\mu+\alpha)\Gamma(1-\nu+\beta)}|x^{\alpha-\mu}y^{\beta-\nu}-x_0^{\alpha-\mu}y_0^{\beta-\nu}|
\end{aligned}
$$

In order to estimate $|x^{\alpha-\mu}y^{\beta-\nu}-x_0^{\alpha-\mu}y_0^{\beta-\nu}|$, we write

$$
\begin{aligned}
|x^{\alpha-\mu}y^{\beta-\nu}-x_0^{\alpha-\mu}y_0^{\beta-\nu}| &= |x^{\alpha-\mu}y^{\beta-\nu}-x_0^{\alpha-\mu}y^{\beta-\nu}+x_0^{\alpha-\mu}y^{\beta-\nu}-x_0^{\alpha-\mu}y_0^{\beta-\nu}|\\
&\leqslant y^{\beta-\nu}|x^{\alpha-\mu}-x_0^{\alpha-\mu}|+x_0^{\alpha-\mu}|y^{\beta-\nu}-y_0^{\beta-\nu}|\\
&\leqslant b^{\beta-\nu}|x^{\alpha-\mu}-x_0^{\alpha-\mu}|+a^{\alpha-\mu}|y^{\beta-\nu}-y_0^{\beta-\nu}|
\end{aligned}
$$

Next, we estimate $|x^{\alpha-\mu}-x_0^{\alpha-\mu}|$. Without loss of generality, we may assume that $x>x_0$. Since, by the triangle inequality, $|x-x_0|\leqslant|(x,y)-(x_0,y_0)|<\theta$, $|y-y_0|\leqslant$

$|(x,y)-(x_0,y_0)| \leqslant \theta$, thus, for $\theta \leqslant x_0 < x \leqslant a$, and by means of mean value theorem of differentiation, we find

$$x^{\alpha-\mu} - x_0^{\alpha-\mu} < (\alpha-\mu)\theta^{\alpha-\mu-1}(x-x_0) < 2\theta^{\alpha-\mu}$$

for $0 \leqslant x_0 < \theta$, $x \leqslant 2\theta$. Also, we find that

$$x^{\alpha-\mu} - x_0^{\alpha-\mu} \leqslant x^{\alpha-\mu} < 2^{\alpha-\mu}\theta^{\alpha-\mu} < 2\theta^{\alpha-\mu},$$

while for $0 \leqslant x_0 < x \leqslant \theta$, we find

$$x^{\alpha-\mu} - x_0^{\alpha-\mu} \leqslant x^{\alpha-\mu} \leqslant \theta^{\alpha-\mu} < 2\theta^{\alpha-\mu}.$$

We can obtain the estimate of $|y^{\beta-\nu} - y_0^{\beta-\nu}|$ by the same way. In consequence, we obtain

$$\begin{aligned}
|Tu(x,y) - Tu(x_0,y_0)| &\leqslant \frac{\varepsilon a^{1+\alpha-\mu} b^{1+\beta-\nu}\Gamma(1-\mu)\Gamma(1-\nu)}{\Gamma(1-\mu+\alpha)\Gamma(1-\nu+\beta)} \\
&\quad + \frac{kab|x^{\alpha-\mu}y^{\beta-\nu} - x_0^{\alpha-\mu}y_0^{\beta-\nu}|\Gamma(1-\mu)\Gamma(1-\nu)}{\Gamma(1-\mu+\alpha)\Gamma(1-\nu+\beta)} \\
&\leqslant \frac{\varepsilon a^{1+\alpha-\mu} b^{1+\beta-\nu}\Gamma(1-\mu)\Gamma(1-\nu)}{\Gamma(1-\mu+\alpha)\Gamma(1-\nu+\beta)} \\
&\quad + \frac{2ka^{1+\alpha-\mu} b^{1+\beta-\nu}\Gamma(1-\mu)\Gamma(1-\nu)}{\Gamma(1-\mu+\alpha)\Gamma(1-\nu+\beta)}(\theta^{\alpha-\mu} + \theta^{\beta-\nu}) \\
&\leqslant \frac{\varepsilon a^{1+\alpha-\mu} b^{1+\beta-\nu}\Gamma(1-\mu)\Gamma(1-\nu)}{\Gamma(1-\mu+\alpha)\Gamma(1-\nu+\beta)} + 2\varepsilon
\end{aligned}$$

Therefore, $Tu(x,y)$ is continuous at the point (x_0,y_0). It follows from the arbitrary choice of (x_0,y_0) that $Tu(x,y)$ is continuous in P, that is, $Tu(x,y) \in C(P)$. On the other hand, by Propositions 1.1 and 1.2, we see that

$$(D_0^{\varrho_i}Tu)(x,y) = (I_0^{r-\varrho_i}f)(x,y,u,(D_0^{\varrho_1}u),\ldots,(D_0^{\varrho_n}u)),\ i=1,\ldots,n.$$

In a similar way, we can obtain that the right-hand side of the above equality belongs to function space $C(P)$. That is, u is a solution of integral equation (2).

For sufficiency, applying D_0^r to both sides of (2), by Proposition 1.2, we obtain that u satisfies the equation in (1), and that, it follows from the necessity proof that $(I_0^r f)(x,y,u,(D_0^{\varrho_1}u),\ldots,(D_0^{\varrho_n}u)) \in C(P)$. Hence, $u(x,0) = u(0,y) = 0$, which implies that u is a solution of problem (1). The proof is complete. □

Next, define the operator $T\colon\ X \to X$ by

$$Tu(x,y) = (I_0^r f)(x,y,u(x,y),(D_0^{\varrho_1}u(x,y),\ldots,(D_0^{\varrho_n}u)(x,y)).$$

Lemma 3.2. *Assume that* (H1) *holds, then operator* $T\colon X\to X$ *is completely continuous.*

Proof. From (H1) and the proof of necessity in Lemma 3.1 and the Arzela-Ascoli Theorem, we can easily obtain that $T\colon X\to X$ is completely continuous. $\square$

Theorem 3.3. *Assume that* (H1) *holds, and* f *is nonnegative, satisfying one of the following conditions:*

(H2) *There exist constants* $c_i\geqslant 0$, $i=-1,0,1,2,\dots,n$ *and* $0<\tau_j<1$, $j=0,1,2,\dots,n$, *such that*

$$x^\mu y^\nu|f(x,y,u(x,y),(D_0^{\varrho_1}u),\dots,(D_0^{\varrho_n}u))|\leqslant c_{-1}+c_0|u|^{\tau_0}+\sum_{i=1}^n c_i|(D_0^{\varrho_i}u)|^{\tau_i}$$

for all $(x,y)\in P$.

(H3) *There exist constants* $d_i\geqslant 0$, $i=0,1,2,\dots,n$ *and* $\eta_j>1$, $j=0,1,2,\dots,n$, *such that*

$$x^\mu y^\nu|f(x,y,u(x,y),(D_0^{\varrho_1}u),\dots,(D_0^{\varrho_n}u))|\leqslant d_0|u|^{\eta_0}+\sum_{i=1}^n d_i|(D_0^{\varrho_i}u)|^{\eta_i}$$

for all $(x,y)\in P$.

(H4) *There exist constants* $c_i\geqslant 0$, $i=-1,0,1,2,\dots,n$, *satisfying*

$$\frac{\Gamma(1-\mu)\Gamma(1-\nu)a^{\alpha-\mu}b^{\beta-\nu}}{\Gamma(1-\mu+\alpha)\Gamma(1-\nu+\beta)}\Big(c_{-1}+\sum_{i=0}^n c_i\Big)\leqslant\frac{1}{n+1}$$
$$\frac{\Gamma(1-\mu)\Gamma(1-\nu)a^{\alpha-\gamma_i-\mu}b^{\beta-\delta_i-\nu}}{\Gamma(1-\mu+\alpha-\gamma_i)\Gamma(1-\nu+\beta-\delta_i)}\Big(c_{-1}+\sum_{i=0}^n c_i\Big)<\frac{1}{n+1}$$

$i=1,2,\dots,n$, *such that*

$$x^\mu y^\nu|f(x,y,u(x,y),(D_0^{\varrho_1}u),\dots,(D_0^{\varrho_n}u))|\leqslant c_{-1}+c_0|u|+\sum_{i=1}^n c_i|(D_0^{\varrho_i}u)|$$

for all $(x,y)\in P$.

Then problem (1) has at least a positive solution.

Proof. By Lemma 3.1, we know that we only need to consider existence of fixed point of operator T in X. It follows from Lemma 3.2 that $T\colon X\to X$ is a completely continuous operator. First, we assume that condition (H2) holds. Let

$\tau = \max\{\tau_0, \tau_1, \ldots, \tau_n\}$, and $B_R = \{u \in X;\ \|u\| < R\}$ be a closed, bounded and convex subset of the function space X, where

$$R > \max\left\{1; \frac{2c_{-1}(n+1)\Gamma(1-\mu)\Gamma(1-\nu)a^{\alpha-\mu}b^{\beta-\nu}}{\Gamma(1-\mu+\alpha)\Gamma(1-\nu+\beta)}; \frac{2c_{-1}(n+1)\Gamma(1-\mu)\Gamma(1-\nu)a^{\alpha-\gamma_i-\mu}b^{\beta-\delta_i-\nu}}{\Gamma(1-\mu+\alpha-\gamma_i)\Gamma(1-\nu+\beta-\delta_i)}; \left(\frac{\Gamma(1-\mu+\alpha)\Gamma(1-\nu+\beta)}{2(n+1)\sum\limits_{i=0}^{n}(c_i+1)\Gamma(1-\mu)\Gamma(1-\nu)a^{\alpha-\mu}b^{\beta-\nu}}\right)^{\frac{1}{1-\tau}}; \left(\frac{\Gamma(1-\mu+\alpha-\gamma_i)\Gamma(1-\nu+\beta-\delta_i)}{2(n+1)\sum\limits_{i=0}^{n}(c_i+1)\Gamma(1-\mu)\Gamma(1-\nu)a^{\alpha-\gamma_i-\mu}b^{\beta-\delta_i-\nu}}\right)^{\frac{1}{1-\tau}}\right\},$$

$i = 1, 2, \ldots, n$.

By (H2), for every $u \in X$, we have

$$\begin{aligned}
|Tu(x,y)| &= |(I_0^r f)(x, y, u, (D_0^{\varrho_1}u), \ldots, (D_0^{\varrho_n}u))| \\
&\leqslant \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^x \int_0^y (x-s)^{\alpha-1}(y-t)^{\beta-1} s^{-\mu} t^{-\nu} \\
&\quad \times \left(c_{-1} + c_0|u|^{\tau_0} + \sum_{i=1}^{n} c_i |(D_0^{\varrho_i}u)|^{\tau_i}\right) \mathrm{d}s\, \mathrm{d}t \\
&\leqslant \frac{\Gamma(1-\mu)\Gamma(1-\nu)a^{\alpha-\mu}b^{\beta-\nu}}{\Gamma(1-\mu+\alpha)\Gamma(1-\nu+\beta)} \left(c_{-1} + c_0\|u\|_0^{\tau_0} + \sum_{i=1}^{n} c_i \|(D_0^{\varrho_i}u)\|_0^{\tau_i}\right) \\
&\leqslant \frac{\Gamma(1-\mu)\Gamma(1-\nu)a^{\alpha-\mu}b^{\beta-\nu}}{\Gamma(1-\mu+\alpha)\Gamma(1-\nu+\beta)} \left(c_{-1} + \sum_{i=0}^{n} (c_i+1)R^{\tau_i}\right) \\
&\leqslant \frac{\Gamma(1-\mu)\Gamma(1-\nu)a^{\alpha-\mu}b^{\beta-\nu}}{\Gamma(1-\mu+\alpha)\Gamma(1-\nu+\beta)} \left(c_{-1} + \sum_{i=0}^{n} (c_i+1)R^{\tau}\right) \\
&= \frac{\Gamma(1-\mu)\Gamma(1-\nu)a^{\alpha-\mu}b^{\beta-\nu}}{\Gamma(1-\mu+\alpha)\Gamma(1-\nu+\beta)} \left(c_{-1} + R^{\tau-1}R\sum_{i=0}^{n} (c_i+1)\right) \\
&\leqslant \frac{R}{2(n+1)} + \frac{R}{2(n+1)} = \frac{R}{n+1}, \\
|(D_0^{\varrho_i}T)(x,y)| &= |(I_0^{r-\varrho_i} f)(x, y, u, (D_0^{\varrho_1}u), \ldots, (D_0^{\varrho_n}u))| \\
&\leqslant \frac{\Gamma(1-\mu)\Gamma(1-\nu)a^{\alpha-\gamma_i-\mu}b^{\beta-\delta_i-\nu}}{\Gamma(1-\mu+\alpha-\gamma_i)\Gamma(1-\nu+\beta-\delta_i)} \left(c_{-1} + \sum_{i=0}^{n} (c_i+1)R^{\tau-1}R\right) \\
&\leqslant \frac{R}{2(n+1)} + \frac{R}{2(n+1)} = \frac{R}{n+1}.
\end{aligned}$$

Hence, $\|Tu\| \leqslant R(n+1)^{-1} + \sum\limits_{i=1}^{n} R(n+1)^{-1} = R$ for $u \in B_R$, that is, $T(B_R) \subseteq B_R$. The Schauder Fixed Point Theorem implies that the operator T has at least a fixed point $u^* \in B_R$. By Lemma 3.1, problem (1) has a solution $u^* \in B_R$. On the other hand, by the nonnegativity of f and the monotonicity of (I_0^r), we obtain that $u^*(x,y) = Tu^*(x,y) = (I_0^r f)(x,y,u^*,(D_0^{\varrho_1}u^*),\ldots,(D_0^{\varrho_n}u^*)) \geqslant 0$, that is, problem (1) has a positive solution $u^* \in B_R$.

Secondly, we assume that condition (H3) holds. In a similar way, we can complete this proof, provided if we take a closed, bounded and convex subset $B_R = \{u \in X\,;\ \|u\| < R\}$ of the function space X, where

$$R < \min\left\{1, \left(\frac{\Gamma(1-\mu+\alpha)\Gamma(1-\nu+\beta)}{(n+1)\sum\limits_{i=0}^{n}(c_i+1)\Gamma(1-\mu)\Gamma(1-\nu)a^{\alpha-\mu}b^{\beta-\nu}}\right)^{1/(1-\eta)},\right.$$
$$\left.\left(\frac{\Gamma(1-\mu+\alpha-\gamma_i)\Gamma(1-\nu+\beta-\delta_i)}{(n+1)\sum\limits_{i=0}^{n}(c_i+1)\Gamma(1-\mu)\Gamma(1-\nu)a^{\alpha-\gamma_i-\mu}b^{\beta-\delta_i-\nu}}\right)^{1/(1-\eta)}\right\}$$

$i = 1,2,\ldots,n$, where $\eta = \min\{\eta_0,\eta_1,\ldots,\eta_n\}$.

For condition (H4), in a similar way, we can also easily complete this proof. □

Theorem 3.4. *Assume that* (H1) *holds, and f is nonnegative, satisfying the following condition:*

(H5) *There exist positive functions $g(x,y), h_i(x,y) \in C(P)$, $i = 1,2,\ldots,n$ satisfying*

$$(I_0^r x^{-\mu}y^{-\nu}g)(x,y) + \sum_{i=1}^{n}(I_0^{r-\varrho_i}x^{-\mu}y^{-\nu}g)(x,y) < \frac{1}{2}$$
$$\sum_{i=1}^{n}(I_0^r x^{-\mu}y^{-\nu}h_i)(x,y) + \sum_{i=1}^{n}\sum_{j=1}^{n}(I_0^{r-\varrho_i}x^{-\mu}y^{-\nu}h_j)(x,y) < \frac{1}{2}$$

such that

$$x^{\mu}y^{\nu}|f(x,y,u_1,(D_0^{\varrho_1}u_1),\ldots,(D_0^{\varrho_n}u_1)) - f(x,y,u_2,(D_0^{\varrho_1}u_2),\ldots,(D_0^{\varrho_n}u_2))|$$
$$\leqslant g(x,y)|u_1-u_2| + \sum_{i=1}^{n}h_i(x,y)|(D_0^{\varrho_i}u_1) - (D_0^{\varrho_i}u_2)|$$

for all $(x,y) \in P$ and $u_1,u_2 \in (-\infty,+\infty)$.

Then problem (1) *has a unique positive solution.*

P r o o f. By Lemma 3.1, we know that we only need to consider the existence of a fixed point of the operator T in X. It follows from the necessity proof of Lemma 3.1 that $T\colon X \to X$ is well defined.

For $\forall u, v \in X$, by assumption (H5), we have

$$
\begin{aligned}
&|Tu(x,y) - Tv(x,y)| \\
&= |(I_0^r f)(x,y,u,(D_0^{\varrho_1}u),\ldots,(D_0^{\varrho_n}u)) - (I_0^r f)(x,y,v,(D_0^{\varrho_1}v),\ldots,(D_0^{\varrho_n}v))| \\
&\leqslant (I_0^r)\Big(x^{-\mu}y^{-\nu}\Big(g(x,y)|u-v| + \sum_{i=1}^n h_i(x,y)|(D_0^{\varrho_i}u) - (D_0^{\varrho_i}v)|\Big)\Big) \\
&\leqslant (I_0^r)\Big(x^{-\mu}y^{-\nu}\Big(g(x,y)\|u-v\|_0 + \sum_{i=1}^n h_i(x,y)\|(D_0^{\varrho_i}u) - (D_0^{\varrho_i}v)\|_0\Big)\Big) \\
&\leqslant (I_0^r x^{-\mu}y^{-\nu}g)(x,y) + \sum_{i=1}^n (I_0^r x^{-\mu}y^{-\nu}h_i)(x,y)\|u-v\| \\
&|(D_0^{\varrho_i}Tu)(x,y) - (D_0^{\varrho_i}Tv)(x,y)| \\
&= |(I_0^{r-\varrho_i} f)(x,y,u,(D_0^{\varrho_1}u),\ldots,(D_0^{\varrho_n}u)) - (I_0^{r-\varrho_i} f)(x,y,(D_0^{\varrho_1}v),\ldots,(D_0^{\varrho_n}v))| \\
&\leqslant (I_0^{r-\varrho_i})\Big(x^{-\mu}y^{-\nu}\Big(g(x,y)|u-v| + \sum_{j=1}^n h_j(x,y)|(D_0^{\varrho_j}u) - (D_0^{\varrho_j}v)|\Big)\Big) \\
&\leqslant (I_0^{r-\varrho_i})\Big(x^{-\mu}y^{-\nu}\Big(g(x,y)\|u-v\|_0 + \sum_{j=1}^n h_j(x,y)\|(D_0^{\varrho_j}u) - (D_0^{\varrho_j}v)\|_0\Big)\Big) \\
&\leqslant (I_0^{r-\varrho_i} x^{-\mu}y^{-\nu}g)(x,y) + \sum_{j=1}^n (I_0^{r-\varrho_i} x^{-\mu}y^{-\nu}h_j)(x,y)\|u-v\|
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\|Tu - Tv\| &= \|Tu - Tv\|_0 + \sum_{i=1}^n \|(D_0^{\varrho_i}Tu) - (D_0^{\varrho_i}Tv)\|_0 \\
&\leqslant \Big((I_0^r x^{-\mu}y^{-\nu}g)(x,y) + \sum_{i=1}^n (I_0^{r-\varrho_i} x^{-\mu}y^{-\nu}g)(x,y) \\
&\quad + \sum_{i=1}^n (I_0^r x^{-\mu}y^{-\nu}h_i)(x,y) + \sum_{i=1}^n\sum_{j=1}^n (I_0^{r-\varrho_i} x^{-\mu}y^{-\nu}h_j)(x,y)\Big)\|u-v\| \\
&< \|u-v\|
\end{aligned}
$$

which implies that T is a contraction operator. Then the Contraction Principle assures that the operator T has a unique fixed point $u^* \in X$. By Lemma 3.1, problem (1) has a unique solution $u^* \in X$. By the same reason as in the Theorem 3.3, problem (1) has a unique positive solution $u^* \in X$.

R e m a r k 3.5. We can define another function space concerning the mixed Riemann-Liouville fractional derivative, and consider existence and uniqueness of solution of systems of partial differential equations of fractional order, which are analogy with that ones considered by A. N. Vityuk and A. V. Golushkov [5]

$$(D_0^{r_i} u_i)(x,y) = f_i[x, y, u_1(x,y), \ldots, u_n(x,y), (D_0^{\varrho_i} u_1(x,y), \ldots, (D_0^{\varrho_i} u_n(x,y)]$$

with the initial value conditions

$$\begin{aligned} u_{i,1-r_i}(x,0) &= \varphi_i(x), \quad 0 \leqslant x \leqslant a \\ u_{i,1-r_i}(0,y) &= \psi_i(y), \quad 0 \leqslant y \leqslant b, \quad \varphi_i(0) = \psi_i(0) \end{aligned}$$

where $r_i = (\alpha_i, \beta_i)$, $\varrho_i = (\gamma_i, \delta_i)$, $0 < \gamma_i < \alpha_i \leqslant 1$, $0 < \delta_i < \beta_i \leqslant 1$, $u_{i,1-r_i}(x,y) = (I_0^{1-r_i} u_i)(x,y)$, $\varphi_i(x) \in AC([0,a])$ and $\psi_i(y) \in AC([0,b])$, $i = 1, \ldots, n$.

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