Mem. Differential Equations Math. Phys. 13(1997), 136–139

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LINEAR PFAFF SYSTEMS WITH THE LOWER CHARACTERISTIC VECTORS' SET OF POSITIVE LEBESGUE MEASURE

(Reported on October 27, 1997)

In the closed first quarter R^2_+ of R^2, we consider the linear Pfaff system

$$\frac{\partial x}{\partial t_1} = A(t)x, \quad \frac{\partial x}{\partial t_2} = B(t)x, \quad x \in R^n, \quad t = (t_1, t_2) \in R^2_+, \tag{1}$$

where $A(t)$ and $B(t)$ are bounded ($\|A(t)\| \le a$, $\|B(t)\| \le b$ for $t \in \mathbb{R}^2_+$) and continuously differentiable matrices satisfying the following condition of complete integrability

$$\frac{\partial A(t)}{\partial t_2} + A(t)B(t) = \frac{\partial B(t)}{\partial t_1} + B(t)A(t), \quad t \in R^2_+. \tag{2}$$

Let $x : R^2_+ \to R^n$ be a nontrivial solution of (1). By analogy with the characteristic Lyapunov exponent [1, p.27], the lower Perron exponent [2, see also 3], the characteristic Grudo vector [4] and the characteristic Gaishun functional [5, 6], we introduce the lower characteristic vector $p = (p_1, p_2) \in R^2$ defined by the following condition

$$l_x(p) \equiv \varliminf_{t\to\infty} \frac{\ln\|x(t)\| - (p,t)}{\|t\|} = 0, \quad \varliminf_{t\to\infty} \frac{\ln\|x(t)\| - (p,t) - \varepsilon t_i}{\|t\|} < 0 \ \ \forall \varepsilon > 0, \quad i = 1, 2$$

(the vector limit condition $t \to \infty$ is equivalent to the unbounded growth of the norm $\|t\| = \sqrt{t_1^2 + t_2^2} \to +\infty$), and denoteit by $p[x]$. This notion has a direct application to the investigation of the Poisson stability of the trivial solution of the Pfaff system.

It is known [7] that the ordinary linear system $dx/dt = A(t)x$, $x \in R^n$, $t \in [0, +\infty)$, with continuous (and, even, infinitely differentiable) bounded coefficients, can have a segment of positive measure for its set of lower characteristic exponents. It may be assumed with a good reason that the set of characteristic vectors for solutions of the Pfaff system (1) has zero plane Lebegue measure. It is of interest to know whether the Pfaff system (1), with bounded and infinitely differentiable matrices A and B satisfying (2) in $\mathbb{R}^2_+$, exists such that its set $\Pi(A, B)$ of lower characteristic vectors (i.e., the union of lower characteristic vectors $p[x]$ of all nontrivial solutions $x(t, x_0)$) has positive plane Lebegue measure. The positive answer to this question is contained in the following

Theorem 1. *For any $\alpha_1 \le \alpha_2 \le 0$ and positive integer $n \ge 2$ there exists a completely integrable Pfaff system* (1) *such that the coefficient matrices $A(t)$ and $B(t)$ are bounded and infinitely differentiable and its set of lower characteristic vectors is* $\Pi(A, B) = \{p \in R^2_- : \alpha_1 \le p_1 + p_2 \le \alpha_2\}$.

Construction of the Pfaff system. In the trivial case $\alpha_1 = \alpha_2 = \alpha < 0$, the required system (1) is

$$\frac{\partial x}{\partial t_i} = \|\alpha\| E^{-1}(t) \frac{\partial E(t)}{\partial t_i} x, \quad x \in R^n, \quad t \in R^2_+, \qquad i = 1, 2, \quad E(t) \equiv e_{\oplus}^{-t_1} e^{-t_2},$$

1991 *Mathematics Subject Classification.* 35G05.

Key words and phrases. Linear Pfaff system, lower charcteristic vector.

whose general solution is $x(t,c) = (c_1,\dots,c_n) \times E^{|\alpha|}(t)$, $t \in R^2_+$. Moreover, any partial solution $x(t,\alpha) \neq 0$ has the same lower characteristic set (the set of all its lower characteristic vectors) $p_1 + p_2 = \alpha$, $p_i \leq 0$, $i = 1,2$.

Let us consider the nontrivial case $\alpha_1 < \alpha_2$. From our paper [7] we copy the construction of a perfect set $P_0 \subset \Delta = [0,1]$ analogous to the Cantor discontinuum [8, p. 56] and a modified Cantor step-function $\Theta_1(\alpha)$ [8, p. 232], using, as distinct from the one from [7], the value $\varepsilon_n \equiv \exp[(\alpha_1 - \alpha_2)\exp 2^{n+1}]$, $n \in N$.

Let $\Delta_0^{(1)} = \Delta$ be the initial segment, let $\Delta_1^{(i)} \subset \Delta_0^{(1)}$, $i = 1,2$, be segments of the first rank such that their length is equal to ε_1 and the left (right) endpoints of $\Delta_1^{(1)}$ and $\Delta(\Delta_1^{(2)}$ and $\Delta)$ coincide; then $\delta_1^{(1)} = \Delta \setminus \underset{i}{\cup}\Delta_1^{(i)}$ is an interval of the first rank. Similarly we represent an arbitrary segment Δ_n^m of the first rank (its length is equal to ε_n), $m \in \in \{1,\dots,2^n\}$, in the form of union of two nonintersecting segments $\Delta_{n\oplus 1}^{(2m-1)}$ and $\Delta_{n+1}^{(2m)}$ of the rank $(n+1)$ and one interval $\delta_{n+1}^{(m)} = \Delta_n^{(m)} \setminus (\Delta_{n+1}^{(2m-1)} \cup \Delta_{n+1}^{(2m)})$ of the rank $(n+1)$ such that the length of these segments is equal to ε_{n+1} and their left and right endpoints respectively coincide with the corresponding endpoints of Δ. We denote by $\alpha_n^{(m)}$ the midpoint of $\Delta_n^{(m)}$, $m = 1,\dots,2^n$. Finally, we introduce the set $P_0 = \bigcap\limits_{n=1}^{\infty} \bigcup\limits_{i=1}^{2^n} \Delta_n^{(i)}$ with zero linear Lebesgue measure [see 7, p. 56].

On the segment $\Delta = [0,1]$, we set the Cantor step-function $\Theta_1 : \Delta \to [0,1]$ for which the constancy sections are intervals $\delta_n^{(m)}$. It is known [8, p. 232] that this function is continuous on Δ and its range is $[0,1] = \{\Theta_1(\alpha) : \alpha \in P_0\}$. Now we define the new function $\Theta(\alpha) = -\alpha_2 + (\alpha_2 - -\alpha_1)\Theta_1(\alpha) : [0,1] \to [|\alpha_2|, |\alpha_1|]$.

With the help of the straight line $l_k : t_1 + t_2 = e^k$, $t_i \geq 0$, we divide the quarter $R^2_+ = \{t = (t_1,t_2) : t_i \geq 0\}$ into strips $\{t \in \mathbb{R}^2_+ : e^k \leq t_1 + t_2 < e^{k+1}\}$, $k \geq 0$, denoted consecutively by $\Pi_n^{(m)}$ with the closed "left" and open "right" boundaries, so that the right boundary of $\Pi_n^{(m)}$ coincides with the left boundary of $\Pi_n^{(m+1)}$, $m \in \{1,\dots,2^n\}$; besides, the right boundary of $\Pi_n^{(2n)}$ coincides with the left boundary of $\Pi_{n+1}^{(1)}$ and the initial strip $\Pi_1^{(1)}$ is situated between the intercepts of the straight lines l_0 and l_1.

We set $\tau_t \equiv t_1 + t_2$, $t \in R^2_+$, and define the functions $f(\tau_t)$ and $F(\tau_t)$ of the argument $\tau_t \in [0,+\infty)$ as follows. We select respectively the left and right parts $\widetilde{\Pi}_n^{(m)} = \{t \in \Pi_n^{(m)} : \tau(n,m) \leq \tau_t < \tau(n,m)\sqrt{e}\}$ and $\widetilde{\widetilde{\Pi}}_n^{(m)} = \Pi_m^{(m)} \setminus \widetilde{\Pi}_n^m$ of the strip $\Pi_n^{(m)}$ with the left boundary $\tau_t = \tau(n,m)$.

Using the analogue

$$\varphi(\tau_t,\tau(1),\tau(2)) = \begin{cases} \exp\{-\ln^{-2}(\tau_t/\tau(1) \times \exp[-\ln^{-2}(\tau_t/\tau(2))]\}, \tau(1) < \tau_t < \tau(2), \\ i-1 \quad \text{for} \quad \tau_t = \tau(i), \;\; i = 1,2, \end{cases}$$

for the standard function [9, p. 54] infinitely differentiable on the segment $[\tau(1),\tau(2)]$, we define a bounded infinitely differentiable function $f(\tau_t)$ as follows. We set $f(\tau_t) = |\alpha_2|$ if $t \in \widetilde{\Pi}_n^{(m)}$ for any $n \in N$ and admissible m and extend this value to the triangle $\{t \in R^2_+ : 0 \leq \tau_t \leq 1\}$; in $\widetilde{\widetilde{\Pi}}_n^{(m)}$ we define this function by

$$f(\tau_t) = \begin{cases} |\alpha_2| + [\Theta(\alpha_n^{(m)}) - |\alpha_2|]\varphi(\tau_t,\tau(n,m)\sqrt{e},\bar{\tau}(n,m)), & t \in \bar{\Pi}_n^{(m)}, \\ \Theta(\alpha_n^{(m)} + [|\alpha_2| - \Theta(\alpha_n^{(m)})]\varphi(\tau_t,\bar{\tau}(n,m),\tau(n,m)\sqrt{e}), & t \in \bar{\bar{\Pi}}_n^{(m)}. \end{cases}$$

On the closure of $\widetilde{\widetilde{\Pi}}_n^{(m)}$ we set $F(\tau_t) = \alpha_n^{(m)}$ if $t \in \widetilde{\widetilde{\Pi}}_n^{(m)}$, $n \in N$, $m = 1,\dots,2^n$ and if $\tau_t = \tau(n,m)\sqrt{e}$, and in the strip $\widetilde{\Pi}_n^{(m)}$ without its left boundary $\tau_t = \tau(n,m)$ we set

$$F(\tau_t) = F(\tau(n,m)) + [F(\tau(n,m)\sqrt{e}) - F(\tau(n,m))] \times$$
$$\times \varphi(\tau_t,\tau(n,m),\tau(n,m)\sqrt{e}), \quad \tau(n,m) < \tau_t < \tau(n,m)\sqrt{e}.$$

On the remaining triangle $\{t \in R^2_+ : 0 \le \tau_t < 1\}$, we continue the function $F(\tau_t)$ as the constant. This function is also bounded and infinitely differentiable, moreover, it has bounded derivatives $\partial F(\tau_t)/\partial t_i$, $i = 1, 2$.

Let us introduce the function $E(t) = e^{-t_1} + e^{-t_2}$, $t \in R^2_+$, and the two-dimensional vector-function

$$x(t, c) = (C_1[E(t)]^{f(\tau_t)}, \quad [C_1 F(\tau_t) + C_2][E(t)]^{|\alpha_2|}), \quad t \in R^2_+, \tag{3}$$

depending on an arbitrary constant vector $c \in \mathbb{R}^2$ and being the general solution of the following two-dimensional linear system in partial derivatives

$$\frac{\partial x}{\partial t_i} = A_i(t)x, \quad x \in R^2, \quad t = (t_1, t_2) \ge 0, \quad i = 1, 2, \tag{4}$$

with bounded and infinitely differentiable matrices

$$A_i(t) = \begin{pmatrix} \dfrac{\partial f(\tau_t)}{\partial t_i} \ln E(t) + f(\tau_t) \dfrac{\partial E(t)/\partial t_i}{E(t)} & 0 \\ \dfrac{\partial F(\tau_t)}{\partial t_i} [E(t)]^{|\alpha_2| - f(\tau_t)} & |\alpha_2| \dfrac{\partial E(t)/\partial t_i}{E(t)} \end{pmatrix}, \quad t \in R^2_+, \quad i = 1, 2.$$

Since the functions $f(\tau_t)$ and $F(\tau_t)$ are expressed linearly by means of $\varphi(\tau_t, \tau(1), \tau(2))$, the infinite differentiability of the introduced matrices follows from the similar property of $\varphi(\tau_t, \tau(1), \tau(2))$ and $E(t)$, where $E(t)$ is positive for all $t \in \mathbb{R}^2_+$. The boundedness of the matrices $A_1(t)$ and $A_2(t)$ follows from their definition. From the infinite differentiability of the vector-function $x(t, c)$, which is the general solution (3) of (4), it follows that the condition (2) is fulfilled with $A(t) = A_1(t)$ and $B(t) = A_2(t)$. Hence the system (4) is completely integrable.

The main step for construction of the lower characteristic set $\Pi(A_1, A_2)$ is the proof of the following fact: the lower characteristic set of the solution $x(t, a)$ with an initial vector $0 \ne a = (c_1, -\alpha c_1)$, $0 < \alpha \in P_0$, is the intercept of the straight line $p_1 + p_2 = -\Theta_1(\alpha)$ on R^2, defined by $p_i \le 0$, $i = 1, 2$.

To construct the desired system in the case $n > 2$, it is sufficient to complement the system (4) with the general solution (3) by the diagonal system

$$\frac{\partial y}{\partial t_i} = |\alpha_1| E^{-1}(t) \frac{\partial E(t)}{\partial t_i} y, \quad y \in R^{n-2}, \quad t \in R^2_+, \quad i = 1, 2,$$

with bounded and infinitely differentiable matrices $A_{2+i}(t) = |\alpha_1| E^{-1}(t) \times c \partial E(t)/\partial t_i$, $i = 1, 2$. Then the required system is the block-diagonal system (1) with the matrices $A(t) = \operatorname{diag}[A_1(t), A_3(t)]$ and $B(t) = \operatorname{diag}[A_2(t), A_4(t)]$.

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