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ON NONNEGATIVE SOLUTIONS OF FIRST ORDER SCALAR FUNCTIONAL DIFFERENTIAL EQUATIONS

Abstract. Nonimprovable effective sufficient conditions are established for the existence and uniqueness of a nonnegative solution of the problem

$$u'(t) = \ell(u)(t) + q(t), \qquad u(a) = c \qquad (u(b) = c),$$

where $\ell : C([a,b];R) \to L([a,b];R)$ is a linear bounded operator, $q \in L([a,b];R_+)$ ($q \in L([a,b];R_-)$) and $c \in R_+$.

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Introduction

The following notation is used throughout.

R is the set of all real numbers, $R_+ = [0, +\infty[$.

$C([a,b];R)$ is the Banach space of continuous functions $u : [a,b] \to R$ with the norm $\|u\|_C = \max\{|u(t)| : a \le t \le b\}$.

$C([a,b];R_+) = \{u \in C([a,b];R) : u(t) \ge 0 \text{ for } t \in [a,b]\}$.

$C_{t_0}([a,b];R_+) = \{u \in C([a,b];R_+) : u(t_0) = 0\}$, where $t_0 \in [a,b]$.

$\widetilde{C}([a,b];D)$, where $D \subseteq R$, is the set of absolutely continuous functions $u : [a,b] \to D$.

$L([a,b];R)$ is the Banach space of Lebesgue integrable functions $p : [a,b] \to R$ with the norm $\|p\|_L = \int\limits_a^b |p(s)|ds$.

$L([a,b];D)$, where $D \subseteq R$, is the set of Lebesgue integrable functions $p : [a,b] \to D$.

$\mathcal{M}_{ab}$ is the set of measurable functions $\tau : [a,b] \to [a,b]$.

$\widetilde{\mathcal{L}}_{ab}$ is the set of linear bounded operators $\ell : C([a,b];R) \to L([a,b];R)$.

$\mathcal{P}_{ab}$ is the set of linear operators $\ell \in \widetilde{\mathcal{L}}_{ab}$ transforming the set $C([a,b];R_+)$ into the set $L([a,b];R_+)$.

We will say that $\ell \in \widetilde{\mathcal{L}}_{ab}$ is a t_0-Volterra operator, where $t_0 \in [a,b]$, if for arbitrary $a_1 \in [a,t_0]$, $b_1 \in [t_0,b]$, $a_1 \ne b_1$, and $v \in C([a,b];R)$ satisfying the condition

$$v(t) = 0 \quad \text{for } \ t \in [a_1,b_1],$$

we have

$$\ell(v)(t) = 0 \quad \text{for } \ t \in [a_1,b_1].$$

$[x]_+ = \frac{1}{2}(|x| + x)$, $[x]_- = \frac{1}{2}(|x| - x)$.

By a solution of the equation

$$u'(t) = \ell(u)(t) + q(t), \tag{0.1}$$

where $\ell \in \widetilde{\mathcal{L}}_{ab}$ and $q \in L([a,b];R)$, we understand a function $u \in \widetilde{C}([a,b];R)$ satisfying the equation (0.1) almost everywhere in $[a,b]$. The special case of the equation (0.1) is the differential equation with deviating arguments

$$u'(t) = p(t)u(\tau(t)) - g(t)u(\mu(t)) + q(t), \tag{0.2}$$

where $p, g \in L([a,b];R_+)$, $q \in L([a,b];R)$, $\tau, \mu \in \mathcal{M}_{ab}$.

Consider the problem on the existence and uniqueness of a nonnegative solution u of (0.1) satisfying the initial condition

$$u(a) = c, \tag{0.3}$$

resp.

$$u(b) = c, \tag{0.4}$$

where $q \in L([a,b];R_+)$, resp. $q \in L([a,b];R_-)$, $c \in R_+$. This problem is equivalent to the problem on the validity of the classical theorem on differential inequalities, i.e., whenever $u, v \in \widetilde{C}([a,b];R)$ satisfy the inequalities

$$u'(t) \le \ell(u)(t) + q(t), \qquad v'(t) \ge \ell(v)(t) + q(t),$$
$$u(a) \le v(a), \quad \text{resp.} \quad u(b) \ge v(b),$$

then the inequality $u(t) \le v(t)$, resp. $u(t) \ge v(t)$ for $t \in [a,b]$ is fulfilled.

Along with the equation (0.1), resp. (0.2), and the condition (0.3), resp. (0.4), consider the corresponding homogeneous equation

$$u'(t) = \ell(u)(t), \tag{0.1$_0$}$$

resp.

$$u'(t) = p(t)u(\tau(t)) - g(t)u(\mu(t)), \tag{0.2$_0$}$$

and the corresponding homogeneous condition

$$u(a) = 0, \tag{0.3$_0$}$$

resp.

$$u(b) = 0. \tag{0.4$_0$}$$

In [3] there are established effective optimal criteria guaranteeing the validity of a theorem on differential inequalities for the monotone operators, i.e., when $\ell \in \mathcal{P}_{ab}$, resp. $-\ell \in \mathcal{P}_{ab}$. In the present paper, these results are formulated more precisely, and, moreover, there are established conditions guaranteeing the validity of a theorem on differential inequalities for a general linear operator $\ell \in \widetilde{\mathcal{L}}_{ab}$. This makes the results in [3] more complete (see also [5]).

From the general theory of linear boundary value problems for functional differential equations, the following result is well–known (see, e.g., [2,10,13]).

Theorem 0.1. *The problem* (0.1)*,* (0.3)*, resp.* (0.1)*,* (0.4) *is uniquely solvable iff the corresponding homogeneous problem* (0.5)*,* (0.5)*, resp.* (0.5)*,* (0.5) *has only the trivial solution.*

Definition 0.1. We will say that an operator $\ell \in \widetilde{\mathcal{L}}_{ab}$ belongs to the set $\mathcal{S}_{ab}(a)$, resp. $\mathcal{S}_{ab}(b)$, if the homogeneous problem (0.5), (0.5), resp. (0.5), (0.5) has only the trivial solution, and for arbitrary $q \in L([a,b];R_+)$, resp. $q \in L([a,b];R_-)$ and $c \in R_+$, the solution of the problem (0.1), (0.3), resp. (0.1), (0.4) is nonnegative.

Remark 0.1. According to Theorem 0.1, if $\ell \in \mathcal{S}_{ab}(a)$, resp. $\ell \in \mathcal{S}_{ab}(b)$, then for every $c \in R_+$ and $q \in L([a,b];R_+)$, resp. $q \in L([a,b];R_-)$, the problem (0.1), (0.3), resp. (0.1), (0.4) has a unique solution, and this solution is nonnegative.

Remark 0.2. From Definition 0.1 it immediately follows that $\ell \in \mathcal{S}_{ab}(a)$, resp. $\ell \in \mathcal{S}_{ab}(b)$, iff for the equation (0.1) the classical theorem on differential inequalities holds (see, e.g., [8]), i.e., whenever $u, v \in \widetilde{C}([a,b];R)$ satisfy the inequalities

$$u'(t) \leq \ell(u)(t) + q(t), \qquad v'(t) \geq \ell(v)(t) + q(t) \quad \text{for } t \in [a,b],$$
$$u(a) \leq v(a), \quad \text{resp.} \quad u(b) \geq v(b),$$

then

$$u(t) \leq v(t), \quad \text{resp.} \quad u(t) \geq v(t) \quad \text{for } t \in [a,b].$$

Thus the theorems formulated below, in fact, are theorems on differential inequalities. On the other hand, due to Theorem 0.1, it is clear that if $\ell \in \mathcal{S}_{ab}(a)$, resp. $\ell \in \mathcal{S}_{ab}(b)$, then the problem (0.1), (0.3), resp. (0.1), (0.4) is uniquely solvable for any $c \in R$ and $q \in L([a,b];R)$. For other effective conditions for the solvability of the Cauchy problem see, e.g., [3,5,6,7,10,11,12].

Remark 0.3. If $\ell \in \mathcal{P}_{ab}$, resp. $-\ell \in \mathcal{P}_{ab}$, then the inclusion $\ell \in \mathcal{S}_{ab}(a)$, resp. $\ell \in \mathcal{S}_{ab}(b)$, holds iff the problem

$$u'(t) \leq \ell(u)(t), \qquad u(a) = 0, \tag{0.5}$$

resp.

$$u'(t) \geq \ell(u)(t), \qquad u(b) = 0, \tag{0.6}$$

has no nontrivial nonnegative solution.

1. Theorems on Differential Inequalities

1.1. Main results.

Theorem 1.1. *Let $\ell \in \mathcal{P}_{ab}$. Then $\ell \in \mathcal{S}_{ab}(a)$ iff there exists $\gamma \in \widetilde{C}([a,b];\,]0,+\infty[)$ satisfying the inequality*

$$\gamma'(t) \geq \ell(\gamma)(t) \quad \textit{for } t \in [a,b]. \tag{1.1}$$

Corollary 1.1. *Let $\ell \in \mathcal{P}_{ab}$ and at least one of the following items be fulfilled:*

a) *ℓ is an $a-$Volterra operator;*

b) *there exist a nonnegative integer k, a natural number $m > k$, and a constant $\alpha \in \,]0,1[$ such that*

$$\rho_m(t) \leq \alpha \rho_k(t) \quad \textit{for } t \in [a,b], \tag{1.2}$$

where

$$\rho_0(t) \stackrel{def}{=} 1, \ \rho_{i+1}(t) \stackrel{def}{=} \int_a^t \ell(\rho_i)(s)ds \ \textit{for } t \in [a,b] \ \ (i=0,1,\dots); \tag{1.3}$$

c) *there exists* $\overline{\ell} \in \mathcal{P}_{ab}$ *such that*

$$\int_a^b \overline{\ell}(1)(s) \exp\left(\int_s^b \ell(1)(\xi)d\xi\right)ds < 1 \tag{1.4}$$

and on the set $C_a([a,b];R_+)$ *the inequality*

$$\ell(\vartheta(v))(t) - \ell(1)(t)\vartheta(v)(t) \leq \overline{\ell}(v)(t) \quad \text{for } t \in [a,b] \tag{1.5}$$

holds, where

$$\vartheta(v)(t) = \int_a^t \ell(v)(s)ds \quad \text{for } t \in [a,b].$$

Then $\ell \in \mathcal{S}_{ab}(a)$.

Remark 1.1. From Corollary 1.1 b) (for $k = 0$ and $m = 1$) it follows that if $\ell \in \mathcal{P}_{ab}$ and $\int_a^b \ell(1)(s)ds < 1$, then $\ell \in \mathcal{S}_{ab}(a)$. Note also that if $\ell \in \mathcal{P}_{ab}$, $\int_a^b \ell(1)(s)ds = 1$ and the problem (0.5), (0.5) has only the trivial solution, then $\ell \in \mathcal{S}_{ab}(a)$ again (see On Remark 1.1 below).

Nevertheless, the assumptions in Corollary 1.1 are nonimprovable. More precisely, the condition $\alpha \in]0,1[$ cannot be replaced by the condition $\alpha \in]0,1]$, and the strict inequality in (1.4) cannot be replaced by the nonstrict one (see Examples 4.1 and 4.2).

Theorem 1.2. *Let* $-\ell \in \mathcal{P}_{ab}$, ℓ *be an* $a-$ *Volterra operator, and there exist a function* $\gamma \in \widetilde{C}([a,b];R_+)$ *such that*

$$\gamma(t) > 0, \quad \text{for } t \in [a,b[, \tag{1.6}$$

$$\gamma'(t) \leq \ell(\gamma)(t) \quad \text{for } t \in [a,b]. \tag{1.7}$$

Then $\ell \in \mathcal{S}_{ab}(a)$.

Theorem 1.3. *Let* $-\ell \in \mathcal{P}_{ab}$, ℓ *be an* $a-$ *Volterra operator, and*

$$\int_a^b |\ell(1)(s)|ds \leq 1. \tag{1.8}$$

Then $\ell \in \mathcal{S}_{ab}(a)$.

Corollary 1.2. *Let* $-\ell \in \mathcal{P}_{ab}$, ℓ *be an* $a-$ *Volterra operator, and*

$$\int_a^b |\widetilde{\ell}(1)(s)| \exp\left(\int_a^s |\ell(1)(\xi)|d\xi\right)ds \leq 1, \tag{1.9}$$

where

$$\widetilde{\ell} = \ell(\widetilde{\theta}(v))(t) - \ell(1)(t)\widetilde{\theta}(v)(t) \quad \textit{for } \ t \in [a,b],$$

$$\widetilde{\theta}(v)(t) = \int_a^t \ell(\widetilde{v})(s)ds, \ \widetilde{v}(t) = v(t)\exp\left(\int_a^t \ell(1)(s)ds\right) \ \textit{for } \ t \in [a,b]. \tag{1.10}$$

Then $\ell \in \mathcal{S}_{ab}(a)$.

Remark 1.2. Theorems 1.2 and 1.3, and Corollary 1.2 are nonimprovable. More precisely, the condition (1.6) cannot be replaced by the condition

$$\gamma(t) > 0 \quad \text{for } \ t \in [a, b_1[,$$

where $b_1 \in]a,b[$, the condition (1.8) cannot be replaced by the condition

$$\int_a^b |\ell(1)(s)|ds \le 1 + \varepsilon,$$

no matter how small $\varepsilon > 0$ would be, and the condition (1.9) cannot be replaced by the condition

$$\int_a^b |\widetilde{\ell}(1)(s)| \exp\left(\int_a^s |\ell(1)(\xi)|d\xi\right)ds \le 1 + \varepsilon,$$

no matter how small $\varepsilon > 0$ would be (see Examples 4.3 and 4.4).

Remark 1.3. In [4] there is proved that the condition in Theorems 1.2 and 1.3 on an operator ℓ to be $a-$Volterra's type is necessary for ℓ to belong to the set $\mathcal{S}_{ab}(a)$.

Theorem 1.4. *Let* $\ell = \ell_0 - \ell_1$, *where* $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, $\ell_0 \in \mathcal{S}_{ab}(a)$, $-\ell_1 \in \mathcal{S}_{ab}(a)$. *Then* $\ell \in \mathcal{S}_{ab}(a)$.

Remark 1.4. Theorem 1.4 is nonimprovable in the sense that the assumption

$$\ell_0 \in \mathcal{S}_{ab}(a), \qquad -\ell_1 \in \mathcal{S}_{ab}(a)$$

cannot be replaced neither by the assumption

$$(1-\varepsilon)\ell_0 \in \mathcal{S}_{ab}(a), \qquad -\ell_1 \in \mathcal{S}_{ab}(a),$$

nor by the assumption

$$\ell_0 \in \mathcal{S}_{ab}(a), \qquad -(1-\varepsilon)\ell_1 \in \mathcal{S}_{ab}(a),$$

no matter how small $\varepsilon > 0$ would be (see Examples 4.5 and 4.6).

Remark 1.5. Let $\ell \in \widetilde{\mathcal{L}}_{ab}$. Put

$$\widehat{\ell}(v)(t) \stackrel{def}{=} -\psi(\ell(\varphi(v)))(t),$$

where $\psi : L([a,b];R) \to L([a,b];R)$ is an operator defined by

$$\psi(v)(t) \stackrel{def}{=} v(a+b-t),$$

and φ is a restriction of the operator ψ into the space $C([a,b];R)$.

It is clear that if $u \in \widetilde{C}([a,b];R)$ satisfies the inequality

$$u'(t) \le \ell(u)(t) \qquad (\, u'(t) \ge \ell(u)(t)\,) \quad \text{for } \ t \in [a,b], \tag{1.11}$$

then the function $v(t) = \varphi(u)(t)$ for $t \in [a,b]$ satisfies the inequality

$$v'(t) \ge \widehat{\ell}(v)(t) \qquad (\, v'(t) \le \widehat{\ell}(v)(t)\,) \quad \text{for } \ t \in [a,b], \tag{1.12}$$

and vice versa, if $v \in \widetilde{C}([a,b];R)$ satisfies the inequality (1.12), then the function $u(t) = \varphi(v)(t)$ for $t \in [a,b]$ satisfies the inequality (1.11). Therefore, $\ell \in \mathcal{S}_{ab}(a)$ ($\ell \in \mathcal{S}_{ab}(b)$) iff $\widehat{\ell} \in \mathcal{S}_{ab}(b)$ ($\widehat{\ell} \in \mathcal{S}_{ab}(a)$).

According to Remark 1.5, from Theorems 1.1-1.4 and Corollaries 1.1 and 1.2 it immediately follows

Theorem 1.5. *Let $-\ell \in \mathcal{P}_{ab}$. Then $\ell \in \mathcal{S}_{ab}(b)$ iff there exists $\gamma \in \widetilde{C}([a,b];$ $]0,+\infty[)$ satisfying the inequality*

$$\gamma'(t) \le \ell(\gamma)(t) \quad \textit{for } \ t \in [a,b].$$

Corollary 1.3. *Let $-\ell \in \mathcal{P}_{ab}$ and at least one of the following items be fulfilled:*

a) *ℓ is a $b-$Volterra operator;*

b) *there exist a nonnegative integer k, a natural number $m > k$, and a constant $\alpha \in]0,1[$ such that*

$$\rho_m(t) \le \alpha \rho_k(t) \quad \textit{for } \ t \in [a,b],$$

where

$$\rho_0(t) \stackrel{def}{=} 1, \ \rho_{i+1}(t) \stackrel{def}{=} -\int_t^b \ell(\rho_i)(s)ds \quad \textit{for } \ t \in [a,b] \ \ (i = 0,1,\dots);$$

c) *there exists $\overline{\ell} \in \mathcal{P}_{ab}$ such that*

$$\int_a^b \overline{\ell}(1)(s) \exp\left(\int_a^s |\ell(1)(\xi)| d\xi \right) ds < 1$$

and on the set $C_b([a,b];R_+)$ the inequality

$$\ell(1)(t)\vartheta(v)(t) - \ell(\vartheta(v))(t) \le \overline{\ell}(v)(t) \quad \textit{for } \ t \in [a,b]$$

holds, where

$$\vartheta(v)(t) = -\int_t^b \ell(v)(s)ds \quad \text{for } t \in [a,b].$$

Then $\ell \in \mathcal{S}_{ab}(b)$.

Theorem 1.6. *Let* $\ell \in \mathcal{P}_{ab}$, ℓ *be a* $b-$ *Volterra operator, and there exist a function* $\gamma \in \widetilde{C}([a,b]; R_+)$ *such that*

$$\gamma(t) > 0, \quad \text{for } t \in]a,b],$$
$$\gamma'(t) \geq \ell(\gamma)(t) \quad \text{for } t \in [a,b].$$

Then $\ell \in \mathcal{S}_{ab}(b)$.

Theorem 1.7. *Let* $\ell \in \mathcal{P}_{ab}$, ℓ *be a* $b-$ *Volterra operator, and*

$$\int_a^b \ell(1)(s)ds \leq 1.$$

Then $\ell \in \mathcal{S}_{ab}(b)$.

Corollary 1.4. *Let* $\ell \in \mathcal{P}_{ab}$, ℓ *be a* $b-$ *Volterra operator, and*

$$\int_a^b \widetilde{\ell}(1)(s) \exp\left(\int_s^b \ell(1)(\xi)d\xi \right) ds \leq 1,$$

where

$$\widetilde{\ell} = \ell(\widetilde{\theta}(v))(t) - \ell(1)(t)\widetilde{\theta}(v)(t) \quad \text{for } t \in [a,b],$$
$$\widetilde{\theta}(v)(t) = -\int_t^b \ell(\widetilde{v})(s)ds, \quad \widetilde{v}(t) = v(t)\exp\left(-\int_t^b \ell(1)(s)ds \right) \text{ for } t \in [a,b].$$

Then $\ell \in \mathcal{S}_{ab}(b)$.

Theorem 1.8. *Let* $\ell = \ell_0 - \ell_1$, *where* $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, $\ell_0 \in \mathcal{S}_{ab}(b)$, $-\ell_1 \in \mathcal{S}_{ab}(b)$. *Then* $\ell \in \mathcal{S}_{ab}(b)$.

Remark 1.6. The nonimprovability of the conditions of Theorems 1.6–1.8 and Corollaries 1.3 and 1.4 follows from Remarks 1.1, 1.2, 1.4 and 1.5.

1.2. Equations with deviating arguments. Theorems 1.1–1.8 imply the following assertions for differential equations with deviating arguments.

Theorem 1.9. *Let $p \in L([a,b];R_+)$, $\tau \in \mathcal{M}_{ab}$, and at least one of the following items be fulfilled:*

a)

$$\int_a^t p(s) \int_a^{\tau(s)} p(\xi)d\xi ds \leq \alpha \int_a^t p(s)ds \quad \textit{for } \; t \in [a,b], \tag{1.13}$$

where $\alpha \in]0,1[$;

b)

$$\int_a^b p(s)\sigma(s)\Big(\int_s^{\tau(s)} p(\xi)d\xi \Big) \exp\Big[\int_s^b p(\eta)d\eta \Big] ds < 1, \tag{1.14}$$

where $\sigma(t) = \frac{1}{2}(1 + \operatorname{sgn}(\tau(t) - t))$ for $t \in [a,b]$;

c) $\int_a^{\tau^*} p(s)ds \neq 0$ *and*

$$\text{ess sup}\Big\{ \int_t^{\tau(t)} p(s)ds : t \in [a,b] \Big\} < \lambda^*, \tag{1.15}$$

where

$$\lambda^* = \sup\left\{ \frac{1}{x} \ln\left(x + \frac{x}{\exp\Big(x \int_a^{\tau^*} p(s)ds\Big) - 1} \right) : x > 0 \right\},$$

$$\tau^* = \text{ess sup}\{\tau(t) : t \in [a,b]\}.$$

Then the operator ℓ defined by

$$\ell(v)(t) \stackrel{def}{=} p(t)v(\tau(t)) \tag{1.16}$$

belongs to the set $\mathcal{S}_{ab}(a)$.

Remark 1.7. The assumptions a) and b) in Theorem 1.9 are nonimprovable. More precisely, the condition $\alpha \in]0,1[$ cannot be replaced by the condition $\alpha \in]0,1]$, and the strict inequality in (1.14) cannot be replaced by the nonstrict one (see Examples 4.1 and 4.2).

Theorem 1.10. *Let $g \in L([a,b];R_+)$, $\mu \in \mathcal{M}_{ab}$, $\mu(t) \leq t$ for $t \in [a,b]$, and either*

$$\int_a^b g(s)ds \leq 1 \tag{1.17}$$

or

$$\int_a^b g(s)\Big(\int_{\mu(s)}^s g(\xi)\exp\Big[\int_{\mu(\xi)}^s g(\eta)d\eta\Big]d\xi\Big)ds \le 1, \tag{1.18}$$

or $g \not\equiv 0$ *and*

$$\operatorname{ess\,sup}\left\{\int_{\mu(t)}^t g(s)ds : t \in [a,b]\right\} < \eta^*, \tag{1.19}$$

where

$$\eta^* = \sup\left\{\frac{1}{x}\ln\left(x + \frac{x}{\exp\Big(x\int_a^b g(s)ds\Big) - 1}\right) : x > 0\right\}.$$

Then the operator ℓ *defined by*

$$\ell(v)(t) \overset{def}{=} -g(t)v(\mu(t)) \tag{1.20}$$

belongs to the set $\mathcal{S}_{ab}(a)$*.*

Remark 1.8. The condition (1.17), resp. (1.18) in Theorem 1.10 cannot be replaced by the condition

$$\int_a^b g(s)ds \le 1 + \varepsilon,$$

resp.

$$\int_a^b g(s)\Big(\int_{\mu(s)}^s g(\xi)\exp\Big[\int_{\mu(\xi)}^s g(\eta)d\eta\Big]d\xi\Big)ds \le 1 + \varepsilon,$$

no matter how small $\varepsilon > 0$ would be (see Example 4.4).

Theorem 1.11. *Let* $p, g \in L([a,b]; R_+)$*,* $\tau, \mu \in \mathcal{M}_{ab}$*,* $\mu(t) \le t$ *for* $t \in [a,b]$*, and the functions* p, τ *satisfy at least one of the conditions* a)*,* b)*,* c) *in Theorem* 1.9*, while the functions* g, μ *satisfy either* (1.17) *or* (1.18)*, or* (1.19) *in Theorem* 1.10*. Then the operator* ℓ *defined by*

$$\ell(v)(t) \overset{def}{=} p(t)v(\tau(t)) - g(t)v(\mu(t)) \tag{1.21}$$

belongs to the set $\mathcal{S}_{ab}(a)$*.*

Remark 1.9. Theorem 1.11 is nonimprovable in a certain sense (see Examples 4.5 and 4.6).

Theorem 1.12. *Let $g \in L([a,b];R_+)$, $\mu \in \mathcal{M}_{ab}$, and at least one of the following items be fulfilled:*

a) $$\int_t^b g(s) \int_{\mu(s)}^b g(\xi)d\xi ds \le \alpha \int_t^b g(s)ds \quad \text{for } t \in [a,b],$$

where $\alpha \in]0,1[$;

b) $$\int_a^b g(s)\sigma(s)\Bigg(\int_{\mu(s)}^s g(\xi)d\xi\Bigg)\exp\Bigg[\int_a^s g(\eta)d\eta\Bigg]ds < 1,$$

where $\sigma(t) = \frac{1}{2}(1 + \operatorname{sgn}(t - \mu(t)))$ for $t \in [a,b]$;

c) $$\int_{\mu_*}^b g(s)ds \ne 0 \quad \text{and} \quad \text{ess sup}\Bigg\{\int_{\mu(t)}^t g(s)ds : t \in [a,b]\Bigg\} < \vartheta^*,$$

where

$$\vartheta^* = \sup\left\{\frac{1}{x}\ln\left(x + \frac{x}{\exp\Big(x\int_{\mu_*}^b g(s)ds\Big) - 1}\right) : x > 0\right\},$$
$$\mu_* = \text{ess inf}\{\mu(t) : t \in [a,b]\}.$$

Then the operator ℓ defined by (1.20) *belongs to the set $\mathcal{S}_{ab}(b)$.*

Theorem 1.13. *Let $p \in L([a,b];R_+)$, $\tau \in \mathcal{M}_{ab}$, $\tau(t) \ge t$ for $t \in [a,b]$, and either*

$$\int_a^b p(s)ds \le 1 \tag{1.22}$$

or

$$\int_a^b p(s)\Bigg(\int_s^{\tau(s)} p(\xi)\exp\Bigg[\int_s^{\tau(\xi)} p(\eta)d\eta\Bigg]d\xi\Bigg)ds \le 1, \tag{1.23}$$

or $p \not\equiv 0$ and

$$\text{ess sup}\Bigg\{\int_t^{\tau(t)} p(s)ds : t \in [a,b]\Bigg\} < \kappa^*, \tag{1.24}$$

where

$$\kappa^* = \sup\left\{\frac{1}{x}\ln\left(x + \frac{x}{\exp\Big(x\int_a^b p(s)ds\Big) - 1}\right) : x > 0\right\}.$$

Then the operator ℓ defined by (1.16) *belongs to the set $\mathcal{S}_{ab}(b)$.*

Theorem 1.14. *Let $p, g \in L([a,b]; R_+)$, $\tau, \mu \in \mathcal{M}_{ab}$, $\tau(t) \geq t$ for $t \in [a,b]$, and the functions g, μ satisfy at least one of the conditions* a), b), c) *in Theorem* 1.12, *while the functions p, τ satisfy either* (1.22) *or* (1.23), *or* (1.24) *in Theorem* 1.13. *Then the operator ℓ defined by* (1.21) *belongs to the set $\mathcal{S}_{ab}(b)$.*

Remark 1.10. The nonimprovability of the conditions of Theorems 1.12–1.14 follows from Remarks 1.7–1.9 and 1.5.

2. On Positive Solutions of the Homogeneous Equation

In this section we shall consider the problem on the existence of a sign constant solution of the homogeneous equation (0.5). As we will see below, this problem is quite close to the problem on the validity of a theorem on differential inequalities. Moreover, for some cases they are equivalent.

Definition 2.1. We will say that an operator $\ell \in \widetilde{\mathcal{L}}_{ab}$ belongs to the set $\widetilde{\mathcal{S}}_{ab}$, if the homogeneous equation (0.5) has at least one positive solution.

Remark 2.1. Let $\ell \in \widetilde{\mathcal{L}}_{ab}$ be a t_0-Volterra operator, where $t_0 \in [a,b]$, and $\ell \in \widetilde{\mathcal{S}}_{ab}$. Evidently, for any $a_1 \in [a, t_0]$ and $b_1 \in [t_0, b]$, $a_1 \neq b_1$, the inclusion $\ell \in \widetilde{\mathcal{S}}_{a_1 b_1}$ holds as well.

2.1. Main results.

Theorem 2.1. *Let $\ell = \ell_0 - \ell_1$, where $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, $\ell_0 \in \mathcal{S}_{ab}(a)$, $-\ell_1 \in \mathcal{S}_{ab}(b)$, and ℓ_1 be an $a-$ Volterra operator. Then $\ell \in \widetilde{\mathcal{S}}_{ab}$.*

Remark 2.2. Theorem 2.1 is nonimprovable in the sense that the assumption

$$\ell_0 \in \mathcal{S}_{ab}(a), \qquad -\ell_1 \in \mathcal{S}_{ab}(b)$$

cannot be replaced neither by the assumption

$$(1-\varepsilon)\ell_0 \in \mathcal{S}_{ab}(a), \qquad -\ell_1 \in \mathcal{S}_{ab}(b),$$

nor by the assumption

$$\ell_0 \in \mathcal{S}_{ab}(a), \qquad -(1-\varepsilon)\ell_1 \in \mathcal{S}_{ab}(b),$$

no matter how small $\varepsilon > 0$ would be (see Examples 4.5 and 4.6).

Moreover, the assumption on the operator ℓ_1 to be $a-$Volterra's type in Theorem 2.1 is important and cannot be omitted (see Example 4.7).

Nevertheless, in Theorem 2.2 and Corollary 2.1, there are established conditions guaranteeing the inclusion $\ell \in \widetilde{\mathcal{S}}_{ab}$, without the assumption on ℓ_1 to be $a-$Volterra's type.

Theorem 2.2. *Let $\ell = \ell_0 - \ell_1$, where $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, and there exist functions $\alpha, \beta \in \widetilde{C}([a,b]; R_+)$ satisfying the inequalities*

$$\beta'(t) \le -\ell_1(\beta)(t) + \ell_0(\alpha)(t) \quad \text{for } t \in [a,b], \tag{2.1}$$

$$\alpha'(t) \ge -\ell_1(\alpha)(t) + \ell_0(\beta)(t) \quad \text{for } t \in [a,b], \tag{2.2}$$

$$\alpha(t) \le \beta(t) \quad \text{for } t \in [a,b]. \tag{2.3}$$

Let, moreover, one of the following conditions hold:

$$\alpha(t) > 0 \quad \text{for } t \in [a,b]; \tag{2.4}$$

$$\alpha(t) > 0 \quad \text{for } t \in]a,b], \qquad \alpha(a) = 0, \quad \text{and} \quad \ell_0 \in \mathcal{S}_{ab}(a); \tag{2.5}$$

$$\alpha(t) > 0 \quad \text{for } t \in [a,b[, \qquad \alpha(b) = 0, \quad \text{and} \quad -\ell_1 \in \mathcal{S}_{ab}(b). \tag{2.6}$$

Then $\ell \in \widetilde{\mathcal{S}}_{ab}$.

Corollary 2.1. *Let $\ell = \ell_0 - \ell_1$, where $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, and there exist a function $\beta \in \widetilde{C}([a,b];]0,+\infty[)$ satisfying the inequalities*

$$\beta'(t) \le -\ell_1(\beta)(t) \quad \text{for } t \in [a,b], \tag{2.7}$$

$$\beta(a) \int_a^b \ell_0(1)(s)ds < \beta(b). \tag{2.8}$$

Then $\ell \in \widetilde{\mathcal{S}}_{ab}$.

According to Remark 1.5, Theorems 2.1, 2.2 and Corollary 2.1 imply

Theorem 2.3. *Let $\ell = \ell_0 - \ell_1$, where $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, $\ell_0 \in \mathcal{S}_{ab}(a)$, $-\ell_1 \in \mathcal{S}_{ab}(b)$, and ℓ_0 be a $b-$Volterra operator. Then $\ell \in \widetilde{\mathcal{S}}_{ab}$.*

Remark 2.3. The nonimprovability of the conditions of Theorem 2.3 follows from Remarks 2.2 and 1.5.

Theorem 2.4. *Let $\ell = \ell_0 - \ell_1$, where $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, and there exist functions $\alpha, \beta \in \widetilde{C}([a,b]; R_+)$ satisfying the inequality* (2.3) *and*

$$\beta'(t) \ge \ell_0(\beta)(t) - \ell_1(\alpha)(t) \quad \text{for } t \in [a,b],$$

$$\alpha'(t) \le \ell_0(\alpha)(t) - \ell_1(\beta)(t) \quad \text{for } t \in [a,b].$$

Let, moreover, at least one of the conditions (2.4), (2.5), (2.6) *be fulfilled. Then $\ell \in \widetilde{\mathcal{S}}_{ab}$.*

Corollary 2.2. *Let $\ell = \ell_0 - \ell_1$, where $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, and there exist a function $\beta \in \widetilde{C}([a,b];]0,+\infty[)$ satisfying the inequalities*

$$\beta'(t) \ge \ell_0(\beta)(t) \quad \text{for } t \in [a,b], \qquad \beta(b) \int_a^b \ell_1(1)(s)ds < \beta(a).$$

Then $\ell \in \widetilde{\mathcal{S}}_{ab}$.

2.2. Equations with deviating arguments. From the above theorems we immediately get the following assertions for differential equations with deviating arguments.

Theorem 2.5. *Let* $p, g \in L([a,b]; R_+)$, $\tau, \mu \in \mathcal{M}_{ab}$, $\mu(t) \le t$ *for* $t \in [a,b]$, *and the functions* p, τ *satisfy one of the items* a), b), c) *in Theorem* 1.9, *while the functions* g, μ *satisfy one of the items* a), b), c) *in Theorem* 1.12. *Then the operator* ℓ *defined by* (1.21) *belongs to the set* $\widetilde{\mathcal{S}}_{ab}$.

Theorem 2.6. *Let* $p, g \in L([a,b]; R_+)$, $\tau, \mu \in \mathcal{M}_{ab}$, *the functions* g, μ *satisfy the item* a), *resp.* c) *in Theorem* 1.12, *and*

$$\left((1-\alpha)\left(1+\int_a^b g(s)ds\right)+\int_a^b g(s)\int_{\mu(s)}^b g(\xi)d\xi ds\right)\int_a^b p(s)ds < 1-\alpha,$$

resp.

$$\left(\exp\left(x_0\int_a^b g(s)ds\right)-1+\delta\right)\int_a^b p(s)ds < \delta,$$

where $x_0 > 0$ *and* $\delta \in]0,1[$ *are such that*

$$\int_{\mu(t)}^t g(s)ds < \frac{1}{x_0}\ln\left(x_0+\frac{x_0(1-\delta)}{\exp\left(x_0\int_{\mu^*}^b g(s)ds\right)-(1-\delta)}\right) \quad \text{for } t \in [a,b].$$

Then the operator ℓ *defined by* (1.21) *belongs to the set* $\widetilde{\mathcal{S}}_{ab}$.

Theorem 2.7. *Let* $p, g \in L([a,b]; R_+)$, $\tau, \mu \in \mathcal{M}_{ab}$, $\tau(t) \ge t$ *for* $t \in [a,b]$, *and the functions* p, τ *satisfy one of the items* a), b), c) *in Theorem* 1.9, *while the functions* g, μ *satisfy one of the items* a), b), c) *in Theorem* 1.12. *Then the operator* ℓ *defined by* (1.21) *belongs to the set* $\widetilde{\mathcal{S}}_{ab}$.

Theorem 2.8. *Let* $p, g \in L([a,b]; R_+)$, $\tau, \mu \in \mathcal{M}_{ab}$, *the functions* p, τ *satisfy the item* a), *resp.* c) *in Theorem* 1.9, *and*

$$\left((1-\alpha)\left(1+\int_a^b p(s)ds\right)+\int_a^b p(s)\int_a^{\tau(s)} p(\xi)d\xi ds\right)\int_a^b g(s)ds < 1-\alpha,$$

resp.

$$\left(\exp\left(x_0\int_a^b p(s)ds\right)-1+\delta\right)\int_a^b g(s)ds < \delta,$$

where $x_0 > 0$ *and* $\delta \in]0,1[$ *are such that*

$$\int_t^{\tau(t)} p(s)ds < \frac{1}{x_0} \ln \left(x_0 + \frac{x_0(1-\delta)}{\exp\left(x_0 \int_a^{\tau^*} p(s)ds\right) - (1-\delta)} \right) \quad \textit{for } t \in [a,b].$$

Then the operator ℓ *defined by* (1.21) *belongs to the set* $\widetilde{\mathcal{S}}_{ab}$.

3. Proofs

3.1. Proofs of the theorems on differential inequalities.

Proof of Theorem 1.1. In [3] there is proved that if $\ell \in \mathcal{P}_{ab}$ and there exists a function $\gamma \in \widetilde{C}([a,b];]0,+\infty[)$ satisfying (1.1), then $\ell \in \mathcal{S}_{ab}(a)$. The opposite implication is trivial. $\square$

Proof of Corollary 1.1. a) It is not difficult to verify that the function

$$\gamma(t) = \exp\left(\int_a^t \ell(1)(s)ds \right) \quad \text{for } t \in [a,b]$$

satisfies the assumptions of Theorem 1.1.

b) Put

$$\gamma(t) = (1-\alpha)\sum_{j=0}^{k} \rho_j(t) + \sum_{j=k+1}^{m} \rho_j(t) \quad \text{for } t \in [a,b].$$

Then by virtue of (1.2) and (1.3) the assumptions of Theorem 1.1 are fulfilled and so $\ell \in \mathcal{S}_{ab}(a)$.

c) According to (1.4), we can choose $\varepsilon > 0$ such that

$$\int_a^b \overline{\ell}(1)(s) \exp\left(\int_s^b \ell(1)(\xi)d\xi \right) ds < 1 - \varepsilon \exp\left(\int_a^b \ell(1)(\xi)d\xi \right).$$

Put

$$\gamma(t) = \varepsilon \exp\left(\int_a^t \ell(1)(\xi)d\xi \right) + \int_a^t \overline{\ell}(1)(s) \exp\left(\int_s^t \ell(1)(\xi)d\xi \right) ds \quad \text{for } t \in [a,b].$$

Obviously, $\gamma \in \widetilde{C}([a,b];]0,+\infty[)$, $\gamma(t) < 1$ for $t \in [a,b]$, and since $\overline{\ell} \in \mathcal{P}_{ab}$,

$$\gamma'(t) = \ell(1)(t)\gamma(t) + \overline{\ell}(1)(t) \geq \ell(1)(t)\gamma(t) + \overline{\ell}(\gamma)(t) \quad \text{for } t \in [a,b].$$

Consequently, according to Theorem 1.1,

$$\widetilde{\ell} \in \mathcal{S}_{ab}(a), \tag{3.1}$$

where

$$\widetilde{\ell}(v)(t) \overset{def}{=} \ell(1)(t)v(t) + \overline{\ell}(v)(t) \quad \text{for } t \in [a,b]. \tag{3.2}$$

By Remark 0.3, it is sufficient to show that the problem (0.5) has no nontrivial nonnegative solution. Let the function $u \in \widetilde{C}([a,b];R_+)$ satisfy (0.5). Put

$$w(t) = \vartheta(u)(t) \quad \text{for } t \in [a,b]. \tag{3.3}$$

Obviously, $w'(t) = \ell(u)(t)$ for $t \in [a,b]$, and

$$0 \le u(t) \le w(t), \qquad w(a) = 0. \tag{3.4}$$

On the other hand, by (1.5), (3.3), (3.4), and the condition $\ell \in \mathcal{P}_{ab}$,

$$w'(t) = \ell(u)(t) \le \ell(w)(t) =$$
$$= \ell(1)(t)w(t) + \ell(w)(t) - \ell(1)(t)w(t) \le \ell(1)(t)w(t) + \overline{\ell}(u)(t).$$

However, $\overline{\ell} \in \mathcal{P}_{ab}$, and so by (3.4) and (3.2),

$$w'(t) \le \ell(1)(t)w(t) + \overline{\ell}(w)(t) = \widetilde{\ell}(w)(t).$$

This together with (3.1), (3.4), and Remark 0.3, results in $w \equiv 0$, and consequently, $u \equiv 0$. $\square$

Proof of Theorem 1.2. It is known (see, e.g., [10, Theorem 1.2′]) that if ℓ is an $a-$Volterra operator, then the problem (0.5), (0.5) has only the trivial solution. Consequently, according to Theorem 0.1, the problem (0.1), (0.3) is uniquely solvable.

Let u be a solution of the problem (0.1), (0.3) with $q \in L([a,b];R_+)$ and $c \in R_+$. We show

$$u(t) \ge 0 \quad \text{for } t \in [a,b]. \tag{3.5}$$

Note that if $c = 0$ and $q \not\equiv 0$, then u cannot be still nonpositive in $[a,b]$. Indeed, if $u(t) \le 0$ for $t \in [a,b]$, then by the condition $-\ell \in \mathcal{P}_{ab}$, from (0.1) we get $u'(t) \ge 0$, and so $u(t) \ge 0$ for $t \in [a,b]$, which is a contradiction. Thus in any case we have

$$\max\{u(t) : t \in [a,b]\} > 0.$$

Assume that (3.5) is violated. Then there exists $t_0 \in]a,b[$ such that

$$u(t_0) < 0. \tag{3.6}$$

Put

$$v(t) = \lambda\gamma(t) - u(t) \quad \text{for } t \in [a,b],$$

where

$$\lambda = \max\left\{\frac{u(t)}{\gamma(t)} : t \in [a,t_0]\right\}.$$

Analogously as above, since ℓ is an $a-$Volterra operator, on account of $-\ell \in \mathcal{P}_{ab}$, we get $\max\{u(t) : t \in [a, t_0]\} > 0$, and so $0 < \lambda < +\infty$. Obviously, there exists $t_1 \in [a, t_0[$ such that

$$v(t_1) = 0. \tag{3.7}$$

It is also clear that

$$v(t) \geq 0 \quad \text{for } \; t \in [a, t_0]. \tag{3.8}$$

By virtue of (0.1) and (1.7) we have

$$v'(t) \leq \ell(v)(t) - q(t) \quad \text{for } \; t \in [a, b].$$

Hence, taking into account (3.8), the condition $-\ell \in \mathcal{P}_{ab}$, and the fact that ℓ is an $a-$Volterra operator, we obtain

$$v'(t) \leq 0 \quad \text{for } \; t \in [a, t_0].$$

On account of (3.7),

$$v(t) \leq 0 \quad \text{for } \; t \in [t_1, t_0],$$

whence, in view of (1.6) and (3.6), we get a contradiction $0 < v(t_0) \leq 0$. □

Proof of Theorem 1.3. It is known (see, e.g., [10, Theorem 1.2′]) that if ℓ is an $a-$Volterra operator, then the problem (0.5), (0.5) has only the trivial solution. Consequently, according to Theorem 0.1, the problem (0.1), (0.3) is uniquely solvable.

Let u be a solution of the problem (0.1), (0.3) with $q \in L([a, b]; R_+)$ and $c \in R_+$. Show that (3.5) is fulfilled. Assume the contrary that there exists $t_* \in]a, b]$ such that

$$u(t_*) < 0. \tag{3.9}$$

Note also, as in the proof of Theorem 1.2, that

$$\max\{u(t) : t \in [a, t_*]\} > 0. \tag{3.10}$$

Choose $t^* \in [a, t_*[$ such that

$$u(t^*) = \max\{u(t) : t \in [a, t_*]\}. \tag{3.11}$$

The integration of (0.1) from t^* to t_*, on account of (1.8), (3.10), (3.11), the assumptions $-\ell \in \mathcal{P}_{ab}$, $q \in L([a, b]; R_+)$, and the fact that ℓ is an $a-$Volterra operator, results in

$$u(t^*) - u(t_*) = -\int_{t^*}^{t_*} \ell(u)(s)ds - \int_{t^*}^{t_*} q(s)ds \leq u(t^*) \int_a^b |\ell(1)(s)|ds \leq u(t^*).$$

However, the last inequality contradicts (3.9). □

Proof of Corollary 1.2. It is known that if ℓ is an $a-$Volterra operator, then the problem (0.5), (0.5) has only the trivial solution. Consequently, according to Theorem 0.1, the problem (0.1), (0.3) is uniquely solvable.

Let u be a solution of the problem (0.1), (0.3) with $q \in L([a,b];R_+)$ and $c \in R_+$. Show that (3.5) is fulfilled. From (0.1) we get

$$u'(t) = \ell(1)(t)u(t) + \ell(u)(t) - \ell(1)(t)u(t) + q(t) \quad \text{for } t \in [a,b]. \tag{3.12}$$

On the other hand, the integration of (0.1) from a to t, on account of (0.3), yields

$$u(t) = c + \int_a^t \ell(u)(s)ds + \int_a^t q(s)ds \quad \text{for } t \in [a,b]. \tag{3.13}$$

By virtue of (3.13), from (3.12) we obtain

$$u'(t) = \ell(1)(t)u(t) + \ell(\theta(u))(t) - \ell(1)(t)\theta(u)(t) + q_0(t) \text{ for } t \in [a,b], \tag{3.14}$$

where

$$\begin{gathered} q_0(t) = \ell(q^*)(t) - \ell(1)(t)q^*(t) + q(t) \quad \text{for } t \in [a,b], \\ \theta(v)(t) = \int_a^t \ell(v)(s)ds, \qquad q^*(t) = c + \int_a^t q(s)ds \quad \text{for } t \in [a,b]. \end{gathered} \tag{3.15}$$

In view of the condition $-\ell \in \mathcal{P}_{ab}$ and the fact that ℓ is an $a-$Volterra operator, we have

$$\ell(q^*)(t) - \ell(1)(t)q^*(t) \geq 0 \quad \text{for } t \in [a,b].$$

Thus, due to $q \in L([a,b];R_+)$, (3.15) yields

$$q_0(t) \geq 0 \quad \text{for } t \in [a,b]. \tag{3.16}$$

Put

$$w(t) = u(t)\exp\left(-\int_a^t \ell(1)(s)ds\right) \quad \text{for } t \in [a,b]. \tag{3.17}$$

Then $w(a) = c$, and (3.14) results in

$$w'(t) = \exp\left(-\int_a^t \ell(1)(s)ds\right)\widetilde{\ell}(w)(t) + \widetilde{q}(t) \quad \text{for } t \in [a,b], \tag{3.18}$$

where

$$\begin{gathered} \widetilde{\ell}(v)(t) = \ell(\widetilde{\theta}(v))(t) - \ell(1)(t)\widetilde{\theta}(v)(t) \quad \text{for } t \in [a,b], \\ \widetilde{\theta}(v)(t) = \theta(\widetilde{v})(t), \quad \widetilde{v}(t) = v(t)\exp\left(\int_a^t \ell(1)(s)ds\right) \quad \text{for } t \in [a,b], \end{gathered}$$

$$\widetilde{q}(t) = q_0(t) \exp\left(- \int_a^t \ell(1)(s)ds \right) \quad \text{for } \ t \in [a,b]. \tag{3.19}$$

It is easy to verify that $-\widetilde{\ell} \in \mathcal{P}_{ab}$ and $\widetilde{\ell}$ is an $a-$Volterra operator. By virtue of (1.9), Theorem 1.3, and (3.16), (3.18), (3.19), we have $w(t) \geq 0$ for $t \in [a,b]$. Consequently, in view of (3.17), $u(t) \geq 0$ for $t \in [a,b]$. □

Proof of Theorem 1.4. Let u be a solution of the problem (0.1), (0.3), where $q \in L([a,b]; R_+)$ and $c \in R_+$, and let v be a solution of the problem

$$v'(t) = -\ell_1(v)(t) - \ell_0([u]_-)(t), \qquad v(a) = 0. \tag{3.20}$$

Since $-\ell_1 \in \mathcal{S}_{ab}(a)$ and $\ell_0 \in \mathcal{P}_{ab}$,

$$v(t) \leq 0 \quad \text{for } \ t \in [a,b]. \tag{3.21}$$

Moreover, in view of the assumptions $q \in L([a,b]; R_+)$ and $\ell_0 \in \mathcal{P}_{ab}$, the equality (3.20) results in

$$v'(t) \leq -\ell_1(v)(t) + \ell_0(u)(t) + q(t) \quad \text{for } \ t \in [a,b].$$

Therefore, according to Remark 0.2, by virtue of the assumption $-\ell_1 \in \mathcal{S}_{ab}(a)$,

$$v(t) \leq u(t) \quad \text{for } \ t \in [a,b]. \tag{3.22}$$

Now (3.22) and (3.21) imply

$$v(t) \leq -[u(t)]_- \quad \text{for } \ t \in [a,b]. \tag{3.23}$$

On the other hand, due to (3.20), (3.23), and the condition $\ell_0 \in \mathcal{P}_{ab}$, we have

$$v'(t) \geq \ell_0(v)(t) - \ell_1(v)(t) \quad \text{for } \ t \in [a,b].$$

Hence, in view of $\ell_1 \in \mathcal{P}_{ab}$, (3.21) and Remark 0.2, the inclusion $\ell_0 \in \mathcal{S}_{ab}(a)$ yields

$$v(t) \geq 0 \quad \text{for } \ t \in [a,b],$$

and, consequently, according to (3.22), the inequality (3.5) holds.

We have proved that if u is a solution of (0.1), (0.3) with $q \in L([a,b]; R_+)$ and $c \in R_+$, then the inequality (3.5) is fulfilled. Now show that the homogeneous problem (0.5), (0.5) has only the trivial solution. Indeed, let u be a solution of (0.5), (0.5). Since $-u$ is also a solution of (0.5), (0.5), it follows from the above that

$$u(t) \geq 0, \qquad -u(t) \geq 0 \quad \text{for } \ t \in [a,b],$$

and, consequently, $u \equiv 0$. □

Theorems 1.5–1.8 and Corollaries 1.3 and 1.4 follow from Theorems 1.1–1.4, Corollaries 1.1 and 1.2, and Remark 1.5 .

Proof of Theorem 1.9. a) According to (1.13) we have

$$\rho_2(t) \le \alpha\rho_1(t) \quad \text{for } t \in [a,b],$$

where

$$\rho_1(t) = \int_a^t p(s)ds = \int_a^t \ell(1)(s)ds \quad \text{for } t \in [a,b],$$

$$\rho_2(t) = \int_a^t p(s) \int_a^{\tau(s)} p(\xi)d\xi ds = \int_a^t \ell(\rho_1)(s)ds \quad \text{for } t \in [a,b].$$

Thus for $m = 2$, $k = 1$ the condition (1.2) in Corollary 1.1 b) is fulfilled.

b) Let $\overline{\ell}$ be an operator defined by

$$\overline{\ell}(v)(t) \overset{def}{=} p(t)\sigma(t) \int_t^{\tau(t)} p(s)v(\tau(s))ds. \tag{3.24}$$

Obviously, $\overline{\ell} \in \mathcal{P}_{ab}$, and for any $v \in C_a([a,b];R_+)$,

$$\ell(\vartheta(v))(t) - \ell(1)(t)\vartheta(v)(t) = p(t) \int_t^{\tau(t)} p(s)v(\tau(s))ds \le \overline{\ell}(v)(t) \quad \text{for } t \in [a,b],$$

where

$$\vartheta(v)(t) = \int_a^t \ell(v)(s)ds \quad \text{for } t \in [a,b].$$

On the other hand, from (1.14) it follows the inequality (1.4), and the assumptions of Corollary 1.1 c) are fulfilled.

c) According to (1.15), there exists $\varepsilon > 0$ such that

$$\int_t^{\tau(t)} p(s)ds < \lambda^* - \varepsilon \quad \text{for } t \in [a,b]. \tag{3.25}$$

Choose $x_0 > 0$ and $\delta \in]0,1[$ such that

$$\frac{1}{x_0} \ln\left(x_0 + \frac{x_0(1-\delta)}{\exp\left(x_0 \int_a^{\tau^*} p(s)ds\right) - (1-\delta)}\right) > \lambda^* - \varepsilon, \tag{3.26}$$

and put

$$\gamma(t) = \exp\left(x_0 \int_a^t p(s)ds\right) - 1 + \delta \quad \text{for } t \in [a,b].$$

It can be easily verified that on account of (3.25) and (3.26), the inequlaity (1.1) is fulfilled, and so the assumptions of Theorem 1.1 are satisfied. $\square$

Proof of Theorem 1.10. Obviously, if (1.17) holds, then the operator ℓ defined by (1.20) satisfies the condition (1.8) and, according to Theorem 1.3, $\ell \in \mathcal{S}_{ab}(a)$.

If (1.18) holds, then the operator ℓ defined by (1.20) satisfies the condition (1.9), where $\widetilde{\ell}$ is defined by (1.10), and, according to Corollary 1.2, $\ell \in \mathcal{S}_{ab}(a)$.

Now assume that the inequality (1.19) holds. Then there exists $\varepsilon > 0$ such that

$$\int\limits_{\mu(t)}^{t} g(s)ds < \eta^* - \varepsilon \quad \text{for } \ t \in [a,b]. \tag{3.27}$$

Choose $x_0 > 0$ and $\delta \in]0,1[$ such that

$$\frac{1}{x_0} \ln \left(x_0 + \frac{x_0(1-\delta)}{\exp\left(x_0 \int\limits_a^b g(s)ds \right) - (1-\delta)} \right) > \eta^* - \varepsilon, \tag{3.28}$$

and put

$$\gamma(t) = \exp\left(x_0 \int\limits_t^b g(s)ds \right) - 1 + \delta \quad \text{for } \ t \in [a,b].$$

It can be easily verified that by virtue of (3.27) and (3.28), the inequlaities (1.6) and (1.7) are fulfilled, and so the assumptions of Theorem 1.2 are satisfied. $\square$

Theorem 1.11 immediately follows from Theorems 1.4, 1.9 and 1.10. Theorems 1.12–1.14 follow from Theorems 1.9–1.11 and Remark 1.5.

3.2. Proofs of the theorems on positive solutions of the homogeneous equation. To prove Theorems 2.1 and 2.2 we will need some auxiliary propositions.

Proposition 3.1. *$\mathcal{P}_{ab} \cap \widetilde{\mathcal{S}}_{ab} = \mathcal{P}_{ab} \cap \mathcal{S}_{ab}(a)$.*

Proof. Let $\ell \in \mathcal{P}_{ab} \cap \mathcal{S}_{ab}(a)$, and let u be a solution of

$$u'(t) = \ell(u)(t), \qquad u(a) = 1. \tag{3.29}$$

Since $\ell \in \mathcal{S}_{ab}(a)$, we have $u(t) \geq 0$ for $t \in [a,b]$. By virtue of $\ell \in \mathcal{P}_{ab}$, from (3.29) we get $u'(t) \geq 0$ for $t \in [a,b]$, and, consequently, $u(t) > 0$ for $t \in [a,b]$. Therefore, $\mathcal{P}_{ab} \cap \mathcal{S}_{ab}(a) \subseteq \mathcal{P}_{ab} \cap \widetilde{\mathcal{S}}_{ab}$.

Suppose now $\ell \in \mathcal{P}_{ab} \cap \widetilde{\mathcal{S}}_{ab}$. According to Definition 2.1, the equation (0.5) has a positive solution γ. So, from Theorem 1.1 it follows that $\ell \in \mathcal{S}_{ab}(a)$. Therefore, $\mathcal{P}_{ab} \cap \widetilde{\mathcal{S}}_{ab} \subseteq \mathcal{P}_{ab} \cap \mathcal{S}_{ab}(a)$. $\square$

Proposition 3.2. *Let ℓ be a $b-$Volterra operator, $\ell \in \mathcal{P}_{ab} \cap \widetilde{\mathcal{S}}_{ab}$. Then $\ell \in \mathcal{S}_{ab}(b)$.*

Proof. Since ℓ is a $b-$Volterra operator, the problem (0.5), (0.5) has only the trivial solution. Let u be a solution of (0.1), (0.4) with $c \in R_+$, $q \in L([a,b]; R_-)$. Put

$$v_\varepsilon = \varepsilon + u(t) \quad \text{for} \ \ t \in [a,b], \tag{3.30}$$

where $\varepsilon > 0$, and show that

$$v_\varepsilon(t) > 0 \quad \text{for} \ \ t \in [a,b]. \tag{3.31}$$

Indeed, if (3.31) does not hold, then, in view of $v_\varepsilon(b) > 0$, there exists $t_\varepsilon \in [a,b[$ such that

$$v_\varepsilon(t) > 0 \quad \text{for} \ \ t \in]t_\varepsilon, b], \qquad v_\varepsilon(t_\varepsilon) = 0. \tag{3.32}$$

Obviously,

$$v_\varepsilon'(t) = \ell(v_\varepsilon)(t) + q(t) - \varepsilon\ell(1)(t) \quad \text{for} \ \ t \in [a,b],$$

and by virtue of the assumptions $q \in L([a,b]; R_-)$, $\ell \in \mathcal{P}_{ab}$, $\varepsilon > 0$,

$$v_\varepsilon'(t) \le \ell(v_\varepsilon)(t) \quad \text{for} \ \ t \in [a,b]. \tag{3.33}$$

Since ℓ is a $b-$Volterra operator, due to (3.32), (3.33), Remark 0.3 and Proposition 3.1, we have $\ell \notin \widetilde{\mathcal{S}}_{t_\varepsilon b}$. Hence, according to Remark 2.1, we get a contradiction with the assumption $\ell \in \widetilde{\mathcal{S}}_{ab}$.

Now, in view of arbitrariness of $\varepsilon > 0$, (3.30) and (3.31) result in

$$u(t) \ge 0 \quad \text{for} \ \ t \in [a,b],$$

and consequently, $\ell \in \mathcal{S}_{ab}(b)$. $\square$

According to Remark 1.5, Propositions 3.1 and 3.2 imply

Proposition 3.3. $(-\mathcal{P}_{ab}) \cap \widetilde{\mathcal{S}}_{ab} = (-\mathcal{P}_{ab}) \cap \mathcal{S}_{ab}(b)$, *where* $-\mathcal{P}_{ab} = \{\ell \in \widetilde{\mathcal{L}}_{ab} : -\ell \in \mathcal{P}_{ab}\}$.

Proposition 3.4. *Let ℓ be an $a-$Volterra operator, $\ell \in (-\mathcal{P}_{ab}) \cap \widetilde{\mathcal{S}}_{ab}$. Then $\ell \in \mathcal{S}_{ab}(a)$.*

It is clear that Proposition 3.2, resp. Proposition 3.4, in view of Proposition 3.1, resp. Proposition 3.3, can be formulated in an equivalent form.

Proposition 3.5. *Let ℓ be a $b-$Volterra operator, $\ell \in \mathcal{P}_{ab} \cap \mathcal{S}_{ab}(a)$. Then $\ell \in \mathcal{S}_{ab}(b)$.*

Proposition 3.6. *Let ℓ be an $a-$Volterra operator, $\ell \in (-\mathcal{P}_{ab}) \cap \mathcal{S}_{ab}(b)$. Then $\ell \in \mathcal{S}_{ab}(a)$.*

Lemma 3.1. *Let $\ell = \ell_0 - \ell_1$, where $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, and let there exist functions $\alpha, \beta \in \widetilde{C}([a,b]; R_+)$ satisfying the inequalities* (2.3), *and*

$$\begin{aligned} \beta'(t) &\geq \ell_0(\beta)(t) - \ell_1(\alpha)(t) + q(t) \quad \textit{for } t \in [a,b], \\ \alpha'(t) &\leq \ell_0(\alpha)(t) - \ell_1(\beta)(t) + q(t) \quad \textit{for } t \in [a,b], \end{aligned} \tag{3.34}$$

resp.

$$\begin{aligned} \beta'(t) &\leq -\ell_1(\beta)(t) + \ell_0(\alpha)(t) + q(t) \quad \textit{for } t \in [a,b], \\ \alpha'(t) &\geq -\ell_1(\alpha)(t) + \ell_0(\beta)(t) + q(t) \quad \textit{for } t \in [a,b]. \end{aligned} \tag{3.35}$$

Then for every $c \in [\alpha(a), \beta(a)]$, resp. $c \in [\alpha(b), \beta(b)]$, the equation (0.1) *has at least one solution u satisfying the initial condition* (0.3), *resp.* (0.4), *and inequalities*

$$\alpha(t) \leq u(t) \leq \beta(t) \quad \textit{for } t \in [a,b]. \tag{3.36}$$

Proof. Define operator $\chi : C([a,b]; R) \to C([a,b]; R)$ by

$$\chi(v)(t) \overset{def}{=} \frac{1}{2}\big(|v(t) - \alpha(t)| - |v(t) - \beta(t)| + \alpha(t) + \beta(t)\big). \tag{3.37}$$

Obviously,

$$\alpha(t) \leq \chi(v)(t) \leq \beta(t) \quad \text{for } t \in [a,b], \quad v \in C([a,b]; R). \tag{3.38}$$

Let $T : C([a,b]; R) \to C([a,b]; R)$ be an operator defined by

$$T(v)(t) \overset{def}{=} c + \int_a^t (\widehat{\ell}_0(v)(s) - \widehat{\ell}_1(v)(s))ds + \int_a^t q(s)ds, \tag{3.39}$$

resp.

$$T(v)(t) \overset{def}{=} c - \int_t^b (\widehat{\ell}_0(v)(s) - \widehat{\ell}_1(v)(s))ds - \int_t^b q(s)ds, \tag{3.40}$$

where

$$\widehat{\ell}_0(v)(t) \overset{def}{=} \ell_0(\chi(v))(t), \qquad \widehat{\ell}_1(v)(t) \overset{def}{=} \ell_1(\chi(v))(t).$$

By virtue of (3.38), and the assumptions $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, we have that for each $v \in C([a,b]; R)$, the function $T(v)$ belongs to the set $\widetilde{C}([a,b]; R)$ and

$$|T(v)(t)| \leq M \quad \text{for } t \in [a,b], \tag{3.41}$$

$$\begin{aligned} \ell_0(\alpha)(t) - \ell_1(\beta)(t) + q(t) \leq \frac{d}{dt}T(v)(t) \leq \\ \leq \ell_0(\beta)(t) - \ell_1(\alpha)(t) + q(t) \quad \text{for } t \in [a,b], \end{aligned} \tag{3.42}$$

where

$$M = |c| + \int_a^b \big(\ell_0(|\alpha| + |\beta|)(s) + \ell_1(|\alpha| + |\beta|)(s) + |q(s)|\big)ds.$$

According to (3.41), (3.42), and the Arzelà–Ascoli lemma, it is clear that the operator T transforms the space $C([a,b];R)$ into its relatively compact subset. Therefore, by Schauder's fixed point theorem, there exists $u \in C([a,b];R)$ such that

$$u(t) = T(u)(t) \quad \text{for } \ t \in [a,b]. \tag{3.43}$$

Evidently, $u \in \widetilde{C}([a,b];R)$, and $u(a) = c$, resp. $u(b) = c$, i.e.,

$$u(a) - \beta(a) \le 0, \qquad \text{resp.} \qquad u(b) - \beta(b) \le 0. \tag{3.44}$$

In view of (3.42), (3.43), and (3.34), resp. (3.35), we obtain

$$\begin{aligned}(u(t) - \beta(t))' = & \frac{d}{dt}T(u)(t) - \beta'(t) \le \ell_0(\beta)(t) - \ell_1(\alpha)(t) + q(t) - \beta'(t) \le 0 \\ & \text{for } \ t \in [a,b],\end{aligned}$$

resp.

$$\begin{aligned}(u(t) - \beta(t))' = & \frac{d}{dt}T(u)(t) - \beta'(t) \ge \ell_0(\alpha)(t) - \ell_1(\beta)(t) + q(t) - \beta'(t) \ge 0 \\ & \text{for } \ t \in [a,b].\end{aligned}$$

Thus on account of (3.44) we have $u(t) \le \beta(t)$ for $t \in [a,b]$. Analogously one can prove $u(t) \ge \alpha(t)$ for $t \in [a,b]$, and so (3.36) is fulfilled.

According to (3.36), (3.37), and (3.39), resp. (3.40), from (3.43) it follows that

$$u(t) = c + \int_a^t (\ell_0(u)(s) - \ell_1(u)(s))ds + \int_a^t q(s)ds \quad \text{for } \ t \in [a,b],$$

resp.

$$u(t) = c - \int_t^b (\ell_0(u)(s) - \ell_1(u)(s))ds - \int_t^b q(s)ds \quad \text{for } \ t \in [a,b],$$

i.e., u is a solution of (0.1) satisfying (0.3), resp. (0.4). □

Proof of Theorem 2.1. According to Proposition 3.6, we have $-\ell_1 \in \mathcal{S}_{ab}(a)$, and therefore Theorem 1.4 implies $\ell \in \mathcal{S}_{ab}(a)$. By Definition 0.1, the homogeneous problem (0.5), (0.5) has only the trivial solution, and consequently, the problem (3.29) has a unique solution.

Let u be a solution of (3.29). Then in view of the condition $\ell \in \mathcal{S}_{ab}(a)$ we have

$$u(t) \geq 0 \quad \text{for } t \in [a,b]. \tag{3.45}$$

Therefore, by virtue of the assumption $\ell_0 \in \mathcal{P}_{ab}$,

$$u'(t) \geq -\ell_1(u)(t) \quad \text{for } t \in [a,b]. \tag{3.46}$$

Suppose that there exists $b_1 \in]a,b]$ such that

$$u(b_1) = 0.$$

Then since ℓ_1 is an $a-$Volterra operator, on account of (3.45) and (3.46), from Remark 0.3 and Proposition 3.3 it follows that $-\ell_1 \notin \widetilde{\mathcal{S}}_{ab_1}$. But this, in view of Remark 2.1, contradicts the assumption $-\ell_1 \in \widetilde{\mathcal{S}}_{ab}$. Consequently, u is a positive solution of (0.5), i.e., $\ell \in \widetilde{\mathcal{S}}_{ab}$. □

Proof of Theorem 2.2. According to (2.1)–(2.3), from Lemma 3.1 it follows that (0.5) has a solution u satisfying the initial condition $u(b) = \beta(b)$, and

$$u(t) \geq \alpha(t) \quad \text{for } t \in [a,b]. \tag{3.47}$$

Assume that (2.4) is fulfilled. Then from (3.47) it follows that u is a positive solution of (0.5), and consequently $\ell \in \widetilde{\mathcal{S}}_{ab}$.

Now assume that (2.5), resp. (2.6) is fulfilled. Then from (3.47) it follows that

$$u(t) > 0 \quad \text{for } t \in]a,b], \quad \text{resp.} \quad \text{for } t \in [a,b[. \tag{3.48}$$

By virtue of (3.48), the condition $\ell_1 \in \mathcal{P}_{ab}$, resp. the condition $\ell_0 \in \mathcal{P}_{ab}$, from (0.5) we get

$$u'(t) \leq \ell_0(u)(t), \quad \text{resp.} \quad u'(t) \geq \ell_1(u)(t) \quad \text{for } t \in [a,b].$$

Thus due to $\ell_0 \in \mathcal{S}_{ab}(a)$, resp. $-\ell_1 \in \mathcal{S}_{ab}(b)$, and Remark 0.3, we have $u(a) \neq 0$, resp. $u(b) \neq 0$. Therefore, on account of (3.48), u is a positive solution of (0.5), and consequently, $\ell \in \widetilde{\mathcal{S}}_{ab}$. □

Proof of Corollary 2.1. According to (2.8), there exists $\varepsilon > 0$ such that

$$\varepsilon + \beta(a) \int_a^b \ell_0(1)(s)ds \leq \beta(b). \tag{3.49}$$

Put

$$\alpha(t) = \varepsilon + \beta(a) \int_a^t \ell_0(1)(s)ds \quad \text{for } t \in [a,b].$$

From (2.7) it follows that β is nondecreasing, i.e.,

$$\beta(b) \leq \beta(t) \leq \beta(a) \quad \text{for } t \in [a,b]. \tag{3.50}$$

Consequently, it is obvious that the inequalities (2.1), (2.2), and (2.4) hold.

On the other hand, by (3.49) and (3.50) we have

$$\alpha(t) \leq \varepsilon + \beta(a) \int_a^b \ell_0(1)(s)ds \leq \beta(b) \leq \beta(t) \quad \text{for } t \in [a,b].$$

Thus the assumptions of Theorem 2.2 are fulfilled, and therefore $\ell \in \widetilde{\mathcal{S}}_{ab}$. □

Theorems 2.3 and 2.4, and Corollary 2.2 follow from Theorems 2.1 and 2.2, Corollary 2.1, and Remark 1.5. Theorems 2.5 and 2.7 follow from Theorems 1.9, 1.12, 2.1 and 2.3.

Proof of Theorem 2.6. Put

$$\beta(t) = (1-\alpha)\left(1 + \int_t^b g(s)ds\right) + \int_t^b g(s) \int_{\mu(s)}^b g(\xi)d\xi ds,$$

resp.

$$\beta(t) = \exp\left(x_0 \int_t^b g(s)ds\right) - 1 + \delta.$$

It is not difficult to verify that the function β satisfies the assumptions of Corollary 2.1. □

Theorem 2.8 follows from Theorem 2.6 and Remark 1.5.

4. Examples

On Remark 1.1. If (0.5), (0.5) has only the trivial solution, then according to the Fredholm property, the problem (3.29) has a unique solution u. Suppose that u assumes negative values. Put

$$m = \max\{-u(t) : t \in [a,b]\},$$

and choose $t_0 \in [a,b]$ such that $u(t_0) = -m$. The integration of (0.5) from a to t_0 yields

$$m + 1 = -\int_a^{t_0} \ell(u)(s)ds \leq m \int_a^b \ell(1)(s)ds.$$

On account of the assumption $\int_a^b \ell(1)(s)ds = 1$, we get a contradiction $m < m$. Consequently, $u(t) \geq 0$ for $t \in [a,b]$, and in view of the assumption $\ell \in \mathcal{P}_{ab}$, from (3.29) it follows that $u(t) > 0$ for $t \in [a,b]$. Thus, according to Theorem 1.1, we have $\ell \in \mathcal{S}_{ab}(a)$.

Example 4.1. Let $\tau \equiv b$ and $p \in L([a,b];R_+)$ be such that

$$\int_a^b p(s)ds = 1.$$

Obviously, for $\alpha = 1$ the condition (1.13) is fulfilled, and for every $m > k$ the condition (1.2) is satisfied, where ℓ is defined by (1.16). Furthermore,

$$\int_a^b p(s)\sigma(s)\int_s^{\tau(s)} p(\xi)d\xi \exp\bigg(\int_s^b p(\eta)d\eta\bigg)ds = 1,$$

i.e.,

$$\int_a^b \overline{\ell}(1)(s)\exp\bigg(\int_s^b \ell(1)(\eta)d\eta\bigg)ds = 1,$$

where $\overline{\ell}$ is an operator defined by (3.24), and the inequality (1.5) is fulfilled.

On the other hand, the function

$$u(t) = \int_a^t p(s)ds \quad \text{for} \;\; t \in [a,b]$$

is a nontrivial solution of the problem (0.5), (0.5). Therefore, according to Definition 0.1, $\ell \notin \mathcal{S}_{ab}(a)$.

This example shows that the condition $\alpha \in]0,1[$ in Corollary 1.1 b) and in Theorem 1.9 a) cannot be replaced by the condition $\alpha \in]0,1]$. Also the strict inequalities (1.4) and (1.14) in Corollary 1.1 c) and Theorem 1.9 b) cannot be replaced by the nonstrict ones.

Example 4.2. Let $\tau \equiv b$ and $p \in L([a,b];R_+)$ be such that

$$\int_a^b p(s)ds = 1 + \varepsilon,$$

where $\varepsilon > 0$. Then for $\alpha = 1 + \varepsilon$ the condition (1.13) is fulfilled, and for a natural number m and $k = m - 1$, the condition (1.2) is satisfied, where ℓ is an operator defined by (1.16). Furthermore,

$$\int_a^b p(s)\sigma(s)\int_s^{\tau(s)} p(\xi)d\xi \exp\bigg(\int_s^b p(\eta)d\eta\bigg)ds = 1 + \delta,$$

i.e.,

$$\int_a^b \overline{\ell}(1)(s)\exp\bigg(\int_s^b \ell(1)(\eta)d\eta\bigg)ds = 1 + \delta,$$

where $\delta = \varepsilon e^{1+\varepsilon}$, $\overline{\ell}$ is defined by (3.24), and the inequality (1.5) is fulfilled.

On the other hand, the problem

$$u'(t) = p(t)u(\tau(t)), \qquad u(a) = 0 \tag{4.1}$$

has only the trivial solution. Indeed, the integration of (4.1) from a to b yields $u(b) = (1+\varepsilon)u(b)$, i.e., $u(b) = 0$, and so $u'(t) = 0$ for $t \in [a, b]$, which together with $u(a) = 0$ results in $u \equiv 0$.

However,

$$u(t) = 1 - \frac{1}{\varepsilon} \int_a^t p(s)ds$$

is a solution of (3.29) with ℓ defined by (1.16), and $u(b) = -\frac{1}{\varepsilon} < 0$. Therefore, $\ell \notin \mathcal{S}_{ab}(a)$.

This example shows that if $\ell \in \mathcal{P}_{ab}$ satisfies

$$\int_a^b \ell(1)(s)ds > 1$$

and the problem (0.5), (0.5) has only the trivial solution, then it may happen that the operator ℓ does not belong to the set $\mathcal{S}_{ab}(a)$.

Example 4.3. Let $b_1 \in]a, b[$ and $\varepsilon \in]0, 2[$. Choose $g \in L([a, b]; R_+)$ such that

$$\int_a^{b_1} g(s)ds = \frac{\varepsilon}{2}, \qquad \int_{b_1}^b g(s)ds = 1 + \frac{\varepsilon}{2}.$$

Put

$$\mu(t) = \begin{cases} a & \text{for } t \in [a, b_1[\\ b_1 & \text{for } t \in [b_1, b] \end{cases}, \qquad \gamma(t) = \begin{cases} \frac{\varepsilon}{2} - \int_a^t g(s)ds & \text{for } t \in [a, b_1[\\ 0 & \text{for } t \in [b_1, b] \end{cases}.$$

Obviously, all the assumptions of Theorem 1.2 are fulfilled except of (1.6), where ℓ is defined by (1.20).

On the other hand, since ℓ is an $a-$Volterra operator, the function

$$u(t) = \begin{cases} 1 - \int_a^t g(s)ds & \text{for } t \in [a, b_1[\\ (1 - \frac{\varepsilon}{2})\Big(1 - \int_{b_1}^t g(s)ds\Big) & \text{for } t \in [b_1, b] \end{cases}$$

is a unique solution of (3.29), and $u(b) = -\frac{\varepsilon}{2}(1 - \frac{\varepsilon}{2}) < 0$. Consequently, $\ell \notin \mathcal{S}_{ab}(a)$.

This example shows that the condition (1.6) cannot be replaced by

$$\gamma(t) > 0 \quad \text{for } t \in [a, b_1[,$$

where $b_1 \in]a,b[$ is an arbitrarily fixed point.

Example 4.4. Let $\varepsilon > 0$, $\mu \equiv a$, and $g \in L([a,b];R_+)$ be such that

$$\int_a^b g(s)ds = 1 + \varepsilon.$$

It is clear that the operator ℓ defined by (1.20) satisfies

$$\int_a^b |\ell(1)(s)|ds \leq 1 + \varepsilon. \tag{4.2}$$

Obviously, since ℓ is an $a-$Volterra operator, the function

$$u(t) = 1 - \int_a^t g(s)ds \quad \text{for} \ \ t \in [a,b]$$

is a unique solution of (3.29). On the other hand, $u(b) = -\varepsilon < 0$. Therefore, $\ell \notin \mathcal{S}_{ab}(a)$.

This example shows that the condition (1.8), resp. (1.17) in Theorem 1.3, resp. in Theorem 1.10, cannot be replaced by the condition (4.2), resp.

$$\int_a^b g(s)ds \leq 1 + \varepsilon,$$

no matter how small $\varepsilon > 0$ would be.

This example also shows that the condition (1.9), resp. (1.18) in Corollary 1.2, resp. in Theorem 1.10 cannot be replaced by the condition

$$\int_a^b |\widetilde{\ell}(1)(s)| \exp\left(\int_a^s |\ell(1)(\xi)|d\xi\right) ds \leq 1 + \varepsilon,$$

resp.

$$\int_a^b g(s)\left(\int_{\mu(s)}^s g(\xi)\exp\left[\int_{\mu(\xi)}^s g(\eta)d\eta\right]d\xi\right)ds \leq 1 + \varepsilon,$$

no matter how small $\varepsilon > 0$ would be.

Example 4.5. Let $\varepsilon > 0$, $\tau \equiv b$, $\mu \equiv a$, and $p, g \in L([a,b];R_+)$ be such that

$$\int_a^b p(s)ds = 1 + \varepsilon, \qquad \int_a^b g(s)ds < 1.$$

Obviously, $(1-\varepsilon)\ell_0 \in \mathcal{S}_{ab}(a)$ and $-\ell_1 \in \mathcal{S}_{ab}(a)$, where

$$\ell_0(v)(t) \stackrel{def}{=} p(t)v(\tau(t)), \qquad \ell_1(v)(t) \stackrel{def}{=} g(t)v(\mu(t)). \tag{4.3}$$

Note also that the problem (0.5), (0.5) has only the trivial solution. Indeed, the integration of (0.5) from a to b yields $u(b) = (1+\varepsilon)u(b)$, whence we get $\varepsilon u(b) = 0$, i.e., $u(b) = 0$. Consequently, $u'(t) = 0$, which together with $u(a) = 0$ results in $u \equiv 0$. Therefore, the problem (3.29) with $\ell = \ell_0 - \ell_1$ has a unique solution u.

On the other hand, the integration of (3.29) from a to b yields

$$u(b) - 1 = u(b)(1+\varepsilon) - \int_a^b g(s)ds,$$

whence we get

$$\varepsilon u(b) = \int_a^b g(s)ds - 1 < 0,$$

i.e., $u(b) < 0$. Therefore, $\ell \notin \mathcal{S}_{ab}(a)$.

Example 4.6. Let $\varepsilon > 0$, $\tau \equiv b$, $\mu \equiv a$, and $p, g \in L([a,b]; R_+)$ be such that

$$\int_a^b p(s)ds < 1, \qquad \int_a^b g(s)ds = 1+\varepsilon.$$

Obviously, $\ell_0 \in \mathcal{S}_{ab}(a)$ and $-(1-\varepsilon)\ell_1 \in \mathcal{S}_{ab}(a)$, where ℓ_0 and ℓ_1 are defined by (4.3). Note also that the problem (0.5), (0.5) has only the trivial solution. Therefore, the problem (3.29) with $\ell = \ell_0 - \ell_1$ has a unique solution u.

On the other hand, the integration of (3.29) from a to b yields

$$u(b) - 1 = u(b)\int_a^b p(s)ds - (1+\varepsilon),$$

whence we get

$$\varepsilon = u(b)\Big(\int_a^b p(s)ds - 1\Big),$$

i.e., $u(b) < 0$. Therefore, $\ell \notin \mathcal{S}_{ab}(a)$.

Examples 4.5 and 4.6 show that the assumptions

$$\ell_0 \in \mathcal{S}_{ab}(a), \qquad -\ell_1 \in \mathcal{S}_{ab}(a)$$

in Theorem 1.4 can be replaced neither by

$$(1-\varepsilon)\ell_0 \in \mathcal{S}_{ab}(a), \qquad -\ell_1 \in \mathcal{S}_{ab}(a),$$

nor by

$$\ell_0 \in \mathcal{S}_{ab}(a), \qquad -(1-\varepsilon)\ell_1 \in \mathcal{S}_{ab}(a),$$

no matter how small $\varepsilon > 0$ would be.

Moreover, these examples show, that the assumptions

$$\ell_0 \in \mathcal{S}_{ab}(a), \qquad -\ell_1 \in \mathcal{S}_{ab}(b)$$

in Theorem 2.1 can be replaced neither by

$$(1-\varepsilon)\ell_0 \in \mathcal{S}_{ab}(a), \qquad -\ell_1 \in \mathcal{S}_{ab}(b),$$

nor by

$$\ell_0 \in \mathcal{S}_{ab}(a), \qquad -(1-\varepsilon)\ell_1 \in \mathcal{S}_{ab}(b),$$

no matter how small $\varepsilon > 0$ would be.

Example 4.7. Let $\tau \equiv a$, $\mu \equiv b$, $c \in]a, b[$, and choose $p, g \in L([a, b]; R_+)$ such that

$$\int_a^c p(s)ds = 0, \quad \int_c^b p(s)ds = 1, \quad \int_a^c g(s)ds = 1, \quad \int_c^b g(s)ds = 0.$$

Obviously, $\ell_0 \in \mathcal{S}_{ab}(a)$, $-\ell_1 \in \mathcal{S}_{ab}(b)$, where ℓ_0 and ℓ_1 are defined by (4.3), since ℓ_0, resp. ℓ_1 is an $a-$Volterra operator, resp. a $b-$Volterra operator (see Corollary 1.1 a) and Corollary 1.3 a)).

Now suppose that u is a solution of (0.5), where $\ell = \ell_0 - \ell_1$. Then the integration of (0.5) from a to c and from c to b yields

$$\begin{aligned} u(c) - u(a) &= -u(b), \\ u(b) - u(c) &= u(a). \end{aligned}$$

Hence we obtain $u(c) = 0$, i.e., every solution of (0.5) has a zero at the point c. Consequently, $\ell \notin \widetilde{\mathcal{S}}_{ab}$.

This example shows that the assumption on the operator ℓ_1, resp. ℓ_0, in Theorem 2.1, resp. in Theorem 2.3, to be an $a-$Volterra operator, resp. a $b-$Volterra operator, cannot be omitted.

5. Further Remarks

From theorems on differential inequalities follows the theorems on integral inequalities.

Theorem 5.1. *Let $\ell \in \mathcal{P}_{ab} \cap \mathcal{S}_{ab}(a)$, $c \in R$, $q \in L([a, b]; R)$, and $w \in C([a, b]; R)$ be such that*

$$w(t) \le c + \int_a^t \big(\ell(w)(s) + q(s)\big)ds \quad \text{for } t \in [a, b]. \tag{5.1}$$

Then the inequality

$$w(t) \le u(t) \quad \text{for} \ \ t \in [a,b] \tag{5.2}$$

holds, where u is a solution of the problem (0.1), (0.3).

Proof. Put

$$v(t) = c + \int_a^t \big(\ell(w)(s) + q(s)\big)ds \quad \text{for} \ \ t \in [a,b]. \tag{5.3}$$

Obviously, $v \in \widetilde{C}([a,b];R)$.

In view of (5.1), (5.3), and the condition $\ell \in \mathcal{P}_{ab}$ we have

$$v'(t) = \ell(w)(t) + q(t) \le \ell(v)(t) + q(t) \quad \text{for} \ \ t \in [a,b], \quad v(a) = c.$$

According to Remark 0.2, on account of the condition $\ell \in \mathcal{S}_{ab}(a)$, we have

$$v(t) \le u(t) \quad \text{for} \ \ t \in [a,b].$$

The last inequality, by virtue of (5.1) and (5.3), yields (5.2). □

According to Remark 1.5, Theorem 5.1 implies

Theorem 5.2. *Let $-\ell \in \mathcal{P}_{ab}$, $\ell \in \mathcal{S}_{ab}(b)$, $c \in R$, $q \in L([a,b];R)$, and $w \in C([a,b];R)$ be such that*

$$w(t) \le c - \int_t^b \big(\ell(w)(s) + q(s)\big)ds \quad \text{for} \ \ t \in [a,b].$$

Then the inequality (5.2) *holds, where u is a solution of the problem* (0.1), (0.4).

Remark 5.1. For the case where $\ell(u)(t) = p(t)u(t)$, Theorems 5.1 and 5.2 coincide with the well–known Gronwall–Belman lemma (see, e.g., [8]).

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