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ON ADVANCED FUNCTIONAL DIFFERENTIAL EQUATIONS WITH PROPERTIES A AND B

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Dedicated to the blessed memory of
Professor T. Chanturia

In the present paper we give new results on oscillatory properties of the functional differential equation

$$u^{(n)}(t)=(-1)^k\int\limits_{\tau_0(t)}^{\tau(t)} f(u(s))d_s p(s,t). \tag{1_k}$$

Throughout the paper it will be assumed that $n\geq 2$, $k\in\{1,2\}$ and the following conditions are fulfilled:

(i) $f:R\to R$ is a continuous nondecreasing function such that

$$-f(-x)=f(x)>0,\qquad \int\limits_x^{+\infty}\frac{ds}{f(s)}=+\infty \quad\text{for}\quad x>0,\qquad \lim_{x\to+\infty}f(x)=+\infty;$$

(ii) the functions τ_0 and $\tau:[0,+\infty[\to[0,+\infty[$ are continuous and

$$\tau(t)>\tau_0(t)\geq t \quad\text{for}\quad t\geq 0;$$

(iii) the function $p:[0,+\infty[\times[0,+\infty[\to R$ is nondecreasing in the first argument, and Lebesgue integrable on each finite interval of $[0,+\infty[$ in the second argument.

Particular cases of (1_k) are the following differential equations frequently occurring in the oscillation theory (see [1–17] and the references therein):

$$u^{(n)}(t)=(-1)^k\sum_{j=1}^m p_j(t)|u(\tau_j(t))|^\lambda\operatorname{sgn}(u(\tau_j(t))) \tag{2_k}$$

and

$$u^{(n)}(t)=(-1)^k\sum_{j=1}^m p_j(t)u(\tau_j(t)), \tag{3_k}$$

where $\lambda\in]0,1[$, the functions $p_j:[0,+\infty[\to[0,+\infty[$ $(j=1,\dots,m)$ are Lebesgue integrable on each finite interval of $[0,+\infty[$, and $\tau_j:[0,+\infty[\to[0,+\infty[$ $(j=1,\dots,m)$ are continuous functions satisfying the inequalities

$$\tau_j(t)\geq t \quad (j=1,\dots,m).$$

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By a solution of equation (1_k) on an interval $[a,+\infty[\subset[0,+\infty[$ we understand a function $u:[a,+\infty[\to R$ which is absolutely continuous together with its first $n-1$ derivatives on each finite interval of $[0,+\infty[$ and satisfies (1_k) almost everywhere on $[a,+\infty[$.

A solution u of equation (1_k) is said to be *proper* if it is defined on an interval $[a,+\infty[\subset[0,+\infty[$ and

$$\sup\{|u(s)|:\ s\geq t\}>0 \quad \text{for} \quad t\geq a.$$

A proper solution of equation (1_k) is said to be *oscillatory* if it has a sequence of zeros converging to $+\infty$.

We use the following definitions from [9] and [3].

Definition 1. Equation (1_k) has *property A* if every proper solution of this equation for n even is oscillatory and for n odd either is oscillatory or satisfies the condition

$$|u^{(i)}(t)|\downarrow 0 \quad \text{as} \quad t\to+\infty \quad (i=0,1,\dots,n-1). \tag{4}$$

Definition 2. Equation (1_k) has *property B* if every proper solution of this equation for n even either is oscillatory or satisfies (4) or satisfies the condition

$$|u^{(i)}(t)|\uparrow+\infty \quad \text{as} \quad t\to+\infty \quad (i=0,1,\dots,n-1), \tag{5}$$

and for n odd either is oscillatory or satisfies (5).

We introduce the following notation.

$$q(t)=p(\tau(t),t)-p(\tau_0(t),t), \qquad q_l(t)=t^{n-l}\sum_{j=1}^{m}[\tau_j(t)]^{l-1}p_j(t) \quad (l=1,\dots,n).$$

$\mathcal{N}_{n,k}$ is the set of $l\in\{1,\dots,n-1\}$ for which $l+n+k$ is even.

For any $l\in\{1,\dots,n-1\}$ and $a>0$ the function $v_{a,l}:[a,+\infty[\to[1,+\infty[$ is the lower solution of the Cauchy problem

$$v'(t)=\frac{1}{(n-l)!}t^{n-l}\int\limits_{\tau_0(t)}^{\tau(t)} f\Big(\frac{s^{l-1}}{l!}v(t)\Big)d_s p(s,t), \qquad v(a)=1.$$

Theorem 1. *The condition*

$$\int\limits_0^{+\infty} t^{n-1}q(t)dt=+\infty \tag{6}$$

is necessary for equation (1_1) (*equation* (1_2)) *to have property A* (*property B*). *If along with* (6) *the condition*

$$\int\limits_a^{+\infty} t^{n-l-1}\Bigg[\int\limits_{\tau_0(t)}^{\tau(t)} f\Big(\frac{s^{l-1}}{l!}v_{a,l}(s)\Big)d_s p(s,t)\Bigg]dt=+\infty$$

holds for any $a>0$ *and* $l\in\mathcal{N}_{n,1}$ (*for any* $a>0$ *and* $l\in\mathcal{N}_{n,2}$), *then equation* (1_1) (*equation* (1_2)) *has property A* (*property B*).

Corollary 1. *Let condition* (6) *be fulfilled. Then there exists a continuous function* $\tau_* : [0, +\infty[\to [0, +\infty[$ *such that if*

$$\tau_0(t) \geq \tau_*(t) \quad for \quad t \geq 0,$$

then equation (1_1) (*equation* (1_2)) *has property* A (*property* B).

Theorem 2. *Let* n *be odd* (*even*) *and*

$$\liminf_{t \to +\infty} \frac{f(\tau_0^{l-1}(t))}{t^l} > 0$$

for any $l \in \mathcal{N}_{n,1}$ (*for any* $l \in \mathcal{N}_{n,2}$). *Then condition* (6) *is necessary and sufficient for equation* (1_1) (*equation* (1_2)) *to have property* A (*property* B).

Theorems 1, 2 and Corollary 1 generalize respectively Theorems 1.1, 1.2 and Corollary 1.1 from [7]. For equations (2_k) and (3_k) from these results we have the following statements.

Corollary 2. *The condition*

$$\int_0^{+\infty} t^{n-1} q_1(t)dt = +\infty \tag{7}$$

is necessary for equation (2_1) (*equation* (2_2)) *to have property* A (*property* B). *If along with* (7) *the condition*

$$\int_0^{+\infty} t^{n-l-1} \left[\sum_{j=1}^{m} [\tau_j(t)]^{\lambda(l-1)} p_j(t) \Big(\int_0^{\tau_j(t)} q_l(s)ds \Big)^{\frac{\lambda}{\lambda-1}} \right] dt = +\infty$$

holds for any $l \in \mathcal{N}_{n,1}$ (*for any* $l \in \mathcal{N}_{n,2}$), *then equation* (2_1) (*equation* (2_2)) *has property* A (*property* B).

Corollary 3. *Let condition* (7) *be fulfilled. Then there exists a continuous function* $\tau_* : [0, +\infty[\to [0, +\infty[$ *such that if*

$$\tau_j(t) \geq \tau_*(t) \quad for \quad t \geq 0 \quad (j = 1, \dots, m), \tag{8}$$

then equation (2_1) (*equation* (2_2)) *has property* A (*property* B).

Corollary 4. *Let* n *be odd* (*even*) *and*

$$\liminf_{t \to +\infty} \left[t^{-2/\lambda} \tau_j(t) \right] > 0 \quad (j = 1, \dots, m).$$

Then condition (7) *is necessary and sufficient for equation* (2_1) (*equation* (2_2)) *to have property* A (*property* B).

Corollary 5. *The condition* (7) *is necessary for equation* (3_1) (*equation* (3_2)) *to have property* A (*property* B). *If along with* (7) *the condition*

$$\int_0^{+\infty} t^{n-l-1} \left[\sum_{j=1}^{m} [\tau_j(t)]^{l-1} \exp \left(\frac{1}{(n-l)!\, l!} \int_0^{\tau_j(t)} q_l(s)ds \right) p_j(t) \right] dt = +\infty$$

holds for any $l \in \mathcal{N}_{n,1}$ (*for any* $l \in \mathcal{N}_{n,2}$), *then equation* (3_1) (*equation* (3_2)) *has property* A (*property* B).

Corollary 6. *Let condition* (7) *be fulfilled. Then there exists a continuous function* $\tau_* : [0,+\infty[\to [0,+\infty[$ *such that if inequalities* (8) *hold, then equation* (3_1) (*equation* (3_2)) *has property* A (*property* B).

Corollary 7. *Let* n *be odd* (*even*) *and*

$$\liminf_{t\to+\infty}\Big[t^{-2}\tau_j(t)\Big] > 0 \quad (j=1,\dots,m).$$

Then condition (7) *is necessary and sufficient for equation* (3_1) (*equation* (3_2)) *to have property* A (*property* B).

Note that Corollaries 2–7 take into account the effect of advanvced arguments since, as it is well-known (see [4]), in the case

$$\tau_j(t) \equiv t \quad (j=1,\dots,m)$$

condition (7) does not guarantee that equations (2_1) and (3_1) (equations (2_2) and (3_2)) have property A (property B).

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