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FUNCTIONAL DIFFERENTIAL AND DIFFERENCE INEQUALITIES WITH IMPULSES

Abstract. The following problems for partial functional differential equations are considered: the Cauchy problem on the Haar pyramid for first order equations, mixed problems on bounded domains for Hamilton–Jacobi equations, initial boundary value problem of the Dirichlet type for nonlinear parabolic equations. Impulses depend on functional variable and they are given at fixed points.

The theory of functional differential inequalities is presented in the paper. Moreover, discrete versions of theorems on differential inequalities are presented. The numerical method of lines and difference methods are examples of applications of the theory.

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Introduction

Many real processes and phenomena studied in mechanics, theoretical physics, population dynamics and economy are characterized by the fact that at certain moments of their development the system of parameters undergo rapid changes by jumps. In the mathematical simulation of such processes and phenomena the duration of these changes is usually neglected and the process is assumed to change its state impulsively. A natural tool for the mathematical simulation of such processes are impulsive differential equations. The paper V. Milman and Myshkis [1] initiated the theory of impulsive ordinary differential equations. Up to now numerous papers were published concerning various problems for classes of equations and also dealing with special problems appearing in this theory.

It is not our aim to show a full review of papers concerning ordinary impulsive differential equations. We shall mention only monographs which contain reviews. They are [2]–[5].

Partial differential equations with impulses were first treated in [6]. The authors therein have shown that parabolic equations with impulses provide natural framework for many evolutional processes in the population dynamics. Estimates of solutions of impulsive parabolic equations and applications to the population dynamics were considered in [7]. The first results on impulsive quenching problems for reaction-diffusion equations were given in [8].

Hyperbolic differential equations and inequalities with impulses were considered in [9]–[12]. Estimates of solutions, estimates of the difference between solutions of two problems, uniqueness theory and continuous dependence on given functions were considered. The monotone iterative methods for impulsive nonlinear hyperbolic equations were investigated in [13]. Difference methods for first order partial differential or functional differential equations with impulses were investigated in [14]–[17]. The authors proved that there are natural classes of difference methods for such problems. Theorems on difference inequalities or recurrent inequalities were used in the investigation of the stability of difference schemes.

Almost periodic solutions of hyperbolic systems were considered in [18]

Detailed bibliographical information can be found in [17], [19]. An extensive survey of developments in the area of impulsive partial differential equations was given also in [20], [21].

The classical theory of partial differential inequalities has been developed widely in the monographs [22]–[25]. As it is well known, they found applications in differential problems. The basic examples of such questions are: estimates of solutions of partial equations, estimates of the domain of the existence of solutions, criterion of uniqueness, estimates of the error of approximate solutions. Moreover discrete versions of differential inequalities are frequently used to prove the convergence of approximation methods. The numerical method of lines [25]–[27] and difference methods [28]–[31]

represent classical examples.

Results on parabolic differential and functional differential inequalities with impulses can be found in [32]–[38].

The paper is intended as a self-contained exposition of partial functional differential and difference inequalities with impulses. In the following we describe the topics which are considered in the paper.

Chapter I deals with initial problems for functional differential equations of the Hamilton–Jacobi type on the Haar pyramid. We begin with discussing of functional differential inequalities with impulses and applications. The second part of the Chapter I deals with difference methods for initial problems. The main problem in these investigations is to find a suitable functional difference equation which satisfies a consistency condition with respect to the original problem and it is stable. The method of difference inequalities is used in theorems on the stability. It is important fact in our considerations that the right hand sides of equations satisfy the nonlinear estimates of the Perron type with respect to functional variable.

Functional differential inequalities generated by mixed problems are examined in Chapter II. Uniqueness of solutions and continuous dependence on given functions are consequences of comparison theorems. Discrete versions of functional differential problems are considered also.

The method of lines for partial differential equations consists in replacing spatial derivatives by difference expressions. Then the partial equation is transformed into a system of ordinary differential equations. The numerical method of lines for nonlinear differential problems of parabolic type were examined in [25]–[27], [39], [40]. The method of lines is also treated as a tool for proving of existence theorems for differential problems corresponding to parabolic equations [41]–[43] or first order hyperbolic systems [44]. The method of lines for nonlinear functional differential equations was considered in [45]–[49]. The method for equations of higher orders is studied in [39]. For further bibliography see the references in the papers cited above, especially in papers [47]–[49].

In Chapter II we present a theory of the numerical method of lines for functional differential problems with impulses. The main theorems concerning the numerical method of lines will be based on comparison theorems where a function satisfying some differential difference inequalities with impulses is estimated by a solution of an adequate ordinary functional differential problem with impulses. Next we prove that there are natural classes of difference methods for mixed problems.

In Chapter III we present a theory of parabolic functional differential problems with impulses.

Two types of results on parabolic functional differential inequalities are taken into considerations. The first type allows to estimate a function of several variables satisfying a functional differential inequality by means of an other function of several variables. The second one give estimates of functions of several variables by means of solutions of ordinary functional differential problems.

A number of papers concerned with difference approximations for nonlinear parabolic functional problems were published ([50]–[55]). The method of difference inequalities or simple theorems on recurrent inequalities are used in the investigation of the stability of nonlinear difference functional problems. The authors have been assumed that given functions have partial derivatives with respect to all variables except for (x, y). All the above results deal with equations without impulses.

Our assumption are be more general. We consider nonlinear parabolic functional differential equations with impulses and with initial boundary conditions of the Dirichlet type. We show that there is a general class of difference schemes for such problems. We give sufficient conditions for the convergence of a sequence of approximate solutions under the assumptions that given functions satisfy the nonlinear estimates of the Perron type with respect to the functional variable. The proof of stability is based on a theorem on nonlinear recurrent inequalities for functions of one variable.

Impulsive differential equations with a deviated argument and differential integral equations can be derived from a general model of equation by specializing given functions.

Difference methods for impulsive partial functional differential problems are based on general ideas for finite difference equations which were introduced in [56], [57].

It should be noted that all the problems considered in the paper have the following property: the unknown function is the functional argument in equations. The partial derivatives appear in a classical sense. At the moment there are very few important results for functional differential equations with deviated argument at derivatives ([58]–[61]).

CHAPTER I
INITIAL PROBLEMS ON THE HAAR PYRAMID

1.1. Functional Differential Inequalities

For any two metric spaces X and Y we denote by $C(X,Y)$ the class of all continuous functions from X into Y. We will use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components.

Let $a > 0$, $\tau_0 \in R_+$, $R_+ = [0,+\infty)$, be fixed and suppose that the functions $\alpha,\ \beta : [0,a) \to R^n$, $\alpha = (\alpha_1,\dots,\alpha_n)$, $\beta = (\beta_1,\dots,\beta_n)$, and $\tilde\alpha$, $\tilde\beta : [-\tau_0, 0] \to R^n$, $\tilde\alpha = (\tilde\alpha_1,\dots,\tilde\alpha_n)$, $\tilde\beta = (\tilde\beta_1,\dots,\tilde\beta_n)$ satisfy the conditions:

(i) α and β are of class C^1 on $[0,a)$ and $\alpha(x) < \beta(x)$ for $x \in [0,a)$,

(ii) $\tilde\alpha,\ \tilde\beta \in C([-\tau_0,0],\ R^n)$ and $\tilde\alpha(x) \le \tilde\beta(x)$ for $x \in [-\tau_0,0]$,

(iii) $\tilde\beta(0) = \beta(0) = b$ where $b = (b_1,\dots,b_n)$, $b_i > 0$ for $1 \le i \le n$, and $\tilde\alpha(0) = \alpha(0) = -b$.

Let $E = \{(x,y) \in R^{1+n} :\ x \in (0,a),\ y = (y_1,\dots,y_n),\ y \in [\,\alpha(x),\ \beta(x)]\,\}$,

$$E_0 = \{(x,y) \in R^{1+n} :\ x \in [-\tau_0,0],\ y \in [\,\tilde\alpha(x),\ \tilde\beta(x)\,]\},$$
$$\partial_0 E = \partial E \cap (\,(0,a) \times R^n\,)\,,$$

where ∂E is the boundary of E. We will consider functional differential problem on the set E, whereas E_0 will be an initial set. Suppose that $0 < a_1 < a_2 < \dots < a_k$ are given numbers. Write

$$I = [-\tau_0,0],\quad J = [0,a),\quad J_{\rm imp} = \{\,a_1,\dots,a_k\,\} \tag{1.1}$$

and

$$E_{\rm imp} = \{\,(x,y) \in E :\quad x \in J_{\rm imp}\,\}\,.$$

Let $C_{\rm imp}(E_0 \cup E, R)$ be the class of all functions $z : E_0 \cup E \to R$ such that

(i) the restriction of z to the set $E_0 \cup (E \setminus E_{\rm imp})$ is a continuous function,

(ii) for each $(x,y) \in E_{\rm imp}$ there exist the limits

$$\lim_{(t,s)\to(x,y),\,t<x} z(t,s) = z(x^-,y),\qquad \lim_{(t,s)\to(x,y),\,t>x} z(t,s) = z(x^+,y), \tag{1.2}$$

(iii) $z(x,y) = z(x^+,y)$ for $(x,y) \in E_{\rm imp}$.

For a function $z \in C_{\rm imp}(E_0 \cup E, R)$ and $(x,y) \in E_{\rm imp}$ we put $\Delta z(x,y) = z(x,y) - z(x^-,y)$. Let

$$S_x = [\,\tilde\alpha(x),\ \tilde\beta(x)\,]\ \text{ for }\ x \in [-\tau_0,0],\qquad S_x = [\,\alpha(x),\ \beta(x)\,]\ \text{ for }\ x \in [0,a).$$

Write

$$\Omega = (\,E \setminus E_{\rm imp}\,) \times C_{\rm imp}(E_0 \cup E, R) \times R^n,\quad \Omega_{\rm imp} = E_{\rm imp} \times C_{\rm imp}(E_0 \cup E, R)$$

and suppose that

$$f : \Omega \to R,\ \ g : \Omega_{\rm imp} \to R,\ \ \varphi : E_0 \to R$$

are given functions. In this Chapter we discuss a number of questions referring to the Cauchy problem with impulses

$$D_x z(x,y) = f(\,x,y,z,D_y z(x,y)\,) \;\text{ on }\; E\setminus E_{\rm imp}, \tag{1.3}$$

$$\Delta z(x,y) = g(\,x,y,z\,) \;\text{ on } E_{\rm imp}, \quad z(x,y) = \varphi(x,y) \text{ on } E_0, \tag{1.4}$$

where $D_y z = (D_{y_1} z, \ldots, D_{y_n} z)$.

We consider classical solutions of the above problem. A function $z : E_0 \cup E \to R$ is a solution of (1.3), (1.4) if $z \in C_{\rm imp}(E_0 \cup E, R)$, there exist the derivatives $D_x z$, $D_y z$ on $E \setminus E_{\rm imp}$ and z satisfies (1.3), (1.4).

A function $z \in C_{\rm imp}(E_0 \cup E, R)$ will be called a function of class D if z has partial derivatives D_x, $D_y z$ on $E\setminus E_{\rm imp}$ and there is the total derivative of z on $(\,\partial E \setminus E_{\rm imp}\,) \cap (\,(0,a)\times R^n\,)$. We will consider solutions of (1.3), (1.4) or solutions of functional differential inequalities generated by (1.3), (1.4) which are of class D on $E_0 \cup E$.

Let $C_{\rm imp}(I \cup J, R)$ be the class of all functions $w : I \cup J \to R$ such that $w \in C(\,(I\cup J)\setminus J_{\rm imp}, R\,)$ and for each $x \in J_{\rm imp}$ there exists the limits

$$\lim_{t\to x,\, t<x} w(t) = w(x^-), \qquad \lim_{t\to x,\, t>x} w(t) = w(x^+).$$

We assume also that $w(x) = w(x^+)$ for $x \in J_{\rm imp}$.

Two different types of results on functional differential inequalities will be considered in this Chapter. The first type allows to estimate a function of several variables by means of an other function of several variables, while the second one, the so called comparison theorems, give estimates for functions of several variables satisfying functional differential inequalities with impulses, by means of functions of one variable which are solutions of adequate initial problems with impulses. Let

$$E_x = (\,E_0 \cup E\,) \cap (\,[\,-\tau_0, x\,] \times R^n\,)\,, \quad 0 \le x < a.$$

For every $x \in [0,a)$ and $z \in C_{\rm imp}(E_0 \cup E, R)$ we write

$$\|z\|_x = \sup\{\,|z(t,s)| :\; (t,s) \in E_x\,\}$$

and

$$\|z\|_{x^-} = \sup\{\,|z(t,s)| :\;\; (t,s) \in (\,E_0 \cup E\,) \cap (\,[-\tau_0, x) \times R^n\,)\,\}.$$

If $w \in C_{\rm imp}(I \cup J, R)$ then we write also

$$\|w\|_x = \sup\{\,|w(t)| :\; -\tau_0 \le t \le x\,\} \;\text{ and }\; \|z\|_{x^-} = \sup\{\,|w(t)| :\; -\tau_0 \le t < x\,\}.$$

The function f is called to satisfy the Volterra condition if for each $(x,y) \in E \setminus E_{\rm imp}$ there is a set $E[x,y]$ such that

(i) $E[x,y] \subset E_x$,

(ii) if $z, \bar z \in C_{\rm imp}(E_0 \cup E, R)$ and $z(t,s) = \bar z(t,s)$ for $(t,s) \in E[x,y]$ then $f(x,y,z,q) = f(x,y,\bar z,q)$, $q \in R^n$.

Note that the Volterra condition means that the value of f at the point (x,y,z,q) depends on (x,y,q) and on the restriction of z to the set $E[x,y]$.

The function g is called to satisfy the condition $V^{(-)}$ if for each $(x,y) \in E_{\rm imp}$ there is a set $H[x,y]$ such that

(i) $H[x,y] \subset (\,E_0 \cup E\,)\,, \cap\, (\,[-\tau_0, x) \times R^n\,)$

(ii) if $z,\ \bar{z} \in C_{\rm imp}(E_0 \cup E, R)$ and $z(t,s) = \bar{z}(t,s)$ for $(t,s) \in H[x,y]$ then $g(x,y,z) = g(x,y,\bar{z})$.

We consider differential inequalities generated by (1.3), (1.4).

We define functions $I_0,\ I_+,\ I_-\ : E \to \{1,\dots,n\}$ as follows. For each $(x,y) \in E$ there exist sets (possibly empty) of integers $I_0[x,y],\ I_+[x,y],\ I_-[x,y]$ such that

$$I_+[x,y] \cap I_-[x,y] = \emptyset, \quad I_0[x,y] \cup I_+[x,y] \cup I_-[x,y] = \{1,\dots,n\},$$

and

$$y_i = \alpha_i(x) \quad \text{for} \quad i \in I_-[x,y], \qquad y_i = \beta_i(x) \quad \text{for} \quad i \in I_+[x,y],$$
$$\alpha_i(x) < y_i < \beta_i(x) \quad \text{for} \quad i \in I_0[x,y].$$

Let F and G be the Niemycki operators generated by the problem i. e.

$$F[\,z\,](x,y) = f(\,x,y,z,D_y z(x,y)\,) \quad \text{on} \quad E \setminus E_{\rm imp},$$
$$G[\,z\,](x,y) = g(\,x,y,z\,) \quad \text{on} \quad E_{\rm imp}.$$

Assumption H $[\,f,g\,]$**.** Suppose that

1) $f : \Omega \to R$ satisfies the Volterra condition and for $(x,y,z,q) \in \Omega,\ \bar{q} \in R^n$ we have

$$f(\,x,y,z,q\,) - f(\,x,y,z,\bar{q}\,) + \sum_{i \in I_-[x,y]} \alpha_i'(x)\,(q_i - \bar{q}_i) + \sum_{i \in I_+[x,y]} \beta_i'(x)\,(q_i - \bar{q}_i) \le 0$$

where $q = (q_1,\dots,q_n),\ \bar{q} = (\bar{q}_1,\dots,\bar{q}_n)$, and $q_i \le \bar{q}_i$ for $i \in I_-[x,y],\ q_i \ge 0$ for $i \in I_+[x,y]$, $q_i = \bar{q}_i$ for $i \in I_0[x,y]$,

2) the function $g : \Omega_{\rm imp} \to R$ satisfies the condition $V^{(-)}$ and for fixed $(x,y) \in E_{\rm imp}$ the function $g(\,x,y,\,\cdot\,)$ is nondecreasing on $C_{\rm imp}(E_0 \cup E, R)$,

3) the function f satisfies the following monotonicity condition: if $(x,y) \in E,\ z,\bar{z} \in C_{\rm imp}(E_0 \cup E, R)$, $z(t,s) \le \bar{z}(t,s)$ on E_x and $z(x,y) = \bar{z}(x,y)$ then

$$f(x,y,z,q) \le f(x,y,\bar{z},q) \quad \text{for} \quad q \in R^n.$$

Remark 1.1. Suppose that the function

$$\bar{f} : (E \setminus E_{\rm imp}) \times R \times C_{\rm imp}(E_0 \cup E, R) \times R^n \to R$$

is nondecreasing with respect to the functional variable and

$$f(x,y,z,q) = \bar{f}(x,y,z(x,y),z,q).$$

Then f satisfies the monotonicity condition 3).

Theorem 1.2. *Suppose that Assumption H $[\,f,g\,]$ is satisfied and*
1) *the functions $u,\,v\in C_{\rm imp}(E_0\cup E,R)$ are of class D and*

$$u(x,y)\le v(x,y)\ \ on\ \ E_0,\ \ u(0,y)<v(0,y)\ \ for\ \ y\in[-b,b],$$

2) *denoted*

$$T_+=\{\,(x,y)\in E:\\ u(t,s)<v(t,s)\ for\ \ (t,s)\in E\cap([0,x)\times R^n)\ \ and\ u(x,y)=v(x,y)\,\}\,,$$

we assume that

$$D_xu(x,y)-F[\,u\,](x,y)<D_xv(x,y)-F[\,v\,](x,y)\ \ on\ \ T_+\setminus E_{\rm imp},$$

and

$$\Delta u(x,y)-G[\,u\,](x,y)<\Delta v(x,y)-G[\,v\,](x,y)\ \ \ on\ \ T_+\cap E_{\rm imp}.$$

Then

$$u(x,y)<v(x,y)\ \ on\ \ E. \tag{1.5}$$

Proof. Suppose that assertion (1.5) is false. Then the set

$$J_+=\{\,x\in[0,a):\ \ u(x,y)\ge v(x,y)\ \ \ \text{for some}\ \ y\in[\,\alpha(x),\ \beta(x)\,]\,\}$$

is not empty. Defining $\bar x=\inf\ J_+$ it follows that $\bar x>0$ and that there exists $\bar y\in[\,\alpha(\bar x),\ \beta(\bar x)\,]$ such that $(\bar x,\bar y)\in T_+$. There are two cases to be distinguished.

(i) Suppose that $(\bar x,\bar y)\in E\setminus E_{\rm imp}$. Then $D_{y_i}(u-v)(\bar x,\bar y)\ge 0$ for $i\in I_+[\bar x,\bar y]$, $D_{y_i}(u-v)(\bar x,\bar y)\le 0$ for $i\in I_-[\bar x,\bar y]$, and $D_{y_i}(u-v)(\bar x,\bar y)=0$ for $i\in I_0[\bar x,\bar y]$. For $x\in[0,\bar x]$ we put $\eta(x)=(\,\eta_1(x),\dots,\eta_n(x)\,)$ where

$$\eta_i(x)=\alpha_i(x)\quad\text{for}\ \ i\in I_-[\bar x,\bar y],\qquad \eta_i(x)=\beta_i(x)\quad\text{for}\ \ i\in I_+[\bar x,\bar y], \tag{1.6}$$

$$\eta_i(x)=\bar y_i\qquad\text{for}\ \ i\in I_0[\bar x,\bar y]. \tag{1.7}$$

We consider now the composite function $\gamma(x)=(u-v)(x,\eta(x)),\ \ x\in[0,\bar x]$. It attains maximum at $\bar x$. Since $u-v$ is of class D on $E_0\cup E$ then we have

$$D_x(u-v)(\bar x,\bar y)+\sum_{i\in I_-[\bar x,\bar y]}\alpha_i'(\bar x)\,D_{y_i}(u-v)(\bar x,\bar y)+\\ +\sum_{i\in I_+[\bar x,\bar y]}D_{y_i}(u-v)(\bar x,\bar y)\ge 0. \tag{1.8}$$

From Assumption H $[\,f,g\,]$ and condition 2) it follows that

$$D_x(u-v)(\bar x,\bar y)<-\sum_{i\in I_-[\bar x,\bar y]}\alpha_i'(\bar x)\,D_{y_i}(u-v)(\bar x,\bar y)-\sum_{i\in I_+[\bar x,\bar y]}D_{y_i}(u-v)(\bar x,\bar y)$$

which contradicts (1.8).

(ii) Suppose now that $(\bar{x}, \bar{y}) \in E_{\text{imp}}$. Then

$$u(\bar{x}^-, \bar{y}) \leq v(\bar{x}^-, \bar{y}) \tag{1.9}$$

and

$$u(t,s) \leq v(t,s) \quad \text{for} \ \ (t,s) \in (\, E_0 \cup E\,) \cap (\, [-\tau_0, \bar{x}) \times R^n\,).$$

It follows from Assumption H $[\, f, g\,]$ and condition 2) that

$$(u - v)(\bar{x}, \bar{y}) < u(\bar{x}^-, \bar{y}) + g(\, \bar{x}, \bar{y}, u\,) - v(\bar{x}^-, \bar{y}) - g(\, \bar{x}, \bar{y}, v\,) \leq 0$$

which contradicts the condition $(\bar{x}, \bar{y}) \in T_+$.

Hence J_+ is empty and the statement (1.5) follows. □

Remark 1.3. Suppose that $\bar{g} : E_{\text{imp}} \times R \times C_{\text{imp}}(E_0 \cup E, R) \to R$ is a given function and g is defined by $g(x, y, z) = \bar{g}(x, y, z(x^-, y), z)$. Then condition 2) of Assumption H$[\, f, g\,]$ can be replaced by the following one:

2') the function $\bar{g}$ satisfies condition $V^{(-)}$, it is nondecreasing with respect to the functional variable and for fixed $(x, y, z) \in E_{\text{imp}} \times C_{\text{imp}}(E_0 \cup E, R)$ the function $\gamma(p) = p + \bar{g}(x, y, p, z)$ is nondecreasing on R.

Now we consider weak functional differential inequalities.

Assumption H $[\,\sigma, \sigma_0\,]$**.** Assume that

1) the functions $\sigma : (\, J \setminus J_{\text{imp}}\,) \times R_+ \to R_+$ and $\sigma_0 : J_{\text{imp}} \times R_+ \to R_+$ are continuous and $\sigma(\, x, 0\,) = 0$ for $x \in J \setminus J_{\text{imp}}$, $\sigma_0(\, x, 0\,) = 0$ for $x \in J_{\text{imp}}$,

2) the functions $\sigma(x, \cdot\,)$ and $\sigma_0(x, \cdot\,)$ are nondecreasing and the right hand maximum solution of the problem with impulses

$$\omega'(x) = \sigma(\, x, \omega(x)\,) \quad \text{on} \ \ J \setminus J_{\text{imp}},$$
$$\Delta\omega(x) = \sigma_0(\, x, \omega(x^-)\,), \ \ \text{on} \ J_{\text{imp}}, \quad \omega(0) = 0,$$

is $\bar{\omega}(x) = 0$ on J,

3) the estimate

$$f(\, x, y, z, q\,) - f(\, x, y, \bar{z}, q\,) \geq -\sigma(\, x, \|\, \bar{z} - z\,\|_x\,)$$

is satisfied on Ω for $z \leq \bar{z}$,

4) the inequality

$$g(\, x, y, z\,) - g(\, x, y, \bar{z}\,) \geq -\sigma_0(\, x, \|\bar{z} - z\|_{x^-}\,)$$

is satisfied on Ω_{imp} for $z \leq \bar{z}$.

Theorem 1.4. *Suppose that Assumptions H* $[\, f, g\,]$ *and H* $[\,\sigma, \sigma_0\,]$ *are satisfied and*

1) *the functions* u, $v \in C_{\text{imp}}(E_0 \cup E, R)$ *are of class D and* $u(x, y) \leq v(x, y)$ *on* E_0,

2) *the functional differential inequality*

$$D_x u(x, y) - F[\, u\,](x, y) \leq D_x v(x, y) - F[\, v\,](x, y) \ \ \text{on} \ \ E \setminus E_{\text{imp}}$$

and the inequality for impulses

$$\Delta u(x,y) - G[\,u\,](x,y) \le \Delta v(x,y) - G[\,v\,](x,y) \;\; on \;\; E_{\rm imp}$$

are satisfied.

Then $u(x,y) \le v(x,y)$ *on* E.

Proof. Let $\tilde a \in (a_k, a)$ be fixed. Consider the problem

$$\omega'(x) = \sigma(\,x, \omega(x)\,) + \varepsilon_1 \quad \text{on} \;\; [0,\tilde a) \setminus J_{\rm imp},$$
$$\Delta\omega(x) = \sigma_0(\,x, \omega(x^-)\,) + \varepsilon_2 \quad \text{on} \;\; J_{\rm imp}, \qquad \omega(0) = \varepsilon_0.$$

There exists $\bar\varepsilon > 0$ such that for $0 < \varepsilon_i < \bar\varepsilon$, $i = 0,1,2$, there is the maximum solution $\omega(\,\cdot\,,\varepsilon\,) : [0,\tilde a) \to R_+$, $\varepsilon = (\varepsilon_0, \varepsilon_1, \varepsilon_2)$, of the above problem.

We prove that

$$u(x,y) \le v(x,y) \quad \text{on} \;\; E \cap (\,(0,\tilde a) \times R^n\,)\,. \tag{1.10}$$

Let

$$\tilde v(x,y) = v(x,y) + \varepsilon_0 \quad \text{on} \;\; E_0 \quad \text{and}$$
$$\tilde v(x,y) = v(x,y) + \omega(x,\varepsilon) \quad \text{on} \;\; E \cap (\,[0,\tilde a) \times R^n\,)\,.$$

Then $\tilde v$ is of class D on $(\,E_0 \cup E\,) \cap (\,[-\tau_0, \tilde a) \times R^n\,)$ and $(u - v)(x,y) < 0$ on E_0.

Direct calculations give

$$D_x u(x,y) - F[\,u\,](x,y) < D_x \tilde v(x,y) - F[\,\tilde v\,](x,y) \quad \text{on}$$
$$(\,E \setminus E_{\rm imp}\,) \cap (\,(0,\tilde a) \times R^n\,)$$

and

$$\Delta u(x,y) - G[\,u\,](x,y) < \Delta \tilde v(x,y) - G[\,\tilde v\,](x,y) \quad \text{on} \;\; E_{\rm imp}.$$

It follows from Theorem 1.2 that $u(x,y) < \tilde v(x,y)$ on $E \cap (\,(0,\tilde a) \times R^n\,)$. On the other hand $\lim_{\varepsilon \to 0} \omega(x,\varepsilon) = 0$ uniformly on $[0,\tilde a)$ which leads to (1.10). Finally, the constant $\tilde a \in (a_k, a)$ is arbitrary and therefore the proof of Theorem 1.4 is completed. □

Assumption H $[\,\tilde\sigma, \tilde\sigma_0\,]$**.** Suppose that

1) the functions $\tilde\sigma : (\,J \setminus J_{\rm imp}\,) \times R_- \to R_+$ and $\tilde\sigma : J_{\rm imp} \times R_- \to R_+$, $R_- = (-\infty, 0\,]$, are continuous, $\tilde\sigma(x,0) = 0$ on $J \setminus J_{\rm imp}$, $\tilde\sigma(x,0) = 0$ on $J_{\rm imp}$,

2) the function $f : \Omega \to R$ satisfies the condition: if $(x,y) \in E$, $q \in R^n$, $z, \bar z \in C_{\rm imp}(E_0 \cup E, R)$, $z(t,s) \le \bar z(t,s)$ on E_x and $z(x,y) < \bar z(x,y)$ then

$$f(x,y,z,q) - f(x,y,\bar z,q) \le \tilde\sigma(\,x, z(x,y) - \bar z(x,y)\,),$$

3) if $(x,y) \in E_{\rm imp}$, $z, \bar z \in C_{\rm imp}(E_0 \cup E, R)$, $z(t,s) \le \bar z(t,s)$ on E_x and $z(x^-,y) < \bar z(x^-,y)$ then

$$g(x,y,z,) - g(x,y,\bar z,) \le \tilde\sigma_0(\,x, z(x^-,y) - \bar z(x^-,y)\,),$$

4) the left hand minimum solution of the equation with impulses

$$\omega'(x) = \tilde{\sigma}(\,x, \omega(x)\,) \quad \text{on } J \setminus J_{\text{imp}}, \quad \Delta\omega(x) = \tilde{\sigma}_0(x, \omega(x^-)\,) \quad \text{on } J_{\text{imp}},$$

satisfying the condition $\lim_{x \to a^-} \omega(x) = 0$ is $\bar{\omega}(x) = 0,\ x \in J$.

Theorem 1.5. *Suppose that Assumptions $H\,[\,f, g\,]$, $H\,[\,\tilde{\sigma}, \tilde{\sigma}_0\,]$ are satisfied and*

1) *the functions $u,\ v \in C_{\text{imp}}(E_0 \cup E, R)$ are of class D and $u(x,y) < v(x,y)$ on E_0,*

2) *for $(x,y) \in E \setminus E_{\text{imp}}$ we have*

$$D_x u(x,y) - F[\,u\,](x,y) \le D_x v(x,y) - F[\,v\,](x,y)$$

and

$$\Delta u(x,y) - G[\,u\,](x,y) \le \Delta v(x,y) - G[\,v\,](x,y) \ \ for \ \ (x,y) \in E_{\text{imp}}.$$

Then

$$u(x,y) < v(x,y) \ \ on \ \ E. \tag{1.11}$$

Proof. First we prove inequality (1.11) on $E \cap (\,[0, a - \varepsilon) \times R^n\,)$ where $0 < \varepsilon < a - a_k$.

Let

$$0 < p_0 < \min\{\, v(x,y) - u(x,y) : \ (x,y) \in E_0\,\}\,.$$

For $\delta > 0$ denote by $\omega(\,\cdot\,, \delta)$ the right hand minimum solution of the problem

$$\omega'(x) = -\tilde{\sigma}(\,x, -\omega(x)\,) - \delta, \ \ x \in [0, a - \varepsilon) \setminus J_{\text{imp}},$$
$$\Delta\omega(x) = -\tilde{\sigma}_0(\,x, -\omega(x^-)\,), \ \ x \in J_{\text{imp}}, \quad \omega(0) = p_0.$$

If $p_0 > 0$ is fixed then to every $\varepsilon > 0$ there corresponds $\delta_0 > 0$ such that for $0 < \delta < \delta_0$ the solution $\omega(\,\cdot\,, \delta$ of the above problem exists and it is positive on $[0, a - \varepsilon)$. Suppose that $\delta > 0$ is such a constant that $\omega(\,\cdot\,, \delta$ satisfies the above conditions. Let $\tilde{u}(x,y) = u(x,y) + p_0$ on E_0 and $\tilde{u}(x,y) = u(x,y) + \omega(x, \delta)$ on $E \cap (\,[0, a - \varepsilon) \times R^n\,)$. We will prove that

$$\tilde{u}(x,y) < v(x,y) \quad \text{on } \ E \cap (\,[0, a - \varepsilon) \times R^n\,)\,. \tag{1.12}$$

It follows from Assumption H $[\,\tilde{\sigma}, \tilde{\sigma}_0\ \,]$ that

$$D_x \tilde{u}(x,y) - F[\,\tilde{u}\,](x,y) < D_x v(x,y) - F[\,v\,](x,y)$$

on $(\,E \setminus E_{\text{imp}}\,) \cap (\,[0, a - \varepsilon) \times R^n\,)$ and

$$\Delta\tilde{u}(x,y) - G[\,\tilde{u}\,](x,y) < \Delta v(x,y) - G[\,v\,](x,y)$$

on E_{imp}. Since $\tilde{u}(x,y) < v(x,y)$ on E_0, we have the estimate (1.12) from Theorem 1.2.

Since $0 < \varepsilon < a - a_k$ is arbitrary, inequality (1.11) holds true on E. This completes the proof of Theorem 1.5. □

Remark 1.6. If Assumptions H $[f, g]$, H $[\sigma, \sigma_0]$ are satisfied then the Cauchy problem (1.3), (1.4) admits at most one solution $u : E_0 \cup E \to R$ of class D.

1.2. Comparison Results for Impulsive Problems

We will prove a theorem on estimates of functions satisfying functional differential inequalities with impulses by means of solutions of ordinary equations with impulses. We will use functional differential equations as comparison problems.

For $z \in C_{\text{imp}}(E_0 \cup E, R)$ we define the function $Vz : [-\tau_0, a) \to R_+$ by

$$(Vz)(x) = \max\{\,|z(x,y)| : \; y \in S_x\,\}.$$

Lemma 1.7. *If $z \in C_{\text{imp}}(E_0 \cup E, R)$ then $Vz \in C_{\text{imp}}(I \cup J, R)$.*

We omit the proof of the above lemma.

Assumption H $[\varrho, \varrho_0]$. Suppose that

1) the function $\varrho : (J \setminus J_{\text{imp}}) \times C_{\text{imp}}(I \cup J, R_+) \to R_+$ is continuous and satisfies the Volterra condition,

2) the function $\varrho_0 : J_{\text{imp}} \times C_{\text{imp}}(I \cup J, R_+) \to R_+$ is continuous and satisfies the following condition $V^{(-)}$: if $w, \; \bar{w} \in C_{\text{imp}}(I \cup J, R_+)$ and $w(t) = \bar{w}(t)$ on $[-\tau_0, x)$ then $\varrho_0(x, w) = \varrho_0(x, \bar{w})$,

3) the functions ϱ and ϱ_0 are nondecreasing with respect to the functional variables and for each function $\eta \in C(I, R_+)$ there exists the maximum solution $\omega(\,\cdot\,, \eta) : [-\tau_0, a) \to R_+$ of the Cauchy problem

$$\omega'(x) = \varrho(x, \omega) \quad \text{on } J \setminus J_{\text{imp}}, \tag{1.13}$$

$$\Delta\omega(x) = \varrho(x, \omega) \quad \text{on } J_{\text{imp}}, \quad \omega(x) = \eta(x) \quad \text{for } x \in I. \tag{1.14}$$

In the sequel we will use the following lemma on functional differential inequalities.

Lemma 1.8. *Suppose that Assumption H $[\varrho, \varrho_0]$ is satisfied and*

1) *the function $\varphi \in C_{\text{imp}}(I \cup J, R_+)$ satisfies the initial estimate $\phi(x) \le \eta(x)$ on I with $\eta \in C(I, R_+)$, $\omega(\,\cdot\,, \eta)$ is the maximum solution of* (1.13), (1.14),

2) *denoted*

$$T_+ = \{\, x \in J \setminus J_{\text{imp}} : \; \phi(x) > \omega(x, \eta)\,\}$$

we assume that

$$D_-\phi(x) \le \varrho(x, \phi), \quad \textit{for } \; x \in T_+ \setminus J_{\text{imp}},$$

and

$$\Delta\phi(x) \le \varrho_0(x, \phi) \;\; \textit{for } \; T_+ \cap J_{\text{imp}}.$$

Then $\phi(x) \le \omega(x, \eta)$ on J.

We omit the proof of Lemma 1.8.

Assumption H $[G]$**.** Suppose that the function

$$G : (E \setminus E_{\text{imp}}) \times C_{\text{imp}}(I \cup J, R_+) \times R_+^n \to R_+$$

is continuous and satisfies the Volterra condition,

2) the estimate

$$G(x,y,w,q) - \sum_{i \in I_-[x,y]} \alpha_i'(x)\, q_i + \sum_{i \in I_+[x,y]} \beta_i'(x)\, q_i \le \varrho(x,w)$$

is satisfied on $(E \setminus E_{\text{imp}}) \times C_{\text{imp}}(I \cup J, R_+) \times R_+^n$ with $q_i = 0$ for $i \in I_0[x,y]$.

Remark 1.9. If E is the Haar pyramid

$$E = \{ (x,y) : \ \ x \in (0,a), \ \ y \in [-b + Mx, \, b - Mx \} , \tag{1.15}$$

where $b = (b_1, \ldots, b_n)$, $b_i > 0$ for $1 \le i \le n$, $M = (M_1, \ldots, M_n) \in R_+^n$, $b - Ma > 0$, and

$$G(x,y,w,q) = \varrho(x,w) + \sum_{i=1}^{n} M_i\, q_i$$

then condition 2) of Assumption H $[G]$ is satisfied.

Theorem 1.10. *Suppose that Assumptions H* $[\varrho, \varrho_0]$*, H* $[G]$ *are satisfied and*

1) *the function* $u \in C_{\text{imp}}(E_0 \cup E, R)$ *is of class D and* $|u(x,y)| \le \eta(x)$ *on* E_0 *where* $\eta \in C(I, R_+)$*,*

2) *the differential inequality with impulses is satisfied*

$$|D_x u(x,y)| \le G(x, y, Vu, [|D_y u(x,y)|]) \ \ \text{on} \ \ E \setminus E_{\text{imp}},$$
$$\Delta |u(x,y)| \le \varrho_0(x, Vu) \ \ \text{on} \ \ E_{\text{imp}},$$

where

$$[|D_y u(x,y)|] = (|D_{y_1} u(x,y)|, \ldots, |D_{y_n} u(x,y)|).$$

Then $|u(x,y)| \le \omega(x,\eta)$ *on* E *where* $\omega(\cdot, \eta$ *is the maximum solution of* (1.13)*,* (1.14)*.*

Proof. Let $\phi = Vu$ and $\bar{x} \in J \setminus J_{\text{imp}}$ is such a point that $\phi(\bar{x}) > \omega(\bar{x}, \eta)$.

We prove that

$$D_- \varphi(\bar{x}) \le \varrho(\bar{x}, \varphi). \tag{1.16}$$

There is $\bar{y} \in S_{\bar{x}}$ such that (i) $\varphi(\bar{x}) = u(\bar{x}, \bar{y})$ or (ii) $\varphi(\bar{x}) = -u(\bar{x}, \bar{y})$. Let us consider the case (i). We have $D_{y_i} u(\bar{x}, \bar{y}) \ge 0$ for $\in I_+[\bar{x}, \bar{y}]$, $D_{y_i} u(\bar{x}, \bar{y}) \le 0$ for $i \in I_-[\bar{x}, \bar{y}]$, and $D_{y_i} u(\bar{x}, \bar{y}) = 0$ for $i \in I_0[\bar{x}, \bar{y}]$. Let $\eta = (\eta_1, \ldots, \eta_n) : [0, \bar{x}] \to R^n$ be given by (1.6), (1.7) and $\gamma(x) = u(x, \eta(x))$ for $x \in [0, \bar{x}]$.

We have $\gamma(x) \le \varphi(x)$ for $x \in [0,\bar{x}]$ and $\gamma(\bar{x}) = \varphi(\bar{x})$. It follows that $D_-\varphi(\bar{x}) \le \gamma'(\bar{x})$. The above inequality and assumption 2) imply

$$D_-\varphi(\bar{x}) \le D_x u(\bar{x},\bar{y}) + \sum_{i=1}^{n} D_{y_i} u(\bar{x},\bar{y})\, \eta_i'(\bar{x}) \le G(\,\bar{x},\bar{y}, Vu, [|D_y u(\bar{x},\bar{y})|]\,)$$
$$+ \sum_{i\in I_-[\bar{x},\bar{y}]} \alpha_i'(\bar{x}) D_{y_i} u(\bar{x},\bar{y}) + \sum_{i\in I_+[\bar{x},\bar{y}]} \beta_i'(\bar{x}) D_{y_i} u(\bar{x},\bar{y}) \le \sigma(\bar{x},\varphi)$$

which proves (1.16). If the case (ii) is satisfied then (1.16) can be proved in an analogous way. If $\bar{x} \in J_{\text{imp}}$ then we have

$$\Delta\phi(\bar{x}) \le \varrho_0(\,\bar{x},\phi\,).$$

Then ϕ satisfies all the assumptions of Lemma 1.8 and the assertion follows. □

In the case when E is the Haar pyramid (1.15) and

$$E_0 = \{\,(x,y):\ x \in [-\tau_0, 0],\ \ y \in [-b,b]\,\} \tag{1.17}$$

we have the following results.

Theorem 1.11. *Suppose that Assumption $H\,[\,\varrho,\varrho_0\,]$ is satisfied and*
1) *the function $u \in C_{\text{imp}}(E_0 \cup E, R)$ is of class D and $|\,u(x,y)\,| \le \eta(x)$ on E_0 with $\eta \in C(I, R_+)$,*
2) *the differential inequality with impulses is satisfied*

$$|\,D_x u(x,y)\,| \le \varrho(\,x, Vu\,) + \sum_{i=1}^{n} M_i\,|D_{y_i} u(x,y)\,| \ \text{ on }\ E \setminus E_{\text{imp}},$$
$$\Delta|\,u(x,y)\,| \le \varrho_0(\,x, Vu\,) \ \text{ on }\ E_{\text{imp}}.$$

Under these assumptions we have $|\,u(x,y)\,| \le \omega(x,\eta)$ on E.

Now we consider problem (1.3), (1.4) on the Haar pyramid (1.15) with the initial set (1.17). We start with the theorem on the estimate of solutions of equations with impulses.

Theorem 1.12. *Suppose that Assumption $H\,[\,\varrho,\varrho_0\,]$ is satisfied and*
1) *the function $f:\Omega \to R$ satisfies the Volterra condition and*

$$|\,f(x,y,w,q)\,| \le \varrho(x, Vz) + \sum_{i=1}^{n} M_i\,|\,q_i\,| \ \text{ on }\ \Omega,$$

2) *the function $g:\Omega_{\text{imp}} \to R$ satisfies the condition $V^{(-)}$ and*

$$|\,g(x,y,z)\,| \le \varrho_0(\,x, Vz\,) \ \text{ on }\ \Omega_{\text{imp}},$$

3) *$\varphi \in C(E_0,R)$ and $|\,\varphi(x,y)\,| \le \eta(x)$ on E_0 with $\eta \in C(I,R_+)$,*
4) *the function $u \in C_{\text{imp}}(E_0 \cup E, R)$ is the solution of problem* (1.3), (1.4) *and u is of class D.*

Under these assumptions we have $|\, u(x,y)\,| \le \omega(x,\eta)$ *on* E.

Proof. The function u satisfies all the assumptions of Theorem 1.11 and the statement follows. □

Let us consider now two problems: problem (1.3), (1.4) and the following one

$$D_x z(x,y) = \tilde{f}(\,x,y,z,D_y z(x,y)\,) \quad \text{on} \;\; E \setminus E_{\rm imp}, \tag{1.18}$$

$$\Delta z(x,y) = \tilde{g}(\,x,y,z\,) \;\; \text{on}\; E_{\rm imp}, \quad z(x,y) = \tilde{\varphi}(x,y) \;\; \text{on}\; E_0, \tag{1.19}$$

where $\tilde{f} : \Omega \to R$, $\tilde{g} : \Omega_{\rm imp} \to R$, $\tilde{\varphi} : E_0 \to R$. We give an estimate of the difference for solutions of problems (1.3), (1.4) and (1.18), (1.19).

Theorem 1.13. *Suppose that Assumption* $H\,[\,\varrho,\varrho_0\,]$ *is satisfied and*

1) *the function* f, $\tilde{f}$ *and* g, $\tilde{g}$ *satisfy the Volterra condition and the condition* $V^{(-)}$ *respectively,*

2) *the estimates*

$$\Big|\, f(x,y,z,q) - \tilde{f}(x,y,\bar{z},\bar{q}) \,\Big| \le \varrho(\,x, V(z-\bar{z})\,) + \sum_{i=1}^{n} M_i \,|\, q_i - \bar{q}_i \,| \;\; \text{on} \;\; \Omega,$$

and

$$|\, g(\,x,y,z\,) - \tilde{g}(\,x,y,\bar{z}\,)\,| \le \varrho_0(\,x, V(z-\bar{z}\,)\,) \;\; \text{on} \;\; \Omega_{\rm imp}$$

are satisfied,

3) $\eta \in C(I,R_+)$ *and* $|\,(\varphi - \tilde{\varphi})(x,y)\,| \le \eta(x)$ *on* E_0.

4) u, $\tilde{u} \in C_{\rm imp}C(E_0 \cup E, R)$ *are solutions of* (1.3), (1.4) *and* (1.18), (1.19) *respectively, the functions* u, $\tilde{u}$ *are of class* D.

Under these assumptions we have $|\,(u-\tilde{u})(x,y)\,| \le \omega(x,\eta)$ *on* E.

Proof. The function $u - \tilde{u}$ satisfies all the assumptions of Theorem 1.11 and the statement follows. □

As a consequence of the above theorem we get the following uniqueness criterion.

Theorem 1.14. *Suppose that Assumption* $H\,[\,\varrho,\varrho_0\,]$ *is satisfied and*

1) *the functions* f *and* g *satisfy the Volterra condition and the condition* $V^{(-)}$ *respectively,*

2) *the estimates*

$$|\, f(x,y,z,q) - f(x,y,\bar{z},\bar{q}) \,| \le \varrho(\,x, V(z-\bar{z})\,) + \sum_{i=1}^{n} M_i \,|\, q_i - \bar{q}_i \,| \;\; \text{on} \;\; \Omega,$$

and

$$|\, g(\,x,y,z\,) - g(\,x,y,\bar{z}\,)\,| \le \varrho_0(\,x, V(z-\bar{z}\,)\,) \;\; \text{on} \;\; \Omega_{\rm imp}$$

are satisfied,

3) *the maximum solution of problem* (1.13), (1.14) *with* $\eta = 0$ *is* $\omega(\,\cdot\,,0) = 0$ *on* J.

Then the Cauchy problem (1.3), (1.4) *admits at most one solution of the class* D *on* $E_0 \cup E$.

Theorem 1.14 follows from Theorem 1.13 for $\tilde{f} = f,\ \tilde{g} = g$.

Remark 1.15. Suppose that the functions

$$h : (J \setminus J_{\text{imp}}) \times R_+ \to R_+ \ \text{ and } \ h_0 : J_{\text{imp}} \times R_+ \to R_+$$

are continuous and are nondecreasing with respect to the second variable. Put

$$\varrho(x, w) = h(x, \|w\|_x) \ \text{ and } \ \varrho_0(x, w) = h_0(x, \|w\|_{x^-}).$$

Then

(i) assumption 2) of Theorem 1.11 has the form

$$|D_x u(x,y)| \le h(x, \|u\|_x) + \sum_{i=1}^{n} M_i \, |D_{y_i} u(x,y)| \ \text{ on } \ E \setminus E_{\text{imp}},$$

and

$$\Delta |u(x,y)| \le h_0(x, \|u\|_{x^-}) \ \text{ on } \ E_{\text{imp}},$$

(ii) assumption 2) of Theorem 1.14 has the form

$$|f(x,y,z,q) - f(x,y,\bar{z},\bar{q}| \le h(x, \|z - \bar{z}\|_x) + \sum_{i=1}^{n} M_i \, |q_i - \bar{q}_i| \ \text{ on } \ \Omega,$$

and

$$|g(x,y,z) - g(x,y,\bar{z})| \le h_0(x, \|z - \bar{z}\|_{x^-}) \ \text{ on } \ \Omega_{\text{imp}},$$

(iii) the comparison problem (1.13), (1.14) is the ordinary Cauchy problem with impulses

$$\omega'(x) = h(x, \omega(x)), \quad x \in J \setminus J_{\text{imp}},$$
$$\Delta\omega(x) = h_0(x, \omega(x^-)), \quad x \in J_{\text{imp}}, \quad \omega(0) = \eta,$$

where $\eta \in R_+$.

1.3. Difference Equations for Initial Problems

We denote by $F(X, Y)$ the class of all functions defined on X and taking values in Y, where X and Y are arbitrary sets. For E and E_0 defined by (1.15) and (1.17) we put

$$\Omega = (E \setminus E_{\text{imp}}) \times C_{\text{imp}}(E_0 \cup E, R) \times R^n, \quad \Omega_{\text{imp}} = E_{\text{imp}} \times C_{\text{imp}}(E_0 \cup E, R),$$

and suppose that $f : \Omega \to R,\ g : \Omega_{\text{imp}} \to R,\ \varphi : E_0 \to R$ are given functions. We take into considerations the Cauchy problem with impulses (1.3), (1.4). We will approximate classical solutions of the problem by solutions of adequate difference problems.

We define a mesh on the set $E_0 \cup E$ in the following way. Let **N** and **Z** be the sets of natural numbers and integers respectively. For $y, \bar{y} \in$

R^n, $y = (y_1, \ldots, y_n)$, $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_n)$, we write $y * \bar{y} = (y_1\bar{y}_1, \ldots, y_n\bar{y}_n)$. Suppose that (h_0, h') where $h' = (h_1, \ldots, h_n)$ stand for steps of the mesh. For $h = (h_0, h')$ and $(i, m) \in Z^{1+n}$ where $m = (m_1, \ldots, m_n)$ we define nodal points as follows:

$$x^{(i)} = ih_0, \ \ y^{(m)} = m * h', \ \ y^{(m)} = (y_1^{(m_1)}, \ldots, y_n^{(m_n)}).$$

Denote by Θ the set of all $h = (h_0, h')$ such that there are

$$\tilde{N}_0 \in Z, \ N = (N_1, \ldots, N_n) \in N^n$$

with the properties: $\tilde{N}_0 h_0 = \tau_0$ and $N * h' = b$. We assume that $\Theta \neq \emptyset$ and that there is a sequence $\{h^{(j)}\}$, $h^{(j)} \in \Theta$, and $\lim_{j\to\infty} h^{(j)} = 0$. For $h \in \Theta$ we put $|h| = h_0 + h_1 + \cdots + h_n$. There is $N_0 \in N$ such that $N_0 h_0 < a \leq (N_0 + 1)h_0$. For $h \in \Theta$ we define

$$R_h^{1+n} = \{\, (x^{(i)}, y^{(m)}) : \ (i, m) \in Z^{1+n} \,\}$$

and

$$E_{0.h} = E_0 \cap R_h^{1+n}, \ \ E_h = E \cap R_h^{1+n}.$$

In this Section we assume that $h' \leq h_0 M$. For a function $z : E_{0.h} \cup E_h \to R$ we write $z^{(i,m)} = z(x^{(i)}, y^{(m)})$. Elements of the set $E_{0.h} \cup E_h$ will be denoted by $(x^{(i)}, y^{(m)})$ or (x, y). Put

$$E_{j.h} = \Big\{ \, (x^{(i)}, y^{(m)}) \in E_{0.h} \cup E_h : \ \ i \leq j \, \Big\}$$

and

$$\| \, z \, \|_{j.h} = \max \Big\{ \, | \, z^{(i,m)} \, | : \ \ (x^{(i)}, y^{(m)}) \in E_{j.h} \Big\}$$

where $0 \leq j \leq N_0$. Let

$$E'_h = \{\, (x^{(i)}, y^{(m)}) \in E_h : \ \ (x^{(i)} + h_0, y^{(m)}) \in E_h \,\}.$$

The motivation for the definition of the set E'_h is the following. Approximate solution of problem (1.3), (1.4) are functions u_h defined on E_h. We will write difference equations generated by (1.3), (1.4) at each point of the set E'_h. It follows from condition $h' \leq h_0 M$ that we calculate all the values of u_h on E_h.

Suppose that natural numbers $n_1, \ldots, n_k$ are defined by

$$h_0 \, n_i < a_i \leq h_0 \, (n_i + 1), \quad i = 1, \ldots, k.$$

Let

$$E_h^{\text{imp}} = \Big\{ \, (x^{(i)}, y^{(m)}) : \ \ i \in \{\, n_1, \ldots, n_k \,\}, \ (x^{(i)}, y^{(m)}), \ (x^{(i+1)}, y^{(m)}) \in E_h \, \Big\}$$

and

$$\Omega_h = \Big(\, E'_h \setminus E_h^{\text{imp}} \Big) \times F(E_{0.h} \cup E_h, R) \times R^n, \quad \Omega_h^{\text{imp}} = E'_h \times F(E_{0.h} \cup E_h, R).$$

For $1 \le j \le n$ we write $e_j = (0, \ldots, 0, 1, 0, \ldots, 0) \in R^n$, 1 standing on j-th place. We consider difference operators δ_0, $\delta = (\delta_1, \ldots, \delta_n)$ given by

$$\delta_0 z^{(i,m)} = \frac{1}{h_0}\left[z^{(i+1,m)} - \frac{1}{2n}\sum_{j=1}^{n}\left(z^{(i,m+e_j)} + z^{(i,m-e_j)} \right)\right], \quad (1.20)$$

$$\delta_j z^{(i,m)} = \frac{1}{2h_j}\left(z^{(i,m+e_j)} - z^{(i,m-e_j)} \right), \quad 1 \le j \le n. \quad (1.21)$$

Suppose that for $h \in \Theta$ the functions $f_h : \Omega_h \to R$, $g_h : \Omega_h^{\mathrm{imp}} \to R$, $\varphi_h : E_{0.h} \to R$ are given. The function f_h is said to satisfy the Volterra condition if for each $(x^{(i)}, y^{(m)}) \in E'_h$ there is a set $E[i,m]$ such that

(i) $E[i,m] \subset E_{i.h}$,

(ii) for z, $\bar{z} \in F(E_{0.h} \cup E_h, R)$ such that $z = \bar{z}$ on $E[i,m]$ we have

$$f_h(x^{(i)}, y^{(m)}, z, q) = f_h(x^{(i)}, y^{(m)}, \bar{z}, q), \quad q \in R^n.$$

We will approximate solutions of (1.3), (1.4) by means of solutions of the problem

$$\delta_0 z^{(i,m)} = f_h(\, x^{(i)}, y^{(m)}, z, \delta z^{(i,m)}\,), \;\; (x^{(i)}, y^{(m)}) \in E'_h \setminus E_h^{\mathrm{imp}}, \quad (1.22)$$

$$z^{(i+1,m)} - z^{(i,m)} = g_h(\, x^{(i)}, y^{(m)}, z\,), \;\; (x^{(i)}, y^{(m)}) \in E_h^{\mathrm{imp}}, \quad (1.23)$$

$$z^{(i,m)} = \varphi_h^{(i,m)} \quad \text{on} \quad E_{0.h}. \quad (1.24)$$

If f_h and g_h satisfy the Volterra condition and $h' \le M h_0$ then there exists a unique solution $u_h : E_{0.h} \cup E_h \to R$ of problem (1.22)–(1.24).

Let

$$I_h = \left\{ x^{(i)} : \;\; i = -\tilde{N}_0, \ldots, -1, 0 \right\},$$

$$J_h = \left\{ x^{(i)} : \;\; i = 0, 1, \ldots, N_0, \right\}, \quad J'_h = \left\{ x^{(i)} : \;\; i = 0, 1, \ldots N_0 - 1 \right\},$$

and

$$J_{h.j} = \left\{ x^{(i)} \in I_h \cup J_h : \quad i \le j \right\}, \quad 0 \le j \le N_0.$$

We consider two comparison functions

$$\sigma_h : (\, J'_h \setminus J_h^{\mathrm{imp}}\,) \times F(I_h \cup J_h, R_+) \to R_+ \quad \text{and} \quad \tilde{\sigma}_h : J_h^{\mathrm{imp}} \times F(I_h \cup J_h) \to R_+$$

corresponding to f_h and g_h respectively. For simplify notation, we write

$$\sigma_h[\, i, \eta\,] \;\; \text{instead of} \;\; \sigma_h(\, x^{(i)}, \eta\,)$$

and

$$\tilde{\sigma}_h[\, j, \eta\,] \;\; \text{instead of} \;\; \tilde{\sigma}_h(\, x^{(j)}, \eta\,).$$

We will denote by $V_h : F(E_{0.h} \cup E_h) \to F(I_h \cup J_h, R_+)$ the operator given by

$$(V_h z) (x^{(i)}) = \max \Big\{ \, | \, z^{(i,m)} \, | : \;\; y^{(m)} \in [-b, \, b \,] \Big\} \quad \text{for } \; x^{(i)} \in I_h,$$

$$(V_h z) (x^{(i)}) =$$

$$= \max \Big\{ \, | \, z^{(i,m)} \, | : \;\; y^{(m)} \in [-b + M x^{(i)}, \, b - M x^{(i)} \,] \Big\} \quad \text{for } \; x^{(i)} \in J_h.$$

Assumption H $[\, \sigma_h, \tilde{\sigma}_h \,]$. Suppose that

1) the functions σ_h and $\tilde{\sigma}_h$ are nondecreasing with respect to the functional variable and fulfill the Volterra condition,

2) $\sigma[\, \cdot \, , \theta_h \,] = 0$ on $J'_h \setminus J_h^{\text{imp}}$ and $\tilde{\sigma}[\, \cdot \, , \theta_h \,] = 0$ on J_h^{imp} where $\theta_h^{(i)} = 0$ for $x^{(i)} \in I_h \cup J_h$,

3) the difference problem with impulses

$$\eta^{(i+1)} = \eta^{(i)} + h_0 \, \sigma_h[\, i, \eta \,] \;\; \text{for } x^{(i)} \in J'_h \setminus J_h^{\text{imp}}, \tag{1.25}$$

$$\eta^{(i+1)} = \eta^{(i)} + \tilde{\sigma}_h[\, i, \eta \,] \;\; \text{for } \; x^{(i)} \in J_h^{\text{imp}}, \;\; \eta^{(i)} = 0 \;\; \text{ on } \; I_h \tag{1.26}$$

is stable in the following sense: if $\eta_h : I_h \cup J_h \to R_+$ is the solution of the problem

$$\eta^{(i+1)} = \eta^{(i)} + h_0 \, \sigma_h[\, i, \eta \,] + h_0 \gamma(h) \;\; \text{for } x^{(i)} \in J'_h \setminus J_h^{\text{imp}}, \tag{1.27}$$

$$\eta^{(i+1)} = \eta^{(i)} + \tilde{\sigma}_h[i, \eta] + \tilde{\gamma}(h) \text{ for } x^{(i)} \in J_h^{\text{imp}}, \;\; \eta^{(i)} = \alpha_0(h) \;\; \text{on } I_h, \tag{1.28}$$

where γ, $\tilde{\gamma}$, $\alpha_0 : \Theta \to R_+$ and

$$\lim_{h \to 0} \alpha_0(h) = \lim_{h \to 0} \gamma(h) = \lim_{h \to 0} \tilde{\gamma}(h) = 0,$$

then there is a function $\beta : \Theta \to R_+$ such that $\eta_h^{(i)} \le \beta(h), \; x^{(i)} \in J_h$, and $\lim_{h \to 0} \beta(h) = 0$.

Assumption H $[\, f_h, g_h \,]$**.** Suppose that functions f_h and g_h satisfy the Volterra condition and

1) there exist the derivatives

$$(\, D_{q_1} f_h(P), \ldots , D_{q_n} f_h(P) \,) = D_q f_h(P)$$

and $D_q f_h(x, y, z, \, \cdot \,) \in C(R^n, R^n)$,

2) for each $P = (x, y, z, q) \in \Omega_h$ we have

$$1 - \frac{n \, h_0}{h_j} \, \big| \, D_{q_j} f_h(\, P \,) \, \big| \ge 0, \;\; 1 \le j \le n,$$

3) there are functions σ_h and $\tilde{\sigma}_h$ satisfying Assumption H $[\, \sigma_h, \tilde{\sigma}_h \,]$ and such that

$$\Big| \, f_h(\, x^{(i)}, y^{(m)}, z, q \,) - f_h(\, x^{(i)}, y^{(m)}, \bar{z}, q \,) \Big| \le \sigma_h[\, i, V_h(z - \bar{z}) \,] \quad \text{on } \; \Omega_h,$$

and

$$\Big| g_h(x^{(i)}, y^{(m)}, z) - g_h(x^{(i)}, y^{(m)}, \bar{z}) \Big| \le \tilde{\sigma}_h[i, V_h(z - \bar{z})] \quad \text{on } \Omega_h^{\text{imp}},$$

Now we prove a theorem on the convergence of the method (1.22)–(1.24).

Theorem 1.16. *Suppose that Assumptions $H[\sigma_h, \tilde{\sigma}_h]$, $H[f_h, g_h]$ are satisfied and*

1) *the functions* $f \in C(\Omega, R)$, $g \in C(\Omega_{\text{imp}}, R)$ *fulfill the Volterra condition and the condition* $V^{(-)}$ *respectively and* $\varphi \in C(E_0, R)$,

2) $v \in C_{\text{imp}}(E_0 \cup E, R)$ *is the solution of* (1.3), (1.4), *the function* v *is of class* C^2 *on* $E \setminus E_{\text{imp}}$ *and the partial derivatives of the second order of* v *are bounded on* $E \setminus E_{\text{imp}}$,

3) $u_h : E_{0.h} \cup E_h \to R$ *is the solution of* (1.22)–(1.24), v_h *is the restriction of* v *to the set* $E_{0.h} \cup E_h$, *and there exist functions* α_0, $\tilde{\beta}_1$, $\tilde{\beta}_2 : \Theta \to R_+$ *such that*

$$\Big| f_h(x^{(i)}, y^{(m)}, v_h, \delta v_h^{(i,m)}) - f(x^{(i)}, y^{(m)}, v, \delta v^{(i,m)}) \Big| \le \tilde{\beta}_1(h) \quad (1.29)$$

$$\text{on } E'_h \setminus E_h^{\text{imp}},$$

$$\Big| g_h(x^{(i)}, y^{(m)}, u_h) - g(a_{n_i}, y^{(m)}, u) \Big| \le \tilde{\beta}_2(h) \quad \text{on } E_h^{\text{imp}}, \qquad (1.30)$$

$$\Big| \varphi^{(i,m)} - \varphi_h^{(i,m)} \Big| \le \alpha_0(h) \quad \text{on } E_{0.h}$$

and

$$\lim_{h \to 0} \alpha_0(h) = \lim_{h \to 0} \tilde{\beta}_1(h) = \lim_{h \to 0} \tilde{\beta}_2(h) = 0. \qquad (1.31)$$

Then there exists a function $\tilde{\gamma} : \Theta \to R_+$ *such that*

$$\Big| u_h^{(i,m)} - v_h^{(i,m)} \Big| \le \tilde{\gamma}(h) \quad \text{on } E_h \quad \text{and} \quad \lim_{h \to 0} \tilde{\gamma}(h) = 0. \qquad (1.32)$$

Proof. Let the function $\tilde{\Gamma}_h : E'_h \to R$ be defined by

$$\delta_0 v_h^{(i,m)} = f_h(x^{(i)}, y^{(m)}, v_h, \delta v_h^{(i,m)}) + \tilde{\Gamma}_h^{(i,m)} \quad \text{on } E'_h \setminus E_h^{\text{imp}},$$

and

$$v_h^{(i+1,m)} = v_h^{(i,m)} + g_h(x^{(i)}, y^{(m)}, v_h) + \tilde{\Gamma}_h^{(i,m)} \quad \text{on } E_h^{\text{imp}}.$$

It follows from the consistency conditions (1.29)–(1.31) that there exists $\gamma : \Theta \to R_+$ such that $|\tilde{\Gamma}_h^{(i,m)}| \le \gamma(h)$ on E'_h and $\lim_{h\to} \gamma(h) = 0$. Let $\omega_h : I_h \cup J_h \to R_+$ be given by

$$\omega_h^{(i)} = \max\Big\{ |u_h^{(i,m)} - v_h^{(i,m)}| : \; y^{(m)} \in [-b, b] \Big\} \quad \text{for } x^{(i)} \in I_h$$

and

$$\omega_h^{(i)} = \max \Big\{ \,|\, u_h^{(i,m)} - v_h^{(i,m)} \,| : \;\; y^{(m)} \in [\, -b + Mx^{(i)},\, b - Mx^{(i)} \,] \Big\}$$
$$\text{for } \; x^{(i)} \in J_h.$$

It follows that the function ω_h satisfies the difference inequalities

$$\omega_h^{(i+1)} \le \omega_h^{(i)} + h_0 \sigma[\, i, \omega_h \,] + h_0 \gamma(h), \quad x^{(i)} \in J_h' \setminus J_h^{\text{imp}},$$
$$\omega_h^{(i+1)} \le \omega_h^{(i)} + \tilde{\sigma}_h[\, i, \omega_h \,] + \gamma(h), \quad x^{(i)} \in J_h^{\text{imp}}.$$

We have also the initial estimate $\omega_h^{(i)} \le \alpha_0(h)$ on I_h. Consider the problem

$$\eta^{(i+1)} = \eta^{(i)} + h_0 \sigma_h[\, i, \eta \,] + h_0 \gamma(h), \;\; x^{(i)} \in J_h' \setminus J_h^{\text{imp}}, \qquad (1.33)$$
$$\eta^{(i+1)} = \eta^{(i)} + \tilde{\sigma}_h[\, i, \eta \,] + \gamma(h), \; x^{(i)} \in J_h^{\text{imp}}, \quad \eta^{(i)} = \alpha_0(h) \;\text{ on } I_h. \quad (1.34)$$

Denote by $\tilde{\eta}_h : I_h \cup J_h \to R_+$ the solution of (1.33), (1.34). It follows that $\omega_h^{(i)} \le \tilde{\eta}_h^{(i)}$ on J_h and consequently $|u_h^{(i,m)} - v_h^{(i,m)}| \le \tilde{\eta}_h^{(i)}$ on E_h. Now we obtain (1.32) from condition 3) of Assumption H$[\, \sigma_h, \tilde{\sigma}_h \,]$. This completes the proof of Theorem 1.16. $\square$

Now we consider the condition of stability 3) from Assumption H $[\, \sigma_h, \tilde{\sigma}_h \,]$ in the case when f_h and g_h satisfy the Lipschitz condition. Suppose that there are $L_0, \; L \in R_+$ such that

$$\Big| \, f_h(\, x^{(i)}, y^{(m)}, z, q \,) - f_h(\, x^{(i)}, y^{(m)}, \bar{z}, q \,) \Big| \le L \| z - z \|_{h.i} \quad \text{on } \; \Omega_h,$$

and

$$\Big| \, g_h(\, x^{(i)}, y^{(m)}, z \,) - g_h(\, x^{(i)}, y^{(m)}, \bar{z} \,) \Big| \le L_0 \| z - \bar{z} \|_{h.i} \quad \text{on } \; \Omega_h^{\text{imp}}.$$

Then problem (1.33), (1.34) is equivalent to

$$\eta^{(i+1)} = \eta^{(i)}(1 + Lh_0) + h_0 \gamma(h), \;\; x^{(i)} \in J_h' \setminus J_h^{\text{imp}}, \qquad (1.35)$$
$$\eta^{(i+1)} = \eta^{(i)}(1 + L_0) + \tilde{\gamma}(h), \; x^{(i)} \in J_h^{\text{imp}}, \quad \eta^{(0)} = \alpha_0(h). \quad (1.36)$$

Write $n_{k+1} = N_0$. Then the solution η_h of (1.35), (1.36) has the form

$$\eta_h^{(i)} = \alpha_0(h)(1 + Lh_0)^i + h_0 \gamma(h) \sum_{\tau=0}^{i-1} (1 + Lh_0)^\tau, \quad i = 0, 1, \ldots, n_1,$$
$$\eta_h^{(n_j + i)} = \Big[\, (1 + L_0) \eta_h^{(n_j)} + \tilde{\gamma}(h) \Big] \, (\, 1 + Lh_0 \,)^{i-1} +$$
$$+ h_0 \gamma(h) \sum_{\tau=0}^{i-2} (1 + Lh_0)^\tau, \;\; i = 1, 2, \ldots, n_{j+1} - n_j, \;\; j = 1, \ldots, k,$$

where $\sum_{\tau=0}^{-1} = 0$.

We see at once that there is $\tilde{\eta} : \Theta \to R_+$ such that $\eta_h^{(i)} \le \tilde{\eta}(h)$ for $1 \le i \le N_0$ and $\lim_{h \to 0} \tilde{\eta}(h) = 0$.

Remark 1.17. It is easy to prove a theorem on the convergence of the difference method (1.22)–(1.24) with the operators δ_0 and $\delta = (\delta_1, \ldots, \delta_n)$ given by

$$\delta_0 z^{(i,m)} = \frac{1}{h_0} \left(z^{(i+1,m)} - z^{(i,m)} \right) \tag{1.37}$$

and

$$\delta_j z^{(i,m)} = \frac{1}{h_j} \left(z^{(i,m+e_j)} - z^{(i,m)} \right), \quad j = 1, \ldots, \kappa, \tag{1.38}$$

$$\delta_j z^{(i,m)} = \frac{1}{h_j} \left(z^{(i,m)} - z^{(i,m-e_j)} \right), \quad j = \kappa + 1, \ldots, n, \tag{1.39}$$

where $1 \le \kappa \le n$ is fixed. Condition 2) of Assumption H $[f_h, g_h]$ takes the form

$$1 - h_0 \sum_{j=1}^{n} \frac{1}{h_j} \left| D_{q_j} f_h(x, y, z, q) \right| \ge 0,$$

$D_{q_i} f_h(x, y, z, q) \ge 0$ for $1 \le i \le \kappa$, $D_{q_i} f_h(x, y, z, q) \le 0$ for $\kappa + 1 \le i \le n$, where $(x, y, z, q) \in \Omega_h$.

Remark 1.18. All the results of this Chapter can be extended for systems of functional differential equations with impulses

$$D_x z_i(x,y) = f_i(x, y, z(x,y), z, D_y z_i(x,y)), \quad i = 1, \ldots, k, \quad (x,y) \in E \setminus E_{\text{imp}},$$

$$\Delta z(x,y) = g(x, y, z(x^-, y), z) \quad \text{on } E_{\text{imp}}, \quad z(x,y) = \varphi(x,y) \quad \text{on } E_0,$$

where $z = (z_1, \ldots, z_k)$ and

$$f = (f_1, \ldots, f_k) : (E \setminus E_{\text{imp}}) \times R^k \times C_{\text{imp}}(E_0 \cup E, R^k) \times R^n \to R^k$$

$$g = (g_1, \ldots, g_k) : E_{\text{imp}} \times R^k \times C_{\text{imp}}(E_0 \cup R, R^k), \quad \varphi : E_0 \to R^k.$$

Some quasi monotonicity conditions are required for f and g in this case.

CHAPTER II
INITIAL BOUNDARY VALUE PROBLEMS WITH IMPULSES

2.1. Functional Differential Inequalities Generated by Mixed Problems

We formulate the problem. Let $a > 0$, $\tau_0 \in R_+$, $b = (b_1, \dots, b_n) \in R^n$ and $\tau = (\tau_1, \dots, \tau_n) \in R^n_+$ be given where $b_i > 0$ for $1 \le i \le n$. Suppose that $\kappa \in \mathbf{Z}$, $0 \le \kappa \le n$, is fixed. For each $y = (y_1, \dots, y_n) \in R^n$ we write $y = (y', y'')$ where $y' = (y_1, \dots, y_\kappa)$, $y'' = (y_{\kappa+1}, \dots, y_n)$. We have $y' = y$ if $\kappa = n$ and $y'' = y$ if $\kappa = 0$. We define the sets

$$E = (0, a) \times [-b', b') \times (-b'', b''], \quad B = [-\tau_0, 0] \times [0, \tau'] \times [-\tau'', 0].$$

Let $c = (c_1, \dots, c_n) = b + \tau$ and

$$E_0 = [-\tau_0, 0] \times [-b', c'] \times [-c'', b''],$$

$$\partial_0 E = (\, (0, a) \times [-b', c'] \times [-c'', b''] \,) \setminus E, \quad E^* = E_0 \cup E \cup \partial_0 E.$$

If $\tau_0 > 0$ then we put $B^{(-)} = B \cap (\, [-\tau_0, 0) \times R^n)$.

Suppose that $0 < a_1 < a_2 < \dots < a_k < a$ are given numbers. Let $I,\ J,\ J_{\text{imp}}$ be the sets given by (1.1) and

$$E_{\text{imp}} = \{ \, (x, y) \in E : \ x \in J_{\text{imp}} \, \},$$

$$E^*_{\text{imp}} = \{ \, (x, y) \in E^* : \ x \in J_{\text{imp}} \, \}, \quad \partial_0 E_{\text{imp}} = \{ \, (x, y) \in \partial_0 E : \ x \in J_{\text{imp}} \, \}.$$

We denote by $C_{\text{imp}}(E^*, R)$ the class of all functions $z : E^* \to R$ such that

(i) the restriction of z to the set $E^* \setminus E^*_{\text{imp}}$ is a continuous function,

(ii) for each $(x, y) \in E^*_{\text{imp}}$ there are the limits (1.2),

(iii) $z(x, y) = z(x^+, y)$ for $(x, y) \in E^*_{\text{imp}}$.

In the same way we define the set $C_{\text{imp}}(E_0 \cup \partial_0 E, R)$. For a function $z \in C_{\text{imp}}(E^*, R)$ and a point $(x, y) \in E_{\text{imp}}$ we write $\Delta z(x, y) = z(x, y) - z(x^-, y)$.

Suppose that we have a sequence $\{ \, t_1, t_2, \dots, t_p \, \}$ such that $-\tau_0 \le t_1 < t_2 < \dots < t_p \le 0$. Write $\Gamma_i = B \cap (\, (t_i, t_{i+1}) \times R^n \,)$ for $i = 1, \dots, p-1$ and

$$\Gamma_0 = \emptyset \quad \text{if} \quad -\tau_0 = t_1, \quad \Gamma_0 = B \cap (\, (-\tau_0, t_1) \times R^n \,) \quad \text{if} \quad -\tau_0 < t_1,$$

$$\Gamma_p = \emptyset \quad \text{if} \quad t_p = 0, \quad \Gamma_p = B \cap (\, (t_p, 0) \times R^n \,) \quad \text{if} \quad t_p < 0.$$

We denote by $C_{\text{imp}}[B, R]$ the class of all functions $w : B \to R$ such that there is a sequence $\{ \, t_1, \dots, t_p \, \}$ (the numbers p and $t_1, \dots, t_p$ depend on w) such that

(i) the functions $w \,|_{\Gamma_i}$, $i = 0, 1, \dots, p$, are continuous,

(ii) for every $i,\ j,\ 1 \le i,\ j \le p$, $(t_i, s) \in B$, $(t_j, s) \in B$, $t_i > -\tau_0$, $t_j < 0$, there exist the limits

$$\lim_{(t,y) \to (t_i, s),\, t < t_i} w(t, y) = w(t_i^-, s), \qquad \lim_{(t,y) \to (t_j, s),\, t > t_j} w(t, y) = w(t_j^+, s),$$

(iii) $w(t_j, s) = w(t_j^+, s)$ for $(t_j, s) \in B$, $1 \le j \le p-1$ and for $j = p$ if $t_p < 0$,

(iv) the functions $w(t_1, \cdot\,)$ and $w(t_p, \cdot\,)$ are continuous if $t_1 = -\tau_0$ and $t_p = 0$.

We define also in the case $\tau_0 > 0$

$$C_{\text{imp}}[B^{(-)}, R] = \{\, w\,|_{B^{(-)}} : \ \ w \in C_{\text{imp}}[B, R]\,\}\,.$$

Elements of the sets $C_{\text{imp}}[B, R]$ and $C_{\text{imp}}[B^{(-)}, R]$ will be denoted by the same symbols. We denote by $\|\cdot\|_B$ and $\|\cdot\|_{B^{(-)}}$ the supremum norms in the space $C_{\text{imp}}[B, R]$ and $C_{\text{imp}}[B^{(-)}, R]$ respectively.

Suppose that $z : E^* \to R$ and $(x, y) \in [0, a) \times [-b, b]$. Then $z_{(x,y)} : B \to R$ is the function defined by

$$z_{(x,y)}(t, s) = z(x + t, y + s), \ \ (t, s) \in B.$$

The function $z_{(x,y)}$ is the restriction of z to the set $[x - \tau_0, x] \times [y,\, y' + \tau'] \times [y'' - \tau'',\, y'']$ and this restriction is shifted to the set B. If $\tau_0 > 0$ then for the above z and (x, y) we will consider also the function $z_{(x^-,y)} : B^{(-)} \to R$ given by $z_{(x^-,y)}(t, s) = z(x + t, y + s)$, $(t, s) \in B^{(-)}$.

It is easy to see that if $z \in C_{\text{imp}}(E^*, R)$ and $(x, y) \in [0, a) \times [-b, b]$ then $z_{(x,y)} \in C_{\text{imp}}[B, R]$ and $z_{(x^-,y)} \in C_{\text{imp}}[B^{(-)}, R]$ in the case $\tau_0 > 0$.

Put

$$\Omega = (\,E \setminus E_{\text{imp}}) \times C_{\text{imp}}[B, R] \times R^n \quad \text{and} \quad \Omega_{\text{imp}} = E_{\text{imp}} \times C_{\text{imp}}[B^{(-)}, R]$$

and suppose that $f : \Omega \to R$, $g : \Omega_{\text{imp}} \to R$, $\varphi \in C_{\text{imp}}(E_0 \cup \partial_0 E, R)$ are given functions. We assume that g does not depend on the functional variable in the case $\tau_0 = 0$. We take into considerations the functional differential equations with impulses

$$D_x z(x, y) = f(\,x, y, z_{(x,y)}, D_y z(x, y)\,) \quad \text{on} \ \ E \setminus E_{\text{imp}}, \tag{2.1}$$

$$\Delta z(x, y) = g(\,x, y, z_{(x^-,y)}\,) \quad \text{on} \ \ E_{\text{imp}} \tag{2.2}$$

and the initial boundary conditions

$$z(x, y) = \varphi(x, y) \quad \text{on} \ \ E_0 \cup \partial_0 E. \tag{2.3}$$

We consider classical solutions of problem (2.1)–(2.3). A function $\tilde{z} : E^* \to R$ is a solution of (2.1)–(2.3) if $\tilde{z} \in C_{\text{imp}}(E^*, R)$, there are the derivatives $D_x\tilde{z},\ D_y\tilde{z}$ on $E \setminus E_{\text{imp}}$ and $\tilde{z}$ satisfies (2.1)–(2.3).

Example 1. Suppose that $\tilde{f} : (E \setminus E_{\text{imp}}) \times R \times R \times R^n \to R$ is a given function. Put

$$f(\,x, y, w, q\,) = \tilde{f}(\,x, y, w(0, 0), \int_B w(t, s)\, dt\, ds, q\,).$$

Then equation (2.1) is equivalent to

$$D_x z(x, y) = \tilde{f}(\,x, y, z(x, y), \int_B z(x+t, y+s)\, dt\, ds, D_y z(x, y)\,), \ \ (x, y) \in E \setminus E_{\text{imp}}.$$

It is easy to see that differential equations with a deviated argument are also particular cases of (2.1).

We prove theorems on functional differential inequalities with impulses. Suppose that $z \in C_{\text{imp}}(E^*, R)$ and the derivatives $D_x z$, $D_y z$ exists on $E \setminus E_{\text{imp}}$. We consider the Niemycki operators corresponding to (2.1), (2.2)

$$F[\,z\,](x,y) = f(\,x, y, z_{(x,y)}, D_y z(x,y)\,) \quad \text{on} \;\; E \setminus E_{\text{imp}},$$
$$G[\,z\,] = g(\,x, y, z_{(x^-,y)}\,) \quad \text{on} \;\; E_{\text{imp}}.$$

Let I_+, I_-, $I_0 : E \to \{1,\dots,n\}$ be the functions defined in the following way. For each $(x,y) \in E$ there are sets (possibly empty) of integers $I_+[x,y]$, $I_-[x,y]$, $I_0[x,y]$ exist such that

$$I_+[x,y] \cup I_-[x,y] \cup I_0[x,y] = \{1,\dots,n\},$$

and

$$\bar{y}_i = b_i \quad \text{for} \quad i \in I_+[x,y],$$
$$\bar{y}_i = -b_i \quad \text{for} \quad i \in I_-[x,y], \quad -b_i < \bar{y}_i < b_i \quad \text{for} \;\; i \in I_0[x,y].$$

Assumption H $[\,f, g\,]$**.** Suppose that

1) the function f of the variables (x,y,w,q) satisfies the following monotonicity condition: if w, $\bar{w} \in C_{\text{imp}}[B,R]$, $w(t,s) \le \bar{w}(t,s)$ on B and $w(0,0) = \bar{w}(0,0)$ then $f(x,y,w,q) \le f(x,y,\bar{w},q)$,

2) the derivatives $(\,D_{q_1} f, \dots, D_{q_n} f\,)$ exist on Ω and

$$D_{q_i} f(x,y,w,q) \ge 0 \quad \text{on} \;\; \Omega \quad \text{for} \;\; 1 \le i \le \kappa,$$

$$D_{q_i} f(x,y,w,q) \le 0 \quad \text{on} \;\; \Omega \quad \text{for} \;\; \kappa + 1 \le i \le n,$$

3) the function g of the variables (x,y,w) is nondecreasing with respect to w.

Theorem 2.1. *Suppose that Assumption $H\,[\,f, g\,]$ is satisfied and*

1) *u, $v \in C_{\text{imp}}(E^*, R)$, the derivatives $D_x u$, $D_x v$, $D_y u$, $D_y v$ exist on $E \setminus E_{\text{imp}}$,*

2) *the differential inequality*

$$D_x u(x,y) - F[\,u\,](x,y) < D_x v(x,y) - F[\,v\,](x,y) \;\; \textit{on} \;\; E \setminus E_{\text{imp}}, \quad (2.4)$$

and the inequality for impulses

$$\Delta u(x,y) - G[\,u\,](x,y) < \Delta v(x,y - G[\,v\,](x,y) \;\; \textit{on} \;\; \in E_{\text{imp}} \qquad (2.5)$$

are satisfied,

3) *the initial boundary inequality $u(x,y) < v(x,y)$, $(x,y) \in E_0 \cup \partial_0 E$ holds.*

Under these assumptions we have $u(x,y) < v(x,y)$ for $(x,y) \in E$.

Proof. Write

$$I_* = \{x \in [0,a): \quad \text{there is}\, y \ \text{ such that } \ (x,y) \in E \ \text{ and}\, u(x,y) \geq v(x,y)\}\,.$$

Suppose, by contradiction, that assertion fails to be true. Then I_* is not empty. Let $\tilde{x} = \min\ I_*$. It follows that $\tilde{x} > 0$ and there is $\tilde{y}$ such that $(\tilde{x},\tilde{y}) \in E$ and

$$u(\tilde{x},\tilde{y}) = v(\tilde{x},\tilde{y}). \tag{2.6}$$

First suppose that $(\tilde{x},\tilde{y}) \in E \setminus E_{\text{imp}}$. Because $(\tilde{x},\tilde{y}) \notin \partial_0 E$, we have

$$\{\,1,\ldots,\kappa\,\} \cap I_+[\tilde{x},\tilde{y}] = \emptyset \quad \text{and} \quad \{\,\kappa+1,\ldots,n\,\} \cap I_-[\tilde{x},\tilde{y}] = \emptyset.$$

According to the definition of $(\tilde{x},\tilde{y})$ we have

$$D_x(u-v)(\tilde{x},\tilde{y}) \geq 0 \tag{2.7}$$

and

$$\begin{aligned} D_{y_i}(u-v)(\tilde{x},\tilde{y}) &\geq 0 \quad \text{for } \ i \in I_+[\tilde{x},\tilde{y}],\\ D_{y_i}(u-v)(\tilde{x},\tilde{y}) &\leq 0 \quad \text{for } \ i \in I_-[\tilde{x},\tilde{y}],\\ D_{y_i}(u-v)(\tilde{x},\tilde{y}) &= 0 \quad \text{for } \ i \in I_0[\tilde{x},\tilde{y}]. \end{aligned}$$

We also have the inequality

$$u_{(\tilde{x},\tilde{y})}(t,s) \leq v_{(\tilde{x},\tilde{y})}(t,s) \quad \text{for } \ (t,s) \in B.$$

Thus, from assumption 2) and from the above estimates, we deduce that

$$\begin{aligned} D_x(u-v)(\tilde{x},\tilde{y}) < f(\,\tilde{x},\tilde{y},u_{(\tilde{x},\tilde{y})},D_y u(\tilde{x},\tilde{y})\,) - f(\,\tilde{x},\tilde{y},v_{(\tilde{x},\tilde{y})},D_y u(\tilde{x},\tilde{y})\,) + \\ + \sum_{i \in I_+[\tilde{x},\tilde{y}] \cup I_-[\tilde{x},\tilde{y}]} D_{q_i} f(\,\tilde{x},\tilde{y},v_{(\tilde{x},\tilde{y})},\tilde{q}^{(i)}\,)\, D_{y_i}(u-v)(\tilde{x},\tilde{y}) \leq 0, \end{aligned}$$

where $\tilde{q}^{(i)} \in R^n$ are intermediate points. The last inequality contradicts (2.7) and thus the assertion follows.

Now suppose that $(\tilde{x},\tilde{y}) \in E_{\text{imp}}$. Then we have $u(\tilde{x}^-,\tilde{y}) \leq v(\tilde{x}^-,\tilde{y})$ and

$$u_{(\tilde{x}^-,\tilde{y})}(t,s) \leq v_{(\tilde{x}^-,\tilde{y})}(t,s) \quad \text{for } \ (t,s) \in B^{(-)}.$$

It follows from assumption 3) and from the monotonicity of g with respect to the functional variable that

$$u(\tilde{x},\tilde{y}) - v(\tilde{x},\tilde{y}) < u(\tilde{x}^-,\tilde{y}) - v(\tilde{x}^-,\tilde{y}) + g(\,\tilde{x},\tilde{y},u_{(\tilde{x}^-,\tilde{y})}\,) - g(\,\tilde{x},\tilde{y},v_{(\tilde{x}^-,\tilde{y})}\,) \leq 0,$$

which contradicts (2.6). This completes the proof of the Theorem 2.1. □

Remark 2.2. Assumption 2) of Theorem 2.1 can be replaced by the following one. Let

$$T_+ = \{ (x,y) \in E : \ u(t,s) < v(t,s) \ \text{ on } \ E \cap ([0,x) \times R^n) , \ \ u(x,y) = v(x,y) \} .$$

Suppose that

$$D_x u(x,y) - F[\,u\,](x,y) < D_x v(x,y) - F[\,v\,](x,y) \ \ \textit{for} \ \ (x,y) \in T_+ \setminus E_{\text{imp}},$$
$$\Delta u(x,y) - G[\,u\,](x,y) < \Delta v(x,y) - G[\,v\,](x,y) \ \ \textit{for} \ \ (x,y), \in T_+ \cap E_{\text{imp}}.$$

The point $(\tilde{x}, \tilde{y})$ from the proof of Theorem 2.1 is an element of T_+.

Now we will deal with weak functional differential inequalities with impulses. We need more restrictive assumptions on the functions f and g. We will need two comparison functions: σ and σ_0 for f and g respectively.

Assumption H $[\,\sigma, \, \sigma_0\,]$**.** Suppose that

1) the function $\sigma : (J \setminus J_{\text{imp}}) \times R_+ \to R_+$ and $\sigma_0 : J_{\text{imp}} \times R_+ \to R_+$ are continuous and $\sigma(x,0) = 0$ for $x \in J \setminus J_{\text{imp}}$, $\sigma_0(x,0) = 0$ for $x \in J_{\text{imp}}$,

2) the functions $\sigma(x, \cdot\,)$ and $\sigma_0(t, \cdot\,)$ are nondecreasing on R_+ and the function $\tilde{\omega}(x) = 0, \ x \in J$, is the unique solution of the initial problem with impulses

$$\omega'(x) = \sigma(\,x, \omega(x)\,) \quad \text{on} \ \ J \setminus J_{\text{imp}},$$
$$\Delta\omega(x) = \sigma_0(\,x, \omega(x^-)\,) \quad \text{on} \ \ J_{\text{imp}}, \quad \omega(0) = 0.$$

Now we formulate the main theorem of weak functional differential inequalities.

Theorem 2.3. *Suppose that Assumptions* $H\,[\,f,g\,]$*,* $H\,[\,\sigma,\sigma_0\,]$ *are satisfied and*

1) *for every* $w, \, \bar{w} \in C_{\text{imp}}[B,R]$*, if* $\bar{w} \geq w$ *then*

$$f(\,x,y,\bar{w},q\,) - f(\,x,y,w,q\,) \leq \sigma(\,x, \|\bar{w} - w\|_B\,)$$

where $(x,y,q) \in (E \setminus E_{\text{imp}}) \times R^n$*,*

2) *for every* $w, \, \bar{w} \in C_{\text{imp}}[B^{(-)},R]$*, if* $\bar{w} \geq w$*, then*

$$g(\,x,y,\bar{w}\,) - g(\,x,y,w\,) \leq \sigma_0(\,x, \, \|\bar{w} - w\|_{B^{(-)}}\,),$$

where $(x,y) \in E_{\text{imp}}$*,*

3) $u, \, v \in C_{\text{imp}}(E^*, R)$*, the derivatives* $D_x u, \ D_x v, \ D_y u, \ D_y v$ *exist on* $E \setminus E_{\text{imp}}$ *and* $u(x,y) \leq v(x,y)$ *on* $E_0 \cup \partial_0 E$*,*

4) *the differential inequality*

$$D_x u(x,y) - F[\,u\,](x,y) \leq D_x v(x,y) - F[\,v\,](x,y) \ \ \textit{on} \ \ E \setminus E_{\text{imp}},$$

and the inequality for impulses

$$\Delta u(x,y) - G[\,u\,](x,y) \leq \Delta v(x,y) - G[\,v\,](x,y) \ \ \textit{on} \ \ E_{\text{imp}},$$

are satisfied.

Under these assumptions we have

$$u(x,y) \le v(x,y) \;\; for \;\; (x,y) \in E. \tag{2.8}$$

Proof. Let $\tilde{a} \in (a_k, a)$ be fixed. At first we prove that

$$u(x,y) \le v(x,y) \quad \text{for} \;\; (x,y) \in E \cap (\,[0,\tilde{a}) \times R^n\,)\,. \tag{2.9}$$

For $\varepsilon = (\varepsilon_0, \varepsilon_1, \varepsilon_2)$ we consider the maximum solution $\omega(\,\cdot\,,\varepsilon)$ of the problem

$$\omega'(x) = \sigma(\,x, \omega(x)\,) + \varepsilon_0 \;\; \text{on } [0,\tilde{a}) \setminus J_{\text{imp}}, \tag{2.10}$$

$$\Delta\omega(x) = \sigma_0(\,x, \omega(x^-)\,) + \varepsilon_1 \;\; \text{on } \; J_{\text{imp}}, \qquad \omega(0) = \varepsilon_2. \tag{2.11}$$

There is $\bar{\varepsilon} > 0$ such that for $0 < \varepsilon_i < \bar{\varepsilon}$, $i = 0,1,2,$ the solution $\omega(\,\cdot\,,\varepsilon)$ is defined on $[0,\tilde{a})$. Let the function $u^{(\varepsilon)} : E^* \to R$ by given by

$$u^{(\varepsilon)}(x,y) = u(x,y) - \omega(x,\varepsilon) \quad \text{on} \quad (\,E \cup \partial_0 E\,) \cap (\,[0,\tilde{a}) \times R^n\,)\,,$$

$$u^{(\varepsilon)}(x,y) = u(x,y) - \varepsilon_2 \quad \text{on} \;\; E_0.$$

We prove that

$$u^{(\varepsilon)}(x,y) < v(x,y) \quad \text{on} \;\; E \cap (\,[0,\tilde{a}) \times R^n\,)\,. \tag{2.12}$$

It follows from the assumptions of Theorem 2.3 that

$$u^{(\varepsilon)}(x,y) < v(x,y) \quad \text{on} \quad E_0 \cup (\partial_0 E \cap (\,[0,\tilde{a}) \times R^n\,)\,)$$

and

$$D_x u^{(\varepsilon)}(x,y) - F[\,u^{(\varepsilon)}\,](x,y) < D_x v(x,y) - F[\,v\,](x,y)$$
$$\text{on} \;\; (E \cap (\,[0,\tilde{a}) \times R^n\,)\,) \setminus E_{\text{imp}}.$$

$$\Delta u^{(\varepsilon)}(x,y) - G[\,u^{(\varepsilon)}\,](x,y) < \Delta v(x,y) - G[\,v\,](x,y) \quad \text{on} \;\; E_{\text{imp}}.$$

Hence, an application of Theorem 2.1 shows that (2.12) is satisfied. Because

$$\lim_{\varepsilon \to 0} \omega(x,\varepsilon) = 0 \quad \text{uniformly with respect to} x \in [0,\tilde{a}),$$

we get the assertion (2.9). The constant $\tilde{a} \in (a_k, a)$ is arbitrary, then (2.9) implies (2.8). This completes the proof of Theorem 2.3. □

Remark 2.4. Assumption 4) of Theorem 2.3 can be replaced by the following one. Let

$$\tilde{T} = \{\,(x,y) \in E : \;\; u(x,y) > v(x,y)\,\}\,.$$

Suppose that

$$D_x u(x,y) - F[\,u\,](x,y) \le D_x v(x,y) - F[\,v\,](x,y) \text{ on } \tilde{T} \setminus E_{\text{imp}}$$

and

$$\Delta u(x,y) - G[\,u\,](x,y) \le \Delta v(x,y) - G[\,v\,](x,y) \text{ on } \tilde{T} \cap E_{\text{imp}}.$$

In this case we use Theorem 2.1 for $u^{(\varepsilon)}$ and v in the version given in Remark 2.2.

Remark 2.5. Let Assumptions H $[f, g]$, H $[\sigma, \sigma_0]$ and conditions 1), 2) of Theorem 2.3 hold. Then, there is at most one solution of problem (2.1)–(2.3).

2.2. Comparison Theorems for Mixed Problems

In dealing with applications of ordinary differential inequalities to partial differential equations, we have to estimate the solutions of such equations, which are functions of several variables, by functions of one variable. We denote by $C_{\mathrm{imp}}(J, R)$ the class of all functions $\omega : J \to R$ such that the restriction of ω to the set $J \setminus J_{\mathrm{imp}}$ is a continuous function and for every $x \in J_{\mathrm{imp}}$ there are the limits $\lim_{t\to x^+} \omega(t) = \omega(x^+)$, $\lim_{t\to x^-} \omega(t) = \omega(x^-)$. We assume also that $\omega(x) = \omega(x^+)$. Put $\Omega_0 = (E \setminus E_{\mathrm{imp}}) \times C_{\mathrm{imp}}(B, R)$ and suppose that

$$\sigma : (J \setminus J_{\mathrm{imp}}) \times R_+ \to R_+, \quad \sigma_0 : J_{\mathrm{imp}} \to R_+, \quad \lambda = (\lambda_1, \dots, \lambda_n) : \Omega_0 \to R^n$$

are given functions. In this Section we consider the functional differential inequalities of the form

$$\left| D_x z(x, y) - \sum_{i=1}^{n} \lambda_i(x, y, z_{(x,y)})\, D_{y_i} z(x, y) \right| \le \sigma(\, x, \|z_{(x,y)}\|_B\,), \tag{2.13}$$

where $(x, y) \in E \setminus E_{\mathrm{imp}}$, with the inequality for impulses

$$|\, \Delta z(x, y)\, | \le \sigma_0(x, \|z_{(x^-,y)}\|_{B^{(-)}}\,), \quad (x, y) \in E_{\mathrm{imp}}. \tag{2.14}$$

We prove a theorem which allows the estimate of a function satisfying the above inequalities by means of the extremal solution of an adequate differential problem with impulses.

Assumption H $[\lambda, \sigma, \sigma_0]$. Suppose that

1) $\lambda : \Omega_0 \to R^n$ is the function such that

$$\lambda_i(x, y, w) \ge 0 \quad \text{on } \Omega_0 \text{ for } 1 \le i \le \kappa,$$
$$\lambda_i(x, y, w) \le 0 \quad \text{on } \Omega_0 \quad \text{for } \kappa + 1 \le i \le n,$$

2) $\sigma : (J \setminus J_{\mathrm{imp}}) \times R_+ \to R_+$ and $\sigma_0 : J_{\mathrm{imp}} \times R_+ \to R_+$ are continuous and for every $\eta \in R_+$ there exists on J the right hand maximum solution $\omega(\,\cdot\,, \eta)$ of the problem

$$\omega'(x) = \sigma(x, \omega(x)\,) \quad \text{for } x \in J \setminus J_{\mathrm{imp}}, \tag{2.15}$$

$$\Delta\omega(x) = \sigma_0(x, \omega(x^-)\,) \quad \text{for } x \in J_{\mathrm{imp}}, \quad \omega(0) = \eta. \tag{2.16}$$

3) for each $x \in J_{\mathrm{imp}}$ the function $\gamma_0(p) = p + \sigma_0(x, p)$, $p \in R_+$, is nondecreasing on R_+ and σ is nondecreasing with respect to the second variable.

In the sequel we will use the following lemma

Lemma 2.6. *Suppose that*

1) *the functions* $\tilde{\sigma} : (J \setminus J_{\mathrm{imp}}) \times R_+ \to R_+,\ \tilde{\sigma}_0 : J_{\mathrm{imp}} \times R_+ \to R_+$ *are given and* $\tilde{\gamma}(p) = p + \tilde{\sigma}_0(x,p)$ *is nondecreasing on* R *with fixed* $x \in J_{\mathrm{imp}}$,

2) $\alpha,\ \beta \in C_{\mathrm{imp}}(J,R),$ *and* $\alpha(0) < \beta(0)$,

3) *denoted by*

$$\tilde{T}_+ = \{\, x \in (0,a) :\ \alpha(t) < \beta(t)\ for\ t \in [0,x)\ and\ \alpha(x) = \beta(x)\,\}\,,$$

we assume that

$$D_-\alpha(x) - \tilde{\sigma}(x, \alpha(x)\,) < D_-\beta(x) - \tilde{\sigma}(x, \beta(x)\,) \ \ for\ \ x \in \tilde{T}_+ \setminus J_{\mathrm{imp}},$$

and

$$\Delta\alpha(x) - \tilde{\sigma}_0(x, \alpha(x^-)\,) < \Delta\beta(x) - \tilde{\sigma}_0(x, \beta(x^-)\,) \ \ for\ \ x \in J_{\mathrm{imp}}.$$

Then $\alpha(x) < \beta(x)$ *for* $x \in J$.

We omit the proof of the Lemma 2.6; see [24], [25] for analogous results for problems without impulses.

Theorem 2.7. *Suppose that Assumption* $H\,[\,\lambda,\, \sigma,\, \sigma_0\,]$ *is satisfied.*

Let $u \in C_{\mathrm{imp}}(E^*, R)$ *be a function such that*

1) *the derivatives* $D_x u,\ D_y u$ *exist on* $E \setminus E_{\mathrm{imp}}$ *and*

$$|\,u(x,y)\,| \le \eta \ \ on\ \ E_0 \cup \partial_0 E, \tag{2.17}$$

2) u *satisfies the differential inequality (*2.13*) on* $E \setminus E_{\mathrm{imp}}$ *and the estimate for impulses*

$$|\,\Delta u(x,y)\,| \le \sigma_0(\,x, \Gamma u_{(x^-,y)}\,) \ \ on\ \ E_{\mathrm{imp}}$$

holds true, where

$$\Gamma u_{(x^-,y)} = \|u_{(x^-,y)}\|_{B^{(-)}} \ \ if\ \ \tau_0 > 0 \ \ and\ \ \Gamma u_{(x^-,y)} = |\,u(x^-,y)\,| \ \ if\ \ \tau_0 = 0.$$

Under these assumptions we have

$$|\,u(x,y)\,| \le \omega(x,\eta) \ \ on\ \ E \tag{2.18}$$

where $\omega(\,\cdot\,,\eta)$ *is the right hand maximum solution of* (2.15), (2.16).

Proof. Let us define

$$\varphi(x) = \max\{\,|\,u(t,y)\,| : \quad (t,y) \in E^* \cap (\,[-\tau_0, x] \times R^n\,)\,\},\ x \in [0,a).$$

Then $\varphi \in C_{\mathrm{imp}}(J,R)$ and estimation (16) is equivalent to $\varphi(x) \le \omega(x,\eta)$, $x \in J$. Let $\tilde{a} \in (a_k, a)$ be fixed. At first we prove that

$$\varphi(x) \le \omega(x,\eta) \quad \text{on} \ \ [0,\tilde{a}). \tag{2.19}$$

Let $\omega(\,\cdot\,,\eta,\varepsilon)$, $\varepsilon=(\varepsilon_0,\varepsilon_1,\varepsilon_2)$, be the right hand maximum solution of the problem

$$\omega'(x)=\sigma(x,\omega(x))+\varepsilon_0 \quad \text{for } \ x\in J\setminus J_{\text{imp}}, \tag{2.20}$$

$$\Delta\omega(x)=\sigma_0(x,\omega(x^-))+\varepsilon_1 \quad \text{for } x\in J_{\text{imp}}, \quad \omega(0)=\eta+\varepsilon_2, \tag{2.21}$$

where $\varepsilon_i>0$, $i=0,1,2$. There is $\bar\varepsilon>0$ such that for $0<\varepsilon_i<\bar\varepsilon$, $i=0,1,2$, the solution $\omega(\,\cdot\,,\eta,\varepsilon)$ is defined on $[0,\tilde a)$. It is enough to show that

$$\varphi(x)<\omega(x,\eta,\varepsilon), \ \ x\in[0,\tilde a), \tag{2.22}$$

because $\lim_{\varepsilon\to 0}\omega(x,\eta,\varepsilon)=\omega(x,\eta)$ uniformly on $[0,\tilde a)$. We will prove relation (2.22) by using Lemma 2.6. It follows from assumption (2.17) that $\varphi(0)<\omega(0,\eta,\varepsilon)$. Thus, put

$$T_+=\{\,x\in(0,\tilde a): \ \ \varphi(t)<\omega(t,\eta,\varepsilon) \quad \text{for } t\in[0,x), \ \ \varphi(x)=\omega(x,\eta,\varepsilon)\,\}\,.$$

Let $\tilde x\in T_+$. There are two possibilities:
(a) $\tilde x\in T_+\setminus J_{\text{imp}}$, (b) $\tilde x\in T_+\cap J_{\text{imp}}$.

Let us consider the case (a). Then $\varphi(\tilde x)>0$ and there is $\tilde y$ such that $(\tilde x,\tilde y)\in E^*$ and $\varphi(\tilde x)=|\,u(\tilde x,\tilde y)\,|$. It follows from (2.17) that $(\tilde x,\tilde y)\in E\setminus E_{\text{imp}}$. Suppose that $\varphi(\tilde x)=u(\tilde x,\tilde y)$. Then $\|u_{(\tilde x,\tilde y)}\|_B=\varphi(\tilde x)$ and

$$D_{y_i}u(\tilde x,\tilde y)\ge 0 \quad \text{for } i\in I_+[\tilde x,\tilde y], \ \ D_{y_i}u(\tilde x,\tilde y)\le 0 \quad \text{for } i\in I_-[\tilde x,\tilde y],$$
$$D_{y_i}u(\tilde x,\tilde y))=0 \ \ \text{for } i\in I_0[\tilde x,\tilde y].$$

Because $(\tilde x,\tilde y)\notin\partial_0E$, we have $\{1,\dots,\kappa\}\cap I_+[\tilde x,\tilde y]=\emptyset$ and $\{\kappa+1,\dots,n\}\cap I_-[\tilde x,\tilde y]=\emptyset$. Hence, we deduce using assumption 2) that

$$D_-\varphi(\tilde x)\le\sigma(\tilde x,\|u_{(\tilde x,\tilde y)}\|_B)+ \\ +\sum_{i\in I_+[\tilde x,\tilde y]\cup I_-[\tilde x,\tilde y]}\lambda_i(\tilde x,\tilde y,u_{(\tilde x,\tilde y)})D_{y_i}u(\tilde x,\tilde y)\le\sigma(\,\tilde x,\varphi(\tilde x)\,),$$

and consequently

$$D_-\varphi(\tilde x)<\sigma(\tilde x,\varphi(\tilde x))+\varepsilon_0. \tag{2.23}$$

If $\varphi(\tilde x)=-u(\tilde x,\tilde y)$ then we prove (2.23) in a similar way.

Consider the case (b). For each y such that $(\tilde x,y)\in E\setminus E_{\text{imp}}$ we have $|\,u(\tilde x^-,y)\,|\le\varphi(\tilde x^-)$ and $\|u_{(\tilde x,y)}\|_{B^{(-)}}\le\varphi(\tilde x^-)$. There is $\tilde y$ such that $(\tilde x,\tilde y)\in(E\setminus E_{\text{imp}})$ and $\varphi(\tilde x)=|\,u(\tilde x,\tilde y)\,|$. If $r_0=0$ then we have from assumption 2)

$$\varphi(\tilde x)=|\,u(\tilde x,\tilde y)\,|\le|\,u(\tilde x^-,\tilde y)\,|+\sigma_0(\,\tilde x,|u(\tilde x^-,\tilde y)|\,)\le\varphi(\tilde x^-)+\sigma_0(\,\tilde x,\varphi(\tilde x^-)\,),$$

and consequently

$$\Delta\varphi(\tilde x)<\sigma_0(\tilde x,\varphi(\tilde x^-))+\varepsilon_1. \tag{2.24}$$

In a similar way we prove (2.24) in the case $\tau_0>0$. Thus we see that all the assumptions of Lemma 2.6 are satisfied for $\alpha(x)=\varphi(x)$ and $\beta(x)=$

$\omega(x, \eta, \varepsilon)$, $x \in [0, \tilde{a})$. Then inequality (2.22) is satisfied. The constant $\tilde{a} \in (a_k, a)$ is arbitrary then the proof of Theorem 2.7 is completed, as observed earlier. □

Remark 2.8. It is easy to see that assumption (2.17) of Theorem 2.7 can be replaced by the following one. Suppose that $|\, u(x, y) \,| \le \eta$ for $(x, y) \in E_0$ and $|\, u(x, y) \,| \le \omega(x, \eta)$ for $(x, y) \in \partial_0 E$ where $\omega(\,\cdot\,, \eta)$ is the right hand maximum solution of (2.15), (2.16).

The next theorem allows us to estimate the difference of any solution of (2.1)–(2.3) and any solution of

$$D_x z(x, y) = \tilde{f}(x, y, z_{(x,y)}, D_y z(x, y)) \quad \text{on } E \setminus E_{\text{imp}}, \tag{2.25}$$

$$\Delta z(x, y) = \tilde{g}(x, y, z_{(x-,y)}) \quad \text{on } E_{\text{imp}}, \tag{2.26}$$

$$z(x, y) = \tilde{\varphi}(x, y) \quad \text{on } E_0 \cup \partial_0 E, \tag{2.27}$$

where $\tilde{f} : \Omega \to R$, $\tilde{g} : \Omega_{\text{imp}} \to R$ and $\tilde{\varphi} \in C_{\text{imp}}(E_0 \cup \partial_0 E, R)$ are given functions.

Theorem 2.9. *Suppose that*

1) *there are* σ *and* σ_0 *such that conditions* 2), 3) *of Assumption* $H\,[\,\lambda,\, \sigma,\, \sigma_0\,]$ *are satisfied and*

$$\left| f(\, x, y, w, q \,) - \tilde{f}(\, x, y, \bar{w}, q \,) \right| \le \sigma(\, x, \|w - \bar{w}\|_B \,) \ \ \textit{on} \ \ \Omega,$$

$$|\, g(\, x, y, p, w \,) - \tilde{g}(\, x, y, \bar{p}, \bar{w} \,) \,| \le \sigma_0(\, x, \ \|w - \bar{w}\|_{B(-)} \,) \ \ \textit{on} \ \ \Omega_{\text{imp}},$$

2) f *satisfies condition* 2) *of Assumption* $H[\, f, g\,]$ *and*

$$D_q f(x, y, w, \,\cdot\,) \in C(R^n, R^n)$$

for every $(x, y, w) \in (E \setminus E_{\text{imp}}) \times C^*_{\text{imp}}(B, R)$,

3) *the initial-boundary estimates*

$$|\, (\varphi - \tilde{\varphi})(x, y) \,| \le \eta \ \textit{on} \ E_0, \quad |\, (\varphi - \tilde{\varphi})(x, y) \,| \le \omega(x, \eta) \ \ \textit{on} \ \ \partial_0 E$$

are satisfied, where $\omega(\,\cdot\,, \eta)$ *is the right hand maximum solution of* (2.15), (2.16),

4) u *and* v *are solutions of problems* (2.1)–(2.3) *and* (2.25)–(2.27) *respectively.*

Then $|\, u(x, y) - v(x, y) \,| \le \omega(x, \eta)$ *for* $(x, y) \in E$.

Proof. Consider the function $\tilde{z} = u - v$. Then we have on $E \setminus E_{\text{imp}}$

$$\left| D_x \tilde{z}(x, y) - \sum_{i=1}^{n} \int_0^1 D_{q_i} f(x, y, u(x, y), u_{(x,y)}, P(x, y, t)) dt \; D_{y_i} \tilde{z}(x, y) \right|$$

$$\le \sigma(\, x, \|\tilde{z}_{(x,y)}\|_B \,)$$

where $P(x, y, t) = D_y v(x, y) + t[D_y u(x, y) - D_y v(x, y)]$ and

$$|\, \Delta \tilde{z}(x, y) \,| \le \sigma_0(\, x, \Gamma \tilde{z}_{(x-,y)} \,) \quad \text{for } (x, y) \in E_{\text{imp}},$$

where Γ is defined in Theorem 2.7. Hence the assertion follows from Theorem 2.7 which completes the proof of Theorem 2.9. □

As an immediate consequence of Theorem 2.9, we derive uniqueness and continuous dependence results.

Theorem 2.10. *Suppose that*

1) *there are* σ *and* σ_0 *such that condition* 2), 3) *of Assumption* $H[\lambda, \sigma, \sigma_0]$ *are satisfied and*

$$|\, f(x,y,w,q) - f(x,y,\bar{w},q)\,| \leq \sigma(\,x, \|w-\bar{w}\|_B\,) \;\; on \;\; \Omega,$$
$$|\, g(x,y,w) - g(x,y,\bar{w})\,| \leq \sigma_0(\,x,\; \|w-\bar{w}\|_{B(\cdot)}\,) \;\; on \;\; \Omega_{\rm imp},$$

2) f *satisfies condition* 2) *of Assumption* $H[\,f,g\,]$ *and*

$$D_q f(x,y,w,\,\cdot\,) \in C(R^n, R^n)$$

for every $(x,y,w) \in (E \setminus E_{\rm imp}) \times C^*_{\rm imp}(B,R)$,

3) $\omega(x,0) = 0$ *for* $x \in J$ *is the right hand maximum solution of* (2.15), (2.16) *with* $\eta = 0$.

Then, there is at most one solution of problem (2.1)–(2.3).

The above statement follows from Theorem 2.9 with $\tilde{f} = f$.

Remark 2.11. Suppose that the assumptions of Theorem 2.9 are satisfied and the function $\omega(x,0) = 0,\; x \in J$, is the right hand maximum solution of problem (2.15), (2.16) corresponding to $\eta = 0$. Then, for every $\varepsilon > 0$ there is $\delta > 0$ such that if

$$\Big|\, f(x,y,w,q) - \tilde{f}(x,y,\bar{w},q)\,\Big| < \delta \;\; \text{on} \;\; \Omega,$$
$$|\, g(x,y,w) - \tilde{g}(x,y,\bar{w})\,| < \delta \;\; \text{on} \;\; \Omega_{\rm imp},$$

and $|\,(\varphi - \tilde{\varphi})(x,y)\,| < \delta$ on $E_0 \cup \partial_0 E$, then $|\,u(x,y) - v(x,y)\,| < \varepsilon$ for $(x,y) \in E$.

The above statement follows from Theorem 2.9.

2.3. The Lines Method for Impulsive Functional Differential Problems

We define a mesh on E^* with respect to the spatial variable. Assume that for a given $h' = (h_1, \ldots, h_n)$ where $h_i > 0$ for $1 \leq i \leq n$, there exists $(N_1, \ldots, N_n) = N \in Z^n$ such that $N * h' = \tau$. Denote by Θ' the set of all h' having the above property. We assume that $\Theta' \neq \emptyset$ and that there is a sequence

$$\Big\{\; h'^{(j)} \;\Big\}_{j=0}^{\infty}, \quad h'^{(j)} = \Big(\, h_1^{(j)}, \ldots, h_n^{(j)} \Big) \in \Theta',$$

such that $\lim_{j\to\infty} h'^{(j)} = 0$. We define nodal points as follows: let $m = (m_1, \ldots, m_n) \in Z^n$, then

$$y^{(m)} = m * h' \quad \text{and} \quad y^{(m)} = \Big(y_1^{(m_1)}, \ldots, y_n^{(m_n)}\Big)\,.$$

Write

$$R^{1+n}_{x.h'} = \{\,(x, y^{(m)}) : \;\; x \in R, \;\; m \in Z^n\,\}.$$

We define the sets

$$B_{h'} = B \cap R^{1+n}_{x.h'}, \;\; E_{0.h'} = E_0 \cap R^{1+n}_{x.h'}, \;\; E_{h'} = E \cap R^{1+n}_{x.h'}, \;\; \partial_0 E_{h'} = \partial_0 E \cap R^{1+n}_{x.h'},$$

and

$$E^*_{h'} = E_{0.h'} \cup E_{h'} \cup \partial_0 E_{h'}, \;\; E^{\text{imp}}_{h'} = \{\,(x, y^{(m)}) \in E_{h'} \;:\; x \in J_{\text{imp}}\,\},$$

Elements of the set $E^*_{h'}$ will be denoted by $(x, y^{(m)})$ or (x, y). For a function $z : E^*_{h'} \to R$ we write $z^{(m)}(x) = z(x, y^{(m)})$. Let $\delta = (\delta_1, \ldots, \delta_n)$ be the difference operator defined by (1.38), (1.39). We denote by $C_{\text{imp}}(E^*_{h'}, R)$ the class of all functions $z : E^*_{h'} \to R$ such that $z(\,\cdot\,, y) \in C_{\text{imp}}(I \cup J, R)$ for every fixed y.

Suppose that we have a sequence $\{t_1, \ldots, t_p\}$ such that $-\tau_0 \le t_1 < t_2 < \cdots < t_p \le 0$. Let $P_i = (t_i, t_{i+1})$ for $i = 1, \ldots, l-1$ and

$$P_0 = \emptyset \;\text{ if }\; -\tau_0 = t_1, \;\; P_0 = (-\tau_0, t_1)\; z \;\;\text{ if }\; -\tau_0 < t_1,$$
$$P_p = \emptyset \;\text{ if }\; t_p = 0, \;\;\; P_p = (t_p, 0) \;\text{ if }\; t_p < 0.$$

We denote by $C_{\text{imp}}[I, R]$ the class of all functions $\alpha : I \to R$ for which there is a sequence $\{\,t_1, \ldots, t_p\,\}$ (p and $t_1, \ldots, t_p$ depend on α) such that

(i) the functions $\alpha\,|_{P_i}$, $0 \le i \le p$, are continuous;

(ii) for each i, j, $1 \le i, j \le p$, $t_i > -\tau_0$, $t_j < 0$, there exist the limits $\lim_{t \to t_i,\, t < t_i} \alpha(t) = \alpha(t_i^-)$ and $\lim_{t \to t_j,\, t > t_j} \alpha(t) = \alpha(t_j^+)$;

(iii) $\alpha(t_j) = \alpha(t_j^+)$ for $1 \le j \le p-1$ and for $j = p$ if $t_p < 0$.

Let $C_{\text{imp}}[B_{h'}, R]$ be the set of all functions $w : B_{h'} \to R$ such that $w^{(m)}(\,\cdot\,) \in C_{\text{imp}}[I, R]$ for every fixed $y^{(m)}$. In the case $\tau_0 > 0$ we define also

$$B^{(-)}_{h'} = \{\,(x, y^{(m)}) \in B_{h'} \;:\; -\tau_0 \le x < 0\,\}$$

and

$$C_{\text{imp}}[B^{(-)}_{h'}, R] = \{\,w\,|_{B^{(-)}_{h'}} : \; w \in C_{\text{imp}}[B_{h'}, R]\,\}.$$

Elements of the sets $C_{\text{imp}}[B_{h'}, R]$ and $C_{\text{imp}}(B^{(-)}_{h'}, R)$ will be denoted by the same symbols. For $w \in C_{\text{imp}}[B_{h'}, R]$ we write

$$\|\,w\,\|_{B_{h'}} = \sup\{\,|\,w^{(m)}(x)\,| \;:\; (x, y^{(m)}) \in B_{h'}\,\}.$$

We will denote by $\|\cdot\|_{B^{(-)}_{h'}}$ the supremum norm in the space $C_{\text{imp}}[B^{(-)}_{h'}, R]$ in the case $\tau_0 > 0$.

Suppose that $z : E^*_{h'} \to R$ and $(x, y^{(m)}) \in E_{h'}$. We define a function $z_{(x,m)} : B_{h'} \to R$ as follows:

$$z_{(x,m)}(t, y) = z(x + t, y^{(m)} + y), \;\; (t, y) \in B_{h'}.$$

If $\tau_0 > 0$ then we define also $z_{(x^-,m)} : B_{h'}^{(-)} \to R$ by

$$z_{(x^-,m)}(t,y) = z(x+t, y^{(m)} + y), \quad (t,y) \in B_{h'}^{(-)}.$$

Write

$$\Omega_{h'} = (E_{h'} \setminus E_{h'}^{\mathrm{imp}}) \times C_{\mathrm{imp}}[B_{h'}, R] \times R^n, \quad \Omega_{h'}^{\mathrm{imp}} = E_{h'}^{\mathrm{imp}} \times C_{\mathrm{imp}}[B_{h'}^{(-)}, R].$$

Suppose that for $h' \in \Theta'$ the functions $f_{h'} : \Omega_{h'} \to R$, $g_{h'} : \Omega_{h'}^{\mathrm{imp}} \to R$, $\varphi_h : E_{0.h'} \cup \partial_0 E_{h'} \to R$, are given. Applying the method of lines to (2.1)–(2.3) we obtain the system of ordinary functional differential equations with impulses

$$D_x z^{(m)}(x) = f_{h'}(\, x, y^{(m)}, z_{(x,m)}, \delta z^{(m)}(x)\,) \quad \text{on } \; E_{h'} \setminus E_{h'}^{\mathrm{imp}} \tag{2.28}$$

$$\Delta z^{(m)}(x) = g_{h'}(\, x, y^{(m)}, z_{(x^-,m)}\,) \quad \text{on } \; E_{h'}^{\mathrm{imp}} \tag{2.29}$$

$$z^{(m)}(x) = \varphi_{h'}(x, y^{(m)}) \quad \text{on } \; E_{0.h'} \cup \partial_0 E_{h'}. \tag{2.30}$$

A function $\tilde{z} : E_{h'}^{*} \to R$ is a solution of (2.28)–(2.30) if $\tilde{z} \in C_{\mathrm{imp}}(E_{h'}^{*}, R)$, there exists the derivative $D_x \tilde{z}$ on $E_{h'} \setminus E_{h'}^{\mathrm{imp}}$ and $\tilde{z}$ satisfies (2.28)–(2.30).

Let $F_{h'}$, $G_{h'}$ be the Niemycki operators corresponding to (2.28), (2.29), i.e.

$$F_{h'}[\, z\,]^{(m)}(x) = f_{h'}(\, x, y^{(m)}, z_{(x,m)}, \delta z^{(m)}(x)\,) \quad \text{on } \; E_{h'} \setminus E_{h'}^{\mathrm{imp}}$$

and

$$G_{h'}[\, z\,]^{(m)}(x) = g_{h'}(\, x, y^{(m)}, z_{(x^-,m)}\,) \quad \text{on } \; E_{h'}^{\mathrm{imp}}.$$

Now we prove a comparison theorem for differential difference inequalities with impulses. Write

$$S_{h'} = (E_{h'} \setminus E_{h'}^{\mathrm{imp}}) \times C_{\mathrm{imp}}[B_{h'}, R]$$

and suppose that

$$\lambda = (\lambda_1, \ldots, \lambda_n) : S_{h'} \to R, \;\; \sigma : (J \setminus J_{\mathrm{imp}}) \times R_+ \to R_+, \;\; \sigma_0 : J_{\mathrm{imp}} \times R_+ \to R_+$$

are given functions.

We deal with the differential difference inequalities

$$\left| D_x z^{(m)}(x) - \sum_{i=1}^{n} \lambda_i(x, y^{(m)}, z_{(x,m)})\, \delta_i z^{(m)}(x) \right| \le \sigma(\, x, \|z_{(x,m)}\|_{B_{h'}}\,) \tag{2.31}$$

on $E_{h'} \setminus E_{h'}^{\mathrm{imp}}$ with the estimate for impulses

$$\left| \Delta z^{(m)}(x) \right| \le \sigma_0(\, x, \tilde{V}_{h'} z_{(x,m)}\,) \quad \text{on } \; E_{h'}^{\mathrm{imp}}, \tag{2.32}$$

where

$$\tilde{V}_{h'} z_{(x,m)} = |\, u^{(m)}(x^-)\,| \quad \text{if } \tau_0 = 0 \quad \text{and}$$

$$\tilde{V}_{h'} z_{(x,m)} = \|\, u_{(x^-,m)}\,\|_{B_{h'}^{(-)}} \quad \text{if } \; \tau_0 > 0.$$

We prove that the function $z : E^*_{h'} \to R$ satisfying the above inequalities can be estimated by a solution of an adequate ordinary differential equation with impulses. Let us formulate the following assumptions.

Assumption $\mathbf{H}_{\rm imp}[\,\lambda, \sigma, \sigma_0\,]$. Suppose that

1) the function $\lambda : S_{h'} \to R^n$ is such that

$$\lambda_i(x, y, w) \geq 0 \;\text{ on } S_{h'} \;\text{ for } 1 \leq i \leq \kappa \;\text{ and}$$
$$\lambda_i(x, y, w) \leq 0 \;\text{ on } S_{h'} \text{ for } \kappa + 1 \leq i \leq n$$

2) the functions $\sigma : (J \setminus J_{\rm imp}) \times R_+ \to R_+$ and $\sigma_0 : J_{\rm imp} \times R_+ \to R_+$ are continuous and for every $\eta \in R_+$ there exists on J the right hand maximum solution $\omega(\,\cdot\,, \eta)$ of the problem

$$\begin{aligned} \omega'(x) &= \sigma(x, \omega(x)) \;\text{ on }\; J \setminus J_{\rm imp}, \\ \Delta\omega(x) &= \sigma_0(x, \omega(x^-)) \text{ on } \;J_{\rm imp},\;\; \omega(0) = \eta, \end{aligned} \tag{2.33}$$

3) for each $x \in J \setminus J_{\rm imp}$ and $t \in J_{\rm imp}$ the functions $\sigma(x, \,\cdot\,)$ and $\sigma_0(t, \,\cdot\,)$ are nondecreasing on R_+.

Now we prove the following comparison theorem.

Theorem 2.12. *Suppose that Assumption $H_{\rm imp}[\,\lambda, \sigma, \sigma_0\,]$ is satisfied and*

1) *the function $u \in C_{\rm imp}(E^*_{h'}, R)$ has the derivative $D_x u$ on $E_{h'} \setminus E^{\rm imp}_{h'}$, u satisfies differential difference inequality* (2.31) *and the estimate for impulses* (2.32) *holds,*

2) *the initial boundary estimate*

$$\left| u^{(m)}(x) \right| \leq \eta \;\text{ on }\; E_{0.h'} \cup \partial_0 E_{h'},\;\; \eta \in R_+, \tag{2.34}$$

is satisfied.

Under these assumptions we have

$$\left| u^{(m)}(x) \right| \leq \omega(x, \eta) \text{ on } E_{h'}, \tag{2.35}$$

where $\omega(\,\cdot\,, \eta)$ is the right hand maximum solution of (2.33).

Proof. Consider the function

$$\varphi(x) = \max\{\,|\,u^{(m)}(x)\,|\; : \;(x, y^{(m)}) \in E^*_{h'}\,\},\; x \in J.$$

Then $\varphi \in C_{\rm imp}(J, R_+)$ and $\varphi(0) \leq \eta$. We prove that

$$\varphi(x) \leq \omega(x, \eta),\; x \in J. \tag{2.36}$$

Let $\tilde{a} \in (a_k, a)$ be fixed and consider the problem

$$\omega'(x) = \sigma(\,x, \omega(x)\,) + \varepsilon_0,\; x \in J \setminus J_{\rm imp},$$

$$\Delta\omega(x) = \sigma_0(x, \omega(x^-)) + \varepsilon_1,\; x \in J_{\rm imp},\;\; \omega(0) = \eta + \varepsilon_2.$$

Denote by $\omega(\,\cdot\,, \eta, \varepsilon)$, $\varepsilon = (\varepsilon_0, \varepsilon_1, \varepsilon_2)$, the right hand maximum solution of the problem. There is $\tilde{\varepsilon} > 0$ such that for $0 < \varepsilon_i < \tilde{\varepsilon}$, $i = 0, 1, 2$, the solution $\omega(\,\cdot\,,\eta,\varepsilon)$ exists on $[0,\tilde{a})$ and $\lim_{\varepsilon\to 0}\omega(x,\eta,\varepsilon)=\omega(x,\eta)$ uniformly on $[0,\tilde{a})$. We prove that for the above ε we have

$$\varphi(x)<\omega(x,\eta,\varepsilon),\ \ x\in[0,\tilde{a}). \tag{2.37}$$

We will use Lemma 2.6. It follows from (2.34) that $\varphi(0)<\omega(0,\eta,\varepsilon)$. Let

$$T_+=\{\,x\in(0,\tilde{a})\ :\ \varphi(t)<\omega(t,\eta,\varepsilon)\ for\ t\in(0,x),\quad \varphi(x)=\omega(x,\eta,\varepsilon)\}.$$

Now we prove that

$$D_-\varphi(x)<\sigma(\,x,\varphi(x)\,)+\varepsilon_0\ \text{ for } x\in T_+\setminus J_{\text{imp}}. \tag{2.38}$$

Let $x\in T_+\setminus J_{\text{imp}}$. Then $\varphi(x)>0$ and there exists $m\in Z^n$ such that $(x,y^{(m)})\in E^*_{h'}$ and $\varphi(x)=|\,u^{(m)}(x)\,|$. It follows from (2.34) that $(x,y^{(m)})\in E_{h'}\setminus E^{\text{imp}}_{h'}$. Suppose that $\varphi(x)=u^{(m)}(x)$. Then $\|\,u_{(x,m)}\,\|_{B_{h'}}=\varphi(x)$ and

$$D_-\varphi(x)\le\sigma(x,\varphi(x))+\sum_{i=1}^{\kappa}\lambda_i(\,x,y^{(m)},u_{(x,m)})\frac{1}{h_i}\left[u^{(m+e_i)}(x)-u^{(m)}(x)\right]+$$
$$+\sum_{i=\kappa+1}^{n}\lambda_i(\,x,y^{(m)},u_{(x,m)})\frac{1}{h_i}\left[\,u^{(m)}(x)-u^{(m-e_i)}(x)\right]\le\sigma(x,\varphi(x)).$$

Thus, we get

$$D_x\varphi(x)<\sigma(\,x,\varphi(x)\,)+\varepsilon_1. \tag{2.39}$$

In a similar way we prove (2.39) in the case when $\varphi(x)=-u^{(m)}(x)$.

Now we show that

$$\Delta\varphi(x)<\sigma_0(x,\varphi(x^-))+\varepsilon_1\quad\text{for}\ \ x\in T_+\cap J_{\text{imp}}. \tag{2.40}$$

Let $x\in T_+\cap J_{\text{imp}}$. We have $\varphi(x)>0$. There exists $m\in Z^n$ such that $(x,y^{(m)})\in E^{\text{imp}}_h$ and $\varphi(x)=|\,u^{(m)}(x)\,|$. Then we have $\tilde{V}_{h'}u_{(x,m)}=\varphi(x^-)$. If $\tau_0=0$ then we get

$$\varphi(x)=|u^{(m)}(x)|\le|u^{(m)}(x^-)|+\sigma_0(x,|u^{(m)}(x^-)|)\le\varphi(x^-)+\sigma_0(x,\varphi(x^-)),$$

and (2.40) is proved. If $\tau_0>0$ the we obtain

$$\varphi(x)=|\,u^{(m)}(x)\,|\le|\,u^{(m)}(x^-)\,|+\sigma_0(\,x,\|\,u_{(x^-,m)}\,\|_{B^{(-)}_{h'}})\le$$
$$\le\|\,u_{(x^-,m)}\,\|_{B^{(-)}_{h'}}+\sigma_0(\,x,\|u_{(x^-,m)}\,\|_{B^{(-)}_{h'}}\,)\le\varphi(\,x^-)+\sigma_0(\,x,\varphi(x^-)\,).$$

Thus, we get (2.40). We obtain (2.37) by applying Lemma 2.6. Letting $\varepsilon\to 0$ in (2.37) we conclude (2.35) for $x\in[0,\tilde{a})$. Since the number $\tilde{a}\in(a_k,a)$ is arbitrary, then Theorem 2.12 follows. □

Now we prove a theorem on the existence of the approximate solution.

Assumption $\mathbf{H}_{\mathrm{imp}}[\,f_{h'}\,]$. Suppose that

1) the function $f_{h'} : \Omega_{h'} \to R$ of the variables (x, y, w, q) is continuous and it has partial derivatives $(D_{q_1} f_{h'}, \ldots, D_{q_n} f_{h'})$ on $\Omega_{h'}$ satisfying the conditions

$$D_{q_i} f_{h'}(x, y, p, w, q) \geq 0 \quad \text{on} \ \ \Omega_{h'} \quad \text{for} \ \ 1 \leq i \leq \kappa,$$
$$D_{q_i} f_{h'}(x, y, p, w, q) \leq 0 \quad \text{on} \ \ \Omega_{h'} \quad \text{for} \ \ \kappa + 1 \leq i \leq n,$$

2) the functions $\sigma : (J \setminus J_{\mathrm{imp}}) \times R_+ \to R_+$ and $\sigma_0 : J_{\mathrm{imp}} \times R_+ \to R_+$ satisfy conditions 2), 3) of Assumption $\mathrm{H}_{\mathrm{imp}}[\,\lambda, \sigma, \sigma_0\,]$ and

$$|\, f_{h'}(\, x, y, w, q\,) - f_{h'}(\, x, y, \bar{w}, q\,)\,| \leq \sigma(\, x, \|\, w - \bar{w}\,\|_{B_{h'}}\,) \quad \text{on} \ \ \Omega_{h'},$$
$$|\, g_{h'}(\, x, y, w\,) - g_{h'}(\, x, y, , \bar{w}\,)\,| \leq \sigma_0(\, x, \|\, w - \bar{w}\,\|_{B^{(-)}_{h'}}\,) \quad \text{on} \ \ \Omega^{\mathrm{imp}}_{h'},$$

3) the function $\tilde{\omega}(x) = 0$ is the unique solution of the initial problem with impulses

$$\omega'(x) = \sigma(x, \omega(x)), \ \ x \in J \setminus J_{\mathrm{imp}},$$
$$\Delta\omega(x) = \sigma_0(x, \omega(x^-)), \ x \in J_{\mathrm{imp}}, \ \ \omega(0) = 0.$$

Assumption $\mathbf{H}_{\mathrm{imp}}[\,z_0\,]$. Suppose that

1) the function $z_0 \in C(E^*_{h'}, R)$ is such that there exists $D_x z_0$ on $E_{h'} \setminus E^{\mathrm{imp}}_{h'}$ and z_0 satisfies the initial boundary condition (2.30),

2) there are function $\gamma_0, \gamma \in C(J, R_+)$ such that

$$\left| D_x z_0^{(m)}(x) - F_{h'}[\,z_0\,]^{(m)}(x) \right| \leq \gamma(x) \quad \text{on} \ E_{h'} \setminus E^{\mathrm{imp}}_{h'},$$
$$\left| \Delta z_0^{(m)}(x) - G_{h'}[\,z_0\,]^{(m)}(x) \right| \leq \gamma_0(x) \quad \text{on} \ \ E^{\mathrm{imp}}_{h'},$$

and there exists on J the right hand maximum solution w_0 of the problem

$$\omega'(x) = \sigma(x, \omega(x)) + \gamma(x), \ \ x \in J \setminus J_{\mathrm{imp}}, \tag{2.41}$$
$$\Delta\omega(x) = \sigma_0(x, \omega(x^-)) + \gamma_0(x), \ \ x \in J_{\mathrm{imp}}, \ \ \omega(0) = 0. \tag{2.42}$$

Theorem 2.13. *If Assumptions* $H_{\mathrm{imp}}[\,F_{H'}\,]$ *and* $H_{imp}[\,z_0\,]$ *are satisfied, then there exists exactly one solution* $u_{h'} : E^*_{h'} \to R$ *of problem* (2.28)–(2.30).

Proof. In the first step we define the sequence $\{w_k\}_{k=0}^{\infty}$, where $w_k : J \to R_+$, in the following way: w_0 is given in Assumption $H_{\mathrm{imp}}[\,z_0\,]$ and

$$w_{k+1}(x) = \int_0^x \sigma(t, w_k(t))dt, \ \ x \in J \setminus J_{\mathrm{imp}},$$
$$w_{k+1}(x) = w_k(x^-) + \sigma_0(x, w_k(x^-)), \ \ x \in J_{\mathrm{imp}},$$

where $k \geq 0$. It is easy to see that $w_{k+1}(x) \leq w_k(x)$ on J and

$$\lim_{k\to\infty} w_k(x) = 0 \quad \text{uniformly on } J. \tag{2.43}$$

In the second step we define the sequence $\{z_k\}_{k=0}^{\infty}$, $z_k : E^*_{h'} \to R$, in the following way: z_0 is given in Assumption $H_{\mathrm{imp}}[\, z_0\,]$. If z_k is given then z_{k+1} is a solution of the system of differential equations

$$D_x z^{(m)}(x) = f_{h'}(\, x, y^{(m)}, (z_k)_{(x,m)}, \delta z^{(m)}(x)\,) \quad \text{on } E_{h'} \setminus E_{h'}^{\mathrm{imp}},$$

with impulses given by

$$\Delta z^{(m)}(x) = g_{h'}(\, x, y^{(m)}, (z_k)_{(x^-,m)}\,) \;\text{ on }\; E_{h'}^{\mathrm{imp}}$$

and with initial boundary condition (2.30). We prove that

$$\left| (z_{j+i} - z_j)^{(m)}(x) \right| \leq w_j(x) \quad \text{on } E_{h'}, \tag{2.44}$$

for $i,\, j \geq 0$. First we prove (2.44) for $j = 0$ and $i \geq 0$. It follows that estimate (2.44) is satisfied for $j = 0,\ i = 0$. If we assume that

$$\left| (z_j - z_0)^{(m)}(x) \right| \leq w_0(x) \quad \text{on } E_k$$

then using the Hadamard mean value theorem and Assumption $H_{\mathrm{imp}}[f_{h'}]$ we conclude that the functions $z_{j+1} - z_0$ satisfies the differential difference inequality

$$\left| D_x\,(z_{j+1}-z_0)^{(m)}\,(x) - \sum_{l=1}^{n} \int_0^1 D_{q_l} f_{h'}\,(\,Q_m(x,t))\,dt\,\delta_l\,(z_{j+1}-z_0)^{(m)}\,(x) \right| \leq$$

$$\leq \sigma(\, x, w_0(x)\,) + \gamma(x) \quad \text{on } E_{h'} \setminus E_{h'}^{\mathrm{imp}},$$

where

$$Q_m(x,t) = \Big(\, x, y^{(m)}, (z_j)_{(x,m)}, \delta z_0^{(m)}(x) + t\delta(z_{j+1} - z_0)^{(m)}(x) \Big)\,.$$

Now let us consider the estimate for impulses. For $(x, y^{(m)}) \in E_{h'}^{\mathrm{imp}}$ we have

$$\left| \Delta\,(z_{j+1} - z_0)^{(m)}\,(x) \right| \leq$$

$$\leq \left| g_{h'}(\, x, y^{(m)}, (z_j)_{(x^-,m)}\,) - g_{h'}(\, x, y^{(m)}, (z_0)_{(x^-,m)}\,) \right| +$$

$$+ \left| g_{h'}(\, x, y^{(m)}, (z_0)_{(x^-,m)}) - \Delta z_0^{(m)}(x) \right| \leq$$

$$\leq \sigma_0(\, x, \|\, (z_j - z_0)_{(x^-,m)}\, \|_{B_{h'}^{(-)}}\,) + \gamma_0(x) \leq$$

$$\leq \sigma_0(\, x, w_0(x^-)\,) + \gamma_0(x).$$

By virtue of Theorem 2.12 we get

$$\left| (z_{j+1} - z_0)^{(m)}(x) \right| \leq \omega_0(x) \quad \text{on } E_{h'}.$$

Thus (22) is proved for $i = 0$ and for all $j \geq 0$. Now suppose that we have the estimate

$$\left| (z_{j+i} - z_j)^{(m)}(x) \right| \leq w_j(x) \quad \text{on } E_{h'} \quad \text{for all } i \geq 0$$

with fixed $j \in N$. Then, using the Hadamard mean value theorem and Assumption $\mathrm{H}_{\mathrm{imp}}[\, f_{h'}\,]$ we conclude that the function $z_{j+1+i} - z_{j+1}$ satisfies the functional difference inequality

$$\Big| D_x \left(z_{j+1+i} - z_{j+1} \right)^{(m)}(x) -$$

$$- \sum_{l=1}^{n} D_{q_l} f_{h'}(\tilde{Q}_m(x,t))dt\, \delta_l \left(z_{j+1+i} - z_{j+1} \right)^{(m)}(x) \Big| \leq$$

$$\leq \sigma(x, w_j(x)) \quad \text{on } E_{h'} \setminus E_{h'}^{\mathrm{imp}},$$

where $\tilde{Q}$ is an intermediate point given by

$$\tilde{Q}_m(x,t) = \Big(x, y^{(m)}, (z_{j+i})_{(x,m)}, \delta z_{j+1}^{(m)}(x) + t\delta (z_{j+1+i} - z_{j+1})^{(m)}(x) \Big).$$

Now, for $(x, y^{(m)}) \in E_{h'}^{\mathrm{imp}}$, we get

$$\Big| \Delta \left(z_{j+i+1} - z_{j+1} \right)^{(m)}(x) \Big| \leq$$

$$\leq \sigma_0(\, x, \| (z_{j+i} - z_j)_{(x^-,m)} \|_{B_{h'}^{(-)}}) \leq \sigma_0(\, x, w_j(x^-)\,).$$

It follows from Theorem 2.12 that

$$\left| (z_{j+1+i} - z_{j+1})^{(m)}(x) \right| \leq \omega_*(x) \quad \text{on } E_{h'}$$

where $\omega_* : [0, a) \to R_+$ is the right hand maximum solution of the problem with impulses

$$\omega'(x) = \sigma(\, x, \omega_j(x)\,) \quad \text{on } J \setminus J_{\mathrm{imp}},$$
$$\Delta\omega(x) = \sigma_0(\, x, \omega_j(x^-)\,) \quad \text{on } J_{\mathrm{imp}}, \quad \omega(0) = 0.$$

Since $\omega_* = w_{j+1}$ then the proof of (2.44) is completed by induction. The conditions (2.43), (2.44) and the relations

$$z_{k+1}^{(m)}(x) = z_{k+1}^{(m)}(a_i) + \int_{a_i}^{x} f_{h'}(\, t, y^{(m)}, (z_k)_{(t,m)}, \delta z_{k+1}^{(m)}(t)\,)\, dt \quad \text{on } E_{h'} \setminus E_{h'}^{\mathrm{imp}},$$

$$z_{k+1}^{(m)}(x) = z_{k+1}^{(m)}(x^-) + g_{h'}(x, y^{(m)}, (z_k)_{(x^-,m)}) \quad \text{on } E_{h'}^{\mathrm{imp}},$$

where $k \geq 0$, imply that there exists exactly one solution $u_{h'} : E_{h'}^* \to R$ of problem (2.28)–(2.30). This proves the theorem. □

Now we prove the following stability property of problem (2.28)–(2.30).

Theorem 2.14. *Suppose that Assumptions $H_{\rm imp}[\,f_{h'}\,]$, $H_{\rm imp}[\,z_0\,]$ are satisfied and the functions $v_{h'} : E^*_{h'} \to R$ and $\gamma_0, \gamma_1, \gamma_2 : \Theta' \to R_+$ are such that*

$$\Big| D_x v^{(m)}_{h'}(x) - F_{h'}[\,v_{h'}\,] \Big| \le \gamma_1(h') \;\; on \;\; E_{h'} \setminus E^{\rm imp}_{h'}, \tag{2.45}$$

$$\Big| \Delta v^{(m)}_{h'}(x) - G_{h'}[\,v_{h'}\,] \Big| \le \gamma_2(h') \;\; on \;\; E^{\rm imp}_{h'}, \tag{2.46}$$

$$\Big| v^{(m)}_{h'}(x) - \varphi_{h'}(x, y^{(m)}) \Big| \le \gamma_0(h') \;\; on \;\; E_{0.h'} \cup \partial_0 E_{h'}$$

and $\lim_{h'\to 0} \gamma_i(h') = 0, \;\; 1 = 0, 1, 2.$

Then, there exists a function $\omega : J \times \Theta' \to R_+$ such that for $h' \in \Theta'$ we have

$$\Big| u^{(m)}_{h'}(x) - v^{(m)}_{h'}(x) \Big| \le \omega(x, h') \;\; on \;\; E_{h'}$$

where $u_{h'}$ is the solution of problem (2.28) - (2.30) *and* $\lim_{h'\to 0} \omega(x, h') = 0$ *uniformly on* $[0, \tilde a)$ *with arbitrary* $\tilde a \in (a_k, a)$.

Proof. Let $\tilde a \in (a_k, a)$ be fixed. Let $\omega(\,\cdot\,, h') : J \to R_+$ be the maximum solution of the problem with impulses

$$\omega'(x) = \sigma(\, x, \omega(x)\,) + \gamma_1(h'), \;\; x \in [0, \tilde a) \setminus J_{\rm imp}, \tag{2.47}$$

$$\Delta\omega(x) = \sigma_0(\, x, \omega(x^-)\,) + \gamma_2(h'), \;\; x \in J_{\rm imp}, \quad \omega(0) = \gamma_0(h). \tag{2.48}$$

There is $\bar a > 0$ such that for $\|h'\| \in (0, \bar a)$ the maximum solutions of the above problem is defined on $[0, \tilde a)$ and we have $\lim_{h'\to 0} \omega(x, h') = 0$ uniformly with respect to $x \in [0, \tilde a)$. The function $u_{h'} - v_{h'}$ satisfies the differential difference inequality

$$\left| D_x \left(u_{h'} - v_{h'}\right)^{(m)}(x) - \sum_{i=1}^{n} \int_0^1 D_{q_i} f_{h'}(\, P_m(x,t)\,) dt \, \delta_i \left(u_{h'} - v_{h'}\right)^{(m)}(x) \right| \le$$

$$\le \sigma(\, x, \|\, (u_{h'} - v_{h'})_{(x,m)} \,\|_{B_{h'}}) + \gamma_1(h') \quad \text{on} \;\; E^{\rm imp}_{h'},$$

where

$$P_m(x,t) = \Big(x, y^{(m)}, (u_{h'})_{(x,m)}, \delta v^{(m)}_{h'}(x) + t\delta(u_{h'} - v_{h'})^{(m)}(x) \Big)$$

and

$$\Big| \Delta \left(u_{h'} - v_{h'}\right)^{(m)}(x) \Big| \le \sigma_0(\, x, \tilde V_{h'}(u_{h'} - v_{h'})_{(x^-,m)}\,) + \gamma_1(h') \;\; \text{on} \;\; E^{\rm imp}_{h'}.$$

Applying Theorem 2.12 we obtain

$$\Big| (u_{h'} - v_{h'})^{(m)}(x) \Big| \le \omega(x, h') \;\; \text{on} \;\; E_{h'} \cap ([0, \tilde a) \times R^n\,)\,.$$

This completes the proof of Theorem 2.14. □

Now we prove the following convergence theorem.

Theorem 2.15. *Suppose that the Assumptions $H_{\mathrm{imp}}[\,f_{h'}\,]$, $H_{\mathrm{imp}}[\,z_0\,]$ are satisfied and*

1) $f \in C(\Omega, R)$, $g \in C(\Omega_{\mathrm{imp}}, R)$ *and* $v : E^* \to R$ *is a solution of problem* (2.1)–(2.3), *the function* $v\,|_{E\setminus E_{\mathrm{imp}}}$ *is of class* C^1 *and the derivatives* $D_x v$, $D_y v$ *are bounded on* $E \setminus E_{\mathrm{imp}}$,

2) *there exist functions* $\bar{\gamma}_0, \bar{\gamma}_1, \bar{\gamma}_2 : \Theta' \to R_+$ *such that*

$$\Big|\, f(x, y^{(m)}, (v_{h'})_{(x,m)}, \delta v_{h'}^{(m)}(x)) - F_{h'}[\,v_{h'}\,](x, y^{(m)}\,\Big| \leq$$
$$\leq \bar{\gamma}_1(h') \text{ on } E_{h'} \setminus E_{h'}^{\mathrm{imp}}, \tag{2.49}$$

$$\Big|\, g(\,x, y^{(m)}, (v_{h'})_{(x^-,m)}\,) - G_{h'}[\,v_{h'}\,]^{(m)}(x)\,\Big| \leq \bar{\gamma}_2(h') \text{ on } E_{h'}^{\mathrm{imp}}, \tag{2.50}$$

and

$$\Big|\,(\varphi_{h'} - \varphi)^{(m)}(x)\,\Big| \leq \bar{\gamma}_0(h') \quad \text{on } E_{0.h'} \cup \partial E_{h'},$$

where $v_{h'}$ *is the restriction of* v *to the set* $E_{h'}^*$.

Then there exists $\gamma : J \times \Theta' \to R_+$ *such that*

$$\Big|\,(u_{h'} - v_{h'})^{(m)}(x)\,\Big| \leq \gamma(x, h') \quad \text{on } E_{h'}$$

and $\lim_{h'\to 0} \gamma(x, h') = 0$ *uniformly on* $[0, \tilde{a})$ *with arbitrary* $\tilde{a} \in (a_k, a)$.

Proof. It follows from assumption 1) and from compatibility conditions (2.49), (2.50) that there are functions γ_1, $\gamma_2 : \Theta' \to R_+$ such that conditions (2.45), (2.46) are satisfied and $\lim_{h'\to 0} \gamma_i(h') = 0$ for $i = 1, 2$. Then Theorem 2.15 follows from Theorem 2.14. □

Now we give an example of the increment function f_h corresponding to (2.1), (2.2). We define the operator $T_{h'} : C_{\mathrm{imp}}[B_{h'}, R] \to C_{\mathrm{imp}}[B, R]$ as follows. Put

$$S_+ = \{s = (s_1, \ldots, s_n) : \ s_i \in \{\,0,\,1\,\} \text{ for } 1 \leq i \leq n\}.$$

Suppose that $w \in C_{\mathrm{imp}}[B_{h'}, R]$ and $(x, y) \in B$. There is $m \in Z^n$ such that $y^{(m)} \leq y \leq y^{(m+1)}$ where $m + 1 = (m_1 + 1, \ldots, m_n + 1)$ and $(x, y^{(m)})$, $(x, y^{(m+1)}) \in B_{h'}$. We define

$$(T_{h'} w)(x, y) = \sum_{s \in S_+} w^{(m+s)}(x) \left(\frac{y - y^{(m)}}{h'}\right)^s \left(1 - \frac{y - y^{(m)}}{h'}\right)^{1-s}$$

where

$$\left(\frac{y - y^{(m)}}{h'}\right)^s = \prod_{i=1}^{n} \left(\frac{y_i - y_i^{(m_i)}}{h_i}\right)^{s_i},$$
$$\left(1 - \frac{y - y^{(m)}}{h'}\right)^{1-s} = \prod_{i=1}^{n} \left(1 - \frac{y_i - y_i^{(m_i)}}{h_i}\right)^{1-s_i},$$

and we take $0^0 = 1$ in the above formulas. It is easy to see that the function $T_h w \in C_{\text{imp}}[B, R]$. Consider the system of ordinary functional differential equations with impulses

$$D_x z^{(m)}(x) = f\left(x, y^{(m)}, T_{h'} z_{(x,m)}, \delta z^{(m)}(x)\right) \tag{2.51}$$

$$\Delta z^{(m)}(x) = h(\,x, y^{(m)}, T_{h'} z_{(x^-,m)}\,) \tag{2.52}$$

with initial boundary condition (2.30). It is easy to formulate assumptions which are sufficient for the convergence of the method of lines (2.51), (2.52), (2.30). Let us mention that our method satisfies the compatibility condition (2.49), (2.50).

2.4. Functional Difference Equations with Impulses

(2.1)–(2.3) by solutions of adequate difference problems. We formulate a difference problem corresponding to (2.1) - (2.3). We define a mesh in E^* and B. Suppose that $h = (h_0, h')$ where $h' = (h_1, \ldots, h_n)$ stand for steps of the mesh. For $(i, m) \in Z^{1+n}$ where $m = (m_1, \ldots, m_n)$ we define $y^{(m)}$ by $y^{(m)} = (\,y^{(m_1)}, \ldots, y^{(m_n)}\,) = m * h'$ and $x^{(i)} = ih_0$. Denote by Θ the set of all h such that there are $M = (M_1, \ldots, M_n) \in Z^n$ and $M_0 \in Z$ such that $M * h' = \tau$ and $M_0 h_0 = \tau_0$. We assume that $\Theta \neq \emptyset$ and that there is a sequence $\{h^{(j)}\}$, $h^{(j)} \in \Theta$, such that $\lim_{j\to\infty} h^{(j)} = 0$. Let

$$R_h^{1+n} = \left\{\, (x^{(i)}, y^{(m)}) : \quad (i, m) \in Z^{1+n} \right\}.$$

We define the sets

$$E_{0.h} = E \cap R_h^{1+n}, \;\; B_h = B \cap R_h^{1+n}, \;\; E_h = E \cap R_h^{1+n},$$

and

$$\partial_0 E_h = \partial_0 E \cap R_h^{1+n}, \quad E_h^* = E_{0.h} \cup E_h \cup \partial_0 E_h.$$

Elements of the set E_h^* will be denoted by $(x^{(i)}, y^{(m)})$ or (x, y).

For a function $z : E_h^* \to R$ and a point $(x^{(i)}, y^{(m)}) \in E_h^*$ we write $z^{(i,m)} = z(x^{(i)}, y^{(m)})$ and

$$\|z\|_{i.h} = \max\{\, |z^{(j,m)}| : \;\; (x^{(j)}, y^{(m)}) \in E_h^*, \;\; j \le i\,\}.$$

For the above z and for a point $(x^{(i)}, y^{(m)}) \in E_h$ we define the function $z_{(i,m)} : B_h \to R$ by

$$z_{(i,m)}(t, s) = z(x^{(i)} + t, y^{(m)} + s), \;\; (t, s) \in B_h.$$

Write $y^{(m')} = (y^{(m)})'$ and $y^{(m'')} = (y^{(m)})''$. The function $z_{(i,m)}$ is the restriction of z to the set

$$\left([x^{(i)} - \tau_0, x^{(i)}] \times [y^{(m')}, y^{(m')} + \tau'] \times [y^{(m'')} - \tau'', y^{(m'')}]\right) \cap R_h^{1+n}$$

and this restriction is shifted to the set B_h. Let $\{ n_1, \ldots, n_k \}$, $n_i \in N$, be defined by $n_i h_0 < a_i \le (n_i + 1)h_0,\ i = 1, \ldots, k$. Write

$$E_h^{\rm imp} = \{ (x^{(i)}, y^{(m)}) \in E_h : \ \ i \in \{ n_1, \ldots, n_k \} \},$$

$$E'_h = \{ (x^{(i)}, y^{(m)}) \in E_h : \ \ 0 \le i \le N_0 - 1 \}.$$

The motivation for the definition of the set E'_h is the following. Approximate solutions of (2.1)–(2.3) are functions u_h defined on the mesh E_h. For the calculation of all values of u_h on E_h we will write a difference equation or equation for impulses at each point of the set E'_h.

We define difference operators $\delta_0,\ \delta = (\delta_1, \ldots, \delta_n)$ by (1.37)–(1.39). Let

$$\Omega_h = (E'_h \setminus E_h^{\rm imp}) \times F(B_h, R) \times R^n, \quad \Omega_h^{\rm imp} = E_h^{\rm imp} \times F(B_h, R)$$

and assume that for $h \in \Theta$ we have $f_h : \Omega_h \to R$, $g_h : \Omega_h^{\rm imp} \to R$, and $\varphi_h : E_{0.h} \cup \partial_0 E_h \to R$. We consider the initial boundary value problem

$$\delta_0 z^{(i,m)} = f_h(x^{(i)}, y^{(m)}, z_{(i,m)}, \delta z^{(i,m)}), \ \ (x^{(i)}, y^{(m)}) \in E'_h \setminus E_h^{\rm imp}, \quad (2.53)$$

$$z^{(i+i,m)} = z^{(i,m)} + g_h(x^{(i)}, y^{(m)}, z_{(i,m)}), \ \ (x^{(i)}, y^{(m)}) \in E_h^{\rm imp}, \quad (2.54)$$

$$z^{(i,m)} = \varphi_h^{(i,m)} \ \ \text{on} \ \ E_{0.h} \cup \partial_0 E_h. \quad (2.55)$$

We will approximate solutions of (2.1)–(2.3) by means of solutions of (2.53)–(2.55). It is evident that there exists exactly one solutions $u_h : E_h^* \to R$ of (2.53)–(2.54).

In this Section we assume that the functions f_h and g_h of the variables (x, y, w, q) and (x, y, w) respectively, satisfy some nonlinear estimates of the Perron type with respect to the functional variables. Now we formulate adequate comparison problems. Write

$$I_h = \{ x^{(i)} : \ -\tilde{N}_0 \le i \le 0 \}, \ \ J_h = \{ x^{(i)} : \ 0 \le i \le N_0 \},$$
$$J_h^{\rm imp} = \{ x^{(i)} : i \in \{ n_1 \ldots, n_k \} \ \}, \ \ J'_h = \{ x^{(i)} : \ 0 \le i \le N_0 - 1 \},$$

where $\tilde{N}_0 h_0 = \tau_0$, $N_0 h_0 < a \le (N_0{+}1)h_0$. For a function $\alpha : I_h {\cup} J_h \to R$ and for $0 \le i \le N_0$ we define a function $\alpha_{(i)} : I_h \to R$ by $\alpha_{(i)}(x) \ = \ \alpha(x^{(i)} + x)$, $x \in I_h$. Suppose that

$$\sigma_h : (J'_h \setminus J_h^{\rm imp}) \times F(I_h, R_+) \to R_+, \quad \tilde{\sigma}_h : J_h^{\rm imp} \times F(I_h, R_+) \to R_+$$

are given functions. We consider the difference problem with impulses

$$\eta^{(i+1)} = \eta^{(i)} + h_0 \sigma_h(x^{(i)}, \eta_{(i)}) \ \ \text{for} \ x^{(i)} \in J'_h \setminus J_h^{\rm imp}, \quad (2.56)$$

$$\eta^{(i+1)} = \eta^{(i)} + \tilde{\sigma}_h(x^{(i)}, \eta_{(i)}) \ \ \text{for} \ x^{(i)} \in J_h^{\rm imp}, \quad \eta^{(i)} = 0 \ \ \text{on} \ \ I_h. \quad (2.57)$$

For $w \in F(B_h, R)$ we define

$$\|w\|_{B_h} = \max \{ |w^{(i,m)}| : \ (x^{(i)}, y^{(m)}) \in B_h \},$$

and $|h| = h_0 + h_1 + \cdots + h_n$. For $z : E_h^* \to R$ we put

$$F_h[\,(i,m), z\,] \;=\; f_h(\,x^{(i)}, y^{(m)}, z_{(i,m)}, \delta z^{(m)}\,) \quad \text{on} \;\; E_h' \setminus E_h^{\text{imp}},$$
$$G_h[\,(i,m), z\,] \;=\; g_h(\,x^{(i)}, y^{(i,m)}, z_{(m)}\,) \quad \text{on} \;\; E_h^{\text{imp}}.$$

Assumption H $[\,f_h, g_h\,]$**.** Suppose that the functions f_h and g_h satisfy the conditions

1) for $(x, y, w) \in (E_h' \setminus E_h^{\text{imp}}) \times F(B_h, R)$ we have $f_h(x, y, w, \,\cdot\,) \in C(R^n, R)$ and there exist the derivatives

$$(D_{q_1} f_h(P), \ldots, D_{q_n} f_h(P)) = D_q f_h(P), \quad P = (x, y, w, q) \in \Omega_h,$$

2) $D_q f_h(x, y, w, \,\cdot\,) \in C(R^n, R^n)$ and for $P \in \Omega_h$ we have

$$D_{q_i} f_h(P) \geq 0 \quad \text{where} \;\; 1 \leq i \leq \kappa, \quad D_{q_i} f_h(P) \leq 0 \quad \text{where} \;\; \kappa + 1 \leq i \leq n,$$
$$1 \;-\; h_0 \sum_{i=1}^{n} \frac{1}{h_i} \,|\, D_{q_i} f_h(P) \,| \geq 0.$$

We prove a theorem on difference inequalities generated by (2.53)–(2.55).

Theorem 2.16. *Suppose that Assumption* $H[\,f_h, g_h\,]$ *is satisfied and*

1) *the functions* f_h *and* g_h *are nondecreasing with respect to the functional variables,*

2) *the functions* $u, \; v \; : \; E_h^* \to R$ *satisfy*

$$\delta_0 u^{(i,m)} - F_h[\,(i,m), u\,] \leq \delta_0 v^{(i,m)} - F_h[\,(i,m), v\,] \;\; \textit{on} \;\; E_h' \setminus E_h^{\text{imp}},$$
$$u^{(i+1,m)} - u^{(i,m)} - G_h[\,(i,m), u\,] \leq v^{(i+1)} - v^{(i,m)} - F_h[\,(i,m), v\,] \;\; \textit{on} \;\; E_h^{\text{imp}},$$

3) *the initial boundary inequality* $u^{(i,m)} \leq v^{(i,m)}$ *is satisfied on* $E_{0.h} \cup \partial_0 E_h$.

Under these assumptions we have $u^{(i,m)} \leq v^{(i,m)}$ *on* E_h.

Proof. We prove the above inequality by induction with respect to i. It follows from assumption 3) that it is fulfilled for $i = 0$. Suppose that $u^{(j,m)} \leq v^{(j,m)}$ for each $(x^{(j)}, y^{(m)}) \in E_h$ where $0 \leq j \leq i$. Define $\tilde{z} = u - v$. We prove that $\tilde{z}^{(i+1,m)} \leq 0$ for $(x^{(i+1)}, y^{(m)}) \in E_h$. If $(x^{(i)}, y^{(m)}) \in E_h' \setminus E_h^{\text{imp}}$ then

$$\tilde{z}^{(i+1,m)} \leq \tilde{z}^{(i,m))} +$$
$$+ \Big[\, f_h(\,x^{(i)}, y^{(m)}, v_{(i,m)}, \delta u^{(m)}\,) - f_h(\,x^{(i)}, y^{(m)}, v_{(i,m)}, \delta v^{(m)}\,) \Big] =$$
$$= \tilde{z}^{(i,m)} \left[\, 1 - h_0 \sum_{j=1}^{n} \frac{1}{h_j} \,\Big|\, D_{q_j} f_h(\tilde{P} \,\Big|\, \right] +$$
$$+ h_0 \sum_{j=1}^{\kappa} \frac{1}{h_j} D_{q_j} f_h(\tilde{P})\, \tilde{z}^{(i,m+e_j)} - h_0 \sum_{j=\kappa+1}^{n} \frac{1}{h_j} D_{q_j} f_h(\tilde{P})\, \tilde{z}^{(i,m-e_j)},$$

where $\tilde{P} \in \Omega_h$ is an intermediate point. It follows from assumption 2) that $\tilde{z}^{(i+1,m)} \le 0$.

If $(x^{(i)}, y^{(m)}) \in E_h^{\rm imp}$ then

$$\tilde{z}^{(i+1,m)} \le \tilde{z}^{(i,m)} + g_h(\, x^{(i)}, y^{(m)}, u_{(i,m)}\,) - g_h(\, x^{(i)}, y^{(m)}, v_{(i,m)}\,).$$

It follows from the monotonicity of the function g_h with respect to the functional variable that $\tilde{z}^{(i+1,m)} \le 0$. This completes the proof of the Theorem 2.16. □

Remark 2.17. Let $\Gamma_h : E'_h \times F(E^*_h, R) \to R$ be given by

$$\Gamma_h(x^{(i)}, y^{(m)}, z) = z^{(i,m)} + h_0\, F_h[\,(i,m), z\,] \ \text{ on } \ E'_h \setminus E_h^{\rm imp},$$
$$\Gamma_h(x^{(i)}, y^{(m)}, z) = z^{(i,m)} + G_h[\,(i,m), z\,] \ \text{ on } \ E_h^{\rm imp}.$$

The condition 3) of Assumption H $[\, f_h, g_h \,]$ is equivalent to the assumption that Γ_h is nondecreasing with respect to the functional argument. Theorem 2.16 can be proved by the method used in [62].

Now we give a general theorem on the convergence of the difference method for equations with impulses. Let $V_h : F(B_h, R) \to F(I_h, R_+)$ be the operator given by

$$(V_h w)(x^{(i)}) = \max \Big\{ \,|\, z^{(i,m)} \,| : \ \ (x^{(i)}, y^{(m)}) \in B_h \Big\}, \ \ x^{(i)} \in I_h.$$

Assumption H $[\,\sigma_h, \tilde{\sigma}_h\,]$**.** Suppose that

1) the functions σ_h and $\tilde{\sigma}_h$ are nondecreasing with respect to the functional variables and $\sigma_h(x^{(i)}, \theta_h) = 0$ on $J'_h \setminus J_h^{\rm imp}$, $\tilde{\sigma}_h(x^{(i)}, \theta_h) = 0$ on $J_h^{\rm imp}$ where $\theta_h : I_h \to R$ is given by $\theta_h(x) = 0$ for $x \in I_h$,

2) the difference problem with impulses (2.56), (2.57) is stable in the following sense: if $\eta_h : I_h \cup J_h \to R_+$ is the solution of the problem

$$\eta^{(i+1)} = \eta^{(i)} + h_0 \sigma_h(\, x^{(i)}, \eta_{(i)}\,) + h_0 \gamma(h) \ \ for\ x^{(i)} \in J'_h \setminus J_h^{\rm imp}, \tag{2.58}$$

$$\eta^{(i+1)} = \eta^{(i)} + \tilde{\sigma}_h(\, x^{(i)}, \eta_{(i)}\,) + \tilde{\gamma}(h)$$
$$\text{for } x^{(i)} \in J_h^{\rm imp}, \quad \eta^{(i)} = \alpha_0(h) \ on\ I_h, \tag{2.59}$$

where $\gamma,\ \tilde{\gamma},\ \alpha_0 : \Theta \to R_+$ and

$$\lim_{h\to 0} \gamma(h) = \lim_{h\to 0} \tilde{\gamma}(h) = \lim_{h\to 0} \alpha(h) = 0,$$

then there exists a function $\beta : \Theta \to R_+$ such that $\eta_h^{(i)} \le \beta(h),\ 0 \le i \le N_0$ and $\lim_{h\to 0} \beta(h) = 0$.

Assumption $H[\, f_h, g_h, \sigma_h, \tilde{\sigma}_h\,]$. Suppose that the functions f_h and g_h satisfy the conditions: there are σ_h and $\tilde{\sigma}_h$ satisfying Assumption H $[\,\sigma_h, \tilde{\sigma}_h\,]$ and such that

$$\Big|\, f_h(\, x^{(i)}, y^{(m)}, w, q\,) - f_h(\, x^{(i)}, y^{(m)}, \bar{w}, q\,) \Big| \le \sigma_h(\, x^{(i)}, V_h(w - \bar{w})\,) \quad \text{on} \ \ \Omega_h,$$

and

$$\Big| g_h(x^{(i)}, y^{(m)}, w) - g_h(x^{(i)}, y^{(m)}) \Big| \le \tilde{\sigma}_h(x^{(i)}, V_h(w - \bar{w})) \text{ on } \Omega_h^{\rm imp}.$$

The main theorem in this Section is the following.

Theorem 2.18. *Suppose that Assumptions $H[\sigma_h, \tilde{\sigma}_h]$, $H[f_h, g_h]$, $H[f_h, g_h, \sigma_h, \tilde{\sigma}_h]$ are satisfied and*

1) $f \in C(\Omega, R)$, $g \in C(\Omega_{\rm imp}, R)$, $\varphi \in C_{\rm imp}(E_0 \cup \partial_0 E, R)$ *and* $v \in C_{\rm imp}(E^*, R)$ *is the solution of problem* (2.1)–(2.3),

2) *the function* $v\,|_{E \setminus E_{\rm imp}}$ *is of class* C^2 *and partial derivatives of the second order of* v *are bounded on* $E \setminus E_{\rm imp}$,

3) $u_h : E_h^* \to R$ *is the solution of* (2.53)–(2.55) *and there exist* $\alpha : \Theta \to R_+$ *such that* $\Big| \varphi^{(i,m)} - \varphi_h^{(i,m)} \Big| \le \alpha(h)$ *on* $E_{0.h} \cup \partial_0 E_h$, *and* $\lim_{h\to 0} \alpha(h) = 0$,

4) *there exist functions* $\beta_1, \beta_2 : \Theta \to R_+$ *such that the compatibility conditions are satisfied*

$$\Big| F_h[(i,m), v_h] - f(x^{(i)}, y^{(m)}, v_{(P[i,m])}, \delta v^{(m)}) \Big| \le \beta_1(h) \;\; on \;\; E_h' \setminus E_h^{\rm imp} \tag{2.60}$$

and

$$\Big| G_h[(i,m), v_h] - g(x^{(i)}, y^{(m)}, v_{(P[i,m]^-)}) \Big| \le \beta_1(h) \;\; on \;\; E_h^{\rm imp} \tag{2.61}$$

where v_h *is the restriction of* v *to the set* E_h^*, $P[i,m] = (x^{(i)}, y^{(m)})$ *and* $\lim_{h\to 0} \beta_i(h) = 0$, $i = 1, 2$. *The function* $u_{(P[i,m]^-)}$ *is the restriction of* $u_{(P[m])}$ *to the set* $B^{(-)}$. *Then there exists* $\tilde{\beta} : \Theta \to R_+$ *such that*

$$\Big| u_h^{(i,m)} - v_h^{(i,m)} \Big| \le \tilde{\beta}(h) \;\; and \;\; \lim_{h\to 0} \tilde{\beta}(h) = 0. \tag{2.62}$$

Proof. Let the function $\Gamma_h : E_h' \to R$ be defined by

$$\delta_0 v_h^{(i,m)} = F_h[(i,m), v_h] + \Gamma_h^{(i,m)} \quad on \;\; E_h' \setminus E_h^{\rm imp},$$
$$v_h^{(i+1,m)} - v_h^{(i,m)} = G_h[(i,m), v_h] + \Gamma_h^{(i,m)} \quad on \;\; E_h^{\rm imp}.$$

It follows from the compatibility conditions (2.60), (2.61) that there are $\gamma,\ \tilde{\gamma} : \Theta \to R_+$ such that

$$\Big| \Gamma^{(i,m)} \Big| \le \gamma(h) \;\; on \;\; E_h' \setminus E_h^{\rm imp}, \qquad \Big| \Gamma^{(i,m)} \Big| \le \tilde{\gamma}(h) \;\; on \;\; E_h^{\rm imp}$$

and

$$\lim_{h\to 0} \gamma(h) = \lim_{h\to 0} \tilde{\gamma}(h) = 0.$$

Write

$$\bar{\omega}(x^{(i)}) = \max \Big\{ \Big| (u_h - v_h)^{(i,m)} \Big| : \;\; (x^{(i)}, y^{(m)}) \in E_h^* \Big\}, \quad x^{(i)} \in I_h \cup J_h.$$

Then $\bar\omega$ satisfies the difference functional inequalities

$$\bar\omega^{(i+1)} \le \bar\omega^{(i)} + h_0\sigma(\,x^{(i)}, \bar\omega_{(i)}\,) + h_0\gamma(h), \quad x^{(i)} \in J'_h \setminus J_h^{\rm imp},$$

$$\bar\omega^{(i+1)} \le \bar\omega^{(i)} + \tilde\sigma(\,x^{(i)}, \bar\omega_{(i)}\,) + \tilde\gamma(h), \quad x^{(i)} \in J_h^{\rm imp}$$

and $\bar\omega^{(i)} \le \alpha_0(h)$ on I_h.

Consider the difference problem

$$\eta^{(i+1)} = \eta^{(i)} + h_0\sigma_h(x^{(i)}, \eta_{(i)}) + h_0\gamma(h) \;\text{ for }\; x^{(i)} \in J'_h \setminus J_h^{\rm imp},$$
$$\eta^{(i+1)} = \eta^{(i)} + \tilde\sigma_h(x^{(i)}, \eta_{(i)}) + \tilde\gamma(h) = l = l\text{for } \; x^{(i)} \in J_h^{\rm imp},$$
$$\eta^{(i)} = \alpha_0(h) \;\text{ for }\; x^{(i)} \in I_h.$$

Let $\beta_h : I_h \cup J_h \to R_+$ be the solution of the above problem. Then $\bar\omega^{(i)} \le \beta_h^{(i)}$ on J_h. Now we obtain (2.54) from the stability of the solution of problem (2.56), (2.57). This completes the proof of Theorem 2.18. □

2.5. Nonlinear Estimates for Increment Functions

Now we consider examples of nonlinear estimates for f. We will consider the class of operators $T_{h.i} : F(B_h, R) \to F(B, R),\; h \in \Theta,\; 0 \le i \le N_0$, which are given in the following way. Let

$$S_+ = \{\, s = (s_0, s_1, \ldots, s_n) : \; s_i \in \{0,\, 1\} \;\text{ for } 0 \le i \le n \,\}$$

and

$$Y_{h.i}^{\rm imp} \;=\; \left\{ x^{(j)} \in I_h : \;\; x^{(j+i)} \in J_h^{\rm imp} \right\}, \quad i = 0, 1, \ldots, N_0.$$

Suppose that $w \in F(B_h, R),\; (x, y) \in B$ and $0 \le i \le N_0$. There exists $(j, m) \in Z^{1+n}$ such that $(x^{(j)}, y^{(m)}),\; (x^{(j+1)}, y^{(m+1)}) \in B_h$ where $x^{(j)} \le x \le x^{(j+1)},\;\; y^{(m)} \le y \le y^{(m+1)}$. If $x^{(j)} \notin Y_{h.i}^{\rm imp}$ then we define

$$(T_{h.i}w)\,(x, y) = \sum_{s \in S_+} w^{((j,m)+s)} \left(\frac{Y - Y^{(j,m)}}{h} \right)^s \; \left(1 - \frac{Y - Y^{(j,m)}}{h} \right)^{1-s},$$

where

$$\left(\frac{Y \;-\; Y^{(j,m)}}{h} \right)^s \;=\; \left(\frac{x \;-\; x^{(j)}}{h_0} \right)^{s_0} \prod_{\zeta=1}^{n} \left(\frac{y_\zeta \;-\; y_\zeta^{(m_\zeta)}}{h_\zeta} \right)^{s_\zeta}$$

and

$$\left(1 - \frac{Y - Y^{(j,m)}}{h} \right)^{1-s} = \left(1 - \frac{x - x^{(j)}}{h_0} \right)^{1-s_0} \prod_{\zeta=1}^{n} \left(1 - \frac{y_\zeta - y_\zeta^{(m_\zeta)}}{h_\zeta} \right)^{1-s_\zeta}$$

and we take $0^0 = 1$ in the above formulas. If $x^{(j)} \in Y^{\text{imp}}_{h.i}$ then there exists $\zeta,\ 1 \le \zeta \le k$ such that $x \in [(n_\zeta - i)h_0,\ (n_\zeta - i + 1)h_0]$. We define

$$(T_{h.i}w)\,(x,y) = (T_{h.i}w)\,((n_\zeta - i)h_0,\ y) \quad \text{if } x \in [(n_\zeta - i)h_0, a_\zeta - ih_0),$$
$$(T_{h.i}w)\,(x,y) = (T_{h.i}w)\,((n_\zeta - i + 1)h_0,\ y) \quad \text{if } x \in [a_\zeta - ih_0,\ (n_\zeta - i + 1)h_0].$$

Then we have

$$T_{h.i}\ :\ F(B_h, R) \to C_{\text{imp}}[B, R].$$

Consider the initial boundary value problem (2.1)–(2.3) and the difference method

$$\delta_0 z^{(i,m)} = f(\,x^{(i)}, y^{(m)}, T_{h.i}z_{(i,m)},\, \delta z^{(i,m)}\,),\ \ (x^{(i)}, y^{(m)}) \in E'_h \setminus E^{\text{imp}}_h,$$
$$z^{(i+1,m)} = z^{(i,m)} + g(\,x^{(i)}, y^{(m)}, T_{h.i}z_{(i,m)^-}\,),\ \ (x^{(i)}, y^{(m')}) \in E^{\text{imp}}_h,$$
$$z^{(i,m)} = \varphi^{(i,m)} \text{ on } E_{0.h} \cup \partial_0 E_h,$$

where $\delta_0,\ \delta = (\delta_1, \ldots, \delta_n)$ is defined by (1.37)–(1.39) and $T_{h,i}z_{(i,m)^-}$ is the restriction of $T_{h,i}z_{(i,m)}$ to the set $B^{(-)}$. Suppose that

$$\sigma : (J \setminus J_{\text{imp}}) \times R_+ \to R_+,\ \ \tilde\sigma : J_{\text{imp}} \times R_+) \to R_+$$

are given functions. We will consider the following comparison problem

$$\eta'(x) = \sigma(\,x, \eta(x)\,) \ \text{ for } x \in J \setminus J_{\text{imp}}, \tag{2.63}$$
$$\eta(x) = \eta(x^-) + \tilde\sigma(\,x, \eta(x^-)\,)\ for\ x \in\ J_{\text{imp}} \qquad \eta(0) = p_0, \tag{2.64}$$

where $p_0 \in R_+$. Suppose that σ and $\tilde\sigma$ satisfy the conditions

1) σ and $\tilde\sigma$ are continuous, $\sigma(\,x, 0\,) = 0$ for $x \in J \setminus J_{\text{imp}}$ and $\tilde\sigma(\,x, 0\,) = 0$ for $x \in J_{\text{imp}}$,

2) the function σ in nondecreasing with respect to $(x, p) \in (J \setminus J_{\text{imp}}) \times R_+$ and the function $\tilde\sigma$ is nondecreasing with respect to the second variable,

3) the function f and g satisfy the conditions

$$|\,f(\,x, y, w, q\,) - f(\,x, y, \bar w, q\,)\,| \ \le\ \sigma(\,x, \|w - \bar w\|_B\,) \ \text{ on } \Omega, \tag{2.65}$$
$$|\,g(\,x, y, w\,) - g(\,x, y, \bar w\,)\,| \ \le\ \tilde\sigma(\,x, \|w - \bar w\|_{B^{(-)}}\,) \ \text{ on } \Omega_{\text{imp}}. \tag{2.66}$$

We define $f_h : \Omega_h \to R$ and $g_h : \Omega^{\text{imp}}_h \to R$ by

$$f_h(\,x^{(i)}, y^{(m)}, w, q\,) \ = \ f(\,x^{(i)}, y^{(m)}, T_{h.i}w, q\,) \ \text{ on } \Omega_h,$$
$$g_h(\,x^{(i)}, y^{(m)}, w\,) = g(\,x^{(i)}, y^{(m)}, T_{h.i}w^{(-)}\,) \ \text{ on } \Omega^{\text{imp}}_h,$$

where $T_{h.i}w^{(-)}$ is the restriction of $T_{h.i}w$ to the set $B^{(-)}$.

It is easy to prove by induction with respect to n that

$$\sum_{s' \in S'_+} \left(\frac{y\ -\ y^{(m)}}{h'}\right)^{s'} \left(1\ -\ \frac{y\ -\ y^{(m)}}{h'}\right)^{1-s'} \ = \ 1, \quad y^{(m)} \le y \le y^{(m+1)},$$

where

$$S'_+ = \{\, s' = (s_1, \ldots, s_n : \ s_i \in \{0,\ 1\} \ \text{ for } 1 \le i \le n \,\}.$$

Then we have the following estimates:

$$\Big|\, f_h(\, x^{(i)}, y^{(m)}, w, q\,) - f_h(\, x^{(i)}, y^{(m)}, \bar{w}, q\,) \Big| \ \le\ \sigma(\, x^{(i)}, \|w - \bar{w}\|_{B_h}\,) \text{ on } \Omega_h,$$

and

$$\Big|\, g_h(\, x^{(i)}, y^{(m)}, w\,) - g_h(\, x^{(i)}, y^{(m)}, \bar{w}\,) \Big| \ \le\ \tilde{\sigma}(x^{(i)}, \|w - \bar{w}\|_{B_h}\,) \text{ on } \Omega_h^{\text{imp}}.$$

Thus we see that problem (2.58), (2.59) is equivalent to

$$\eta^{(i+1)} \ = \ \eta^{(i)} + h_0\sigma(\, x^{(i)}, \eta^{(i)}\,) + h_0\gamma(h) \ for\ x^{(i)} \in J'_h \setminus J_h^{\text{imp}}, \quad (2.67)$$

$$\eta^{(i+1)} = \eta^{(i)} + \tilde{\sigma}(\, x^{(i)}, \eta^{(i)}\,) + \tilde{\gamma}(h) \ for\ x^{(i)} \in J_h^{\text{imp}}, \quad \eta^{(0)} = \alpha_0(h). \quad (2.68)$$

Denote by $\eta_h : J_h \to R_+$ the solution of the above problem. Let $\tilde{\eta} : J \to R_+$ be the solution of the problem

$$\eta'(x) = \sigma(\, x, \eta(x)\,) + \gamma(h), \ x \in J \setminus J_{\text{imp}}, \quad (2.69)$$

$$\eta(x) = \eta(x^-) + \tilde{\sigma}(\, x, \eta(x^-)\,) + \tilde{\gamma}(h), \ x \in \ J_{\text{imp}}, \quad \eta(0) = \alpha_0(h). \quad (2.70)$$

Then we have

$$\tilde{\eta}^{(i+1)} \ge \tilde{\eta}^{(i)} + h_0\sigma(\, x^{(i)}, \tilde{\eta}^{(i)}\,) + h_0\gamma(h), \ \ x^{(i)} \in J'_k \setminus J_h^{\text{imp}},$$
$$\tilde{\eta}^{(i+1)} \ge \tilde{\eta}^{(i)} + \tilde{\sigma}(\, x^{(i)}, \tilde{\eta}^{(i)}\,), \ \ x^{(i)} \in J_h^{\text{imp}},$$

and consequently $\eta_h^{(i)} \ \le \ \tilde{\eta}^{(i)}$ for $0 \le i \le N_0$. If we assume that problem (2.69), (2.70) with $p_0 = 0$ has the right hand maximum solution $\zeta(x) = 0$ for $x \in J$ then the problem (2.67), (2.68) with $\gamma(h) = \tilde{\gamma}(h) = \alpha_0(h) = 0$ for $h \in \Theta$ is stable in the sense of Assumption H $[\,\sigma_h, \tilde{\sigma}_h\,]$.

Remark 2.19. All the results of this Chapter can be extended for systems of functional differential problems with impulses.

CHAPTER III
PARABOLIC FUNCTIONAL DIFFERENTIAL PROBLEMS WITH IMPULSES

3.1. Functional Differential Inequalities

We formulate the problem. Let $E = (0,a) \times (-b,b)$ where $a > 0$, $b = (b_1, \dots b_n) \in R^n$, and $b_i > 0$ for $1 \le i \le n$. Put $B = [-\tau_0, 0] \times [-\tau, \tau]$ where $\tau_0 \in R_+$, $\tau = (\tau_1, \dots, \tau_n) \in R^n_+$. Write $c = b + \tau$ and

$$E_0 = [-\tau_0, 0] \times [-c, c], \quad \partial_0 E = (0,a) \times (\,[-c,c] \setminus (-b,b)\,), \quad E^* = E \cup E_0 \cup \partial_0 E.$$

If $\tau_0 > 0$ then we put $B^{(-)} = [-\tau_0, 0) \times [-\tau, \tau]$. Suppose that $0 < a_1 < a_2 < \dots < a_k < a$ are given numbers. Let $I,\ J,\ J_{\rm imp}$ be the sets given by (1.1). Suppose that

$$E_{\rm imp},\ \partial_0 E_{\rm imp},\ E^*_{\rm imp},\ \ C_{\rm imp}(E^*, R),\ \ C_{\rm imp}(E_0 \cup \partial_0 E, R)$$

and

$$C_{\rm imp}[B, R],\ C_{\rm imp}[B^{(-)}, R]$$

are the sets defined in Section 2.1 with the above given $E,\ \partial_0 E,\ E_0,\ B,\ B^{(-)}$. Let $M_{n\times n}$ be the class of all $n \times n$ symmetric matrices with real elements. Write

$$\Sigma = (E \setminus E_{\rm imp}) \times R \times C_{\rm imp}[B,R] \times M_{n\times n}, \quad \Sigma_{\rm imp} = E_{\rm imp} \times R \times C_{\rm imp}[B^{(-)}, R]$$

and suppose that $f : \Sigma \to R$, $g : \Sigma_{\rm imp} \to R$, $\varphi : E_0 \cup \partial_0 E \to R$ are given functions. We assume that $\varphi \in C_{\rm imp}(E_0 \cup \partial_0 E, R)$. In this Chapter we consider the initial boundary value problem

$$D_x z(x,y) = f(x, y, z(x,y), z_{(x,y)}, D_y z(x,y), D_{yy} z(x,y)) \ \text{ on } E \setminus E_{\rm imp}, \tag{3.1}$$

$$\Delta z(x,y) = g(\,x, y, z(x^-, y), z_{(x^-,y)}\,) \ \text{ on } E_{\rm imp}, \tag{3.2}$$

$$z(x,y) = \varphi(x,y) \ \text{ on } E_0 \cup \partial_0 E, \tag{3.3}$$

where $D_y z(x,y) = (\,D_{y_1} z(x,y), \dots, D_{y_n} z(x,y)\,)$ and $D_{yy} z(x,y) = \big[\,D_{y_i y_j}\,\big]_{i,j=1,\dots,n}$. A function $z \in C_{\rm imp}(E^*, R)$ will be called the function of class $C^{(1,2)}_{\rm imp}(E^*, R)$ if z has continuous derivatives $D_x z,\ D_y z,\ D_{yy} z$ on $E \setminus E_{\rm imp}$. We consider solutions of class $C^{(1,2)}_{\rm imp}(E^*, R)$ of problem (3.1)–(3.3).

Example 2. Suppose that $\tilde f : (E \setminus E_{\rm imp}) \times R^2 \times R^n \times M_{n\times n}$ and $\tilde g : E_{\rm imp} \times R^2 \to R$ are given functions. We define f and g by

$$f(x,y,p,w,q,r) = \tilde f(x, y, p, \int_B w(t,s)\,dtds, q, r),$$

$$g(x,y,p,w) = \tilde g(x, y, p, \int_{B^{(-)}} w(t,s)\,dtds).$$

Then (3.1), (3.2) is equivalent to the integral differential equation

$$D_x z(x,y) = \tilde{f}(x,y,z(x,y), \int_B z(x+t,y+s)dtds,$$
$$D_y z(x,y),\ D_{yy} z(x,y)) \quad \text{on} \quad E \setminus E_{\text{imp}},$$

with impulses given by the relation

$$\Delta z(x,y) = \tilde{g}(x,y,z(x^-,y), \int_{B^{(-)}} z(x+t,y+s)dtds \ \text{ on } \ E_{\text{imp}}.$$

Example 3. Suppose that $\alpha,\, \bar{\alpha} : B \to R, \quad \beta,\, \bar{\beta} : B \to R^n$ and $\tilde{f},\ \tilde{g}$ are given in Example 2. Assume that

$$-\tau_0 \le \alpha(x,y) - x \le 0, \qquad -\tau_0 \le \bar{\alpha}(x,y) - x < 0,$$
$$-\tau \le \beta(x,y) - y \le \tau, \qquad -\tau \le \bar{\beta}(x,y) - y \le \tau,$$

where $(x,y) \in B$. Put

$$f(x,y,p,w,q,r) = \bar{f}(x,y,p,w(\alpha(x,y)-x,\beta(x,y)-y),q,r),$$
$$g(x,y,p,w) = \bar{g}(x,y,p,w(\bar{\alpha}(x,y)-x,\bar{\beta}(x,y)-y)).$$

Then (3.1), (3.2) reduces to the differential equation with a deviated argument

$$D_x z(x,y) = \bar{f}(\, x,y,z(x,y), z(\alpha(x,y),\beta(x,y)),$$
$$D_y z(x,y), D_{yy} z(x,y)\,) \ \text{ on } \ E \setminus E_{\text{imp}},$$

and with impulses given by

$$\Delta z(x,y) = \bar{g}(\, x,y,z(x-,y), z(\bar{\alpha}(x,y),\bar{\beta}(x,y))\,) \quad \text{on} \quad E_{\text{imp}}.$$

For any matrices

$$r,\, \bar{r} \in M_{n\times n}, \quad r = [\, r_{ij}\,]_{i,j=1,\dots,n}\,, \quad \bar{r} = [\, \bar{r}_{ij}\,]_{i,j=1,\dots,n}\,,$$

we write $r \le \bar{r}$ if

$$\sum_{i,j=1}^{n} (r_{ij} - \bar{r}_{ij})\lambda_i\lambda_j \le 0 \ \text{ for } \ \lambda = (\lambda_1,\dots,\lambda_n) \in R^n,$$

A function $f : \Sigma \to R$ is said to be parabolic with respect to $z \in C^{(1,2)}_{\text{imp}}(E^*,R)$ on $E \setminus E_{\text{imp}}$ and for any $r,\, \bar{r} \in M_{n\times n},\ r \le \bar{r}$, we have

$$f(x,y,z(x,y),z_{(x,y)},D_y z(x,y),r) \le f(x,y,z(x,y),z_{(x,y)},D_y z(x,y),\bar{r}).$$

For $f : \Sigma \to R,\ g : \Sigma_{\text{imp}} \to R$ and $z \in C^{(1,2)}_{\text{imp}}(E^*,R)$ we write

$$F[\,z\,](x,y) = f(\,x,y,z(x,y),z_{(x,y)},D_y z(x,y),D_{yy} z(x,y)\,) \ \ on \ \ E \setminus E_{\text{imp}},$$

and

$$G[z](x,y) = g(x,y,z(x^-,y),z_{(x^-,y)}) \text{ on } E_{\text{imp}}.$$

We start with a theorem on strong inequalities.

Assumption H $[f,g]$**.** Suppose that the functions f and g satisfy the conditions:

1) f is nondecreasing with respect to the functional variable,

2) g is nondecreasing with respect to the functional variable and the function

$$\gamma(p) = p + g(x,y,p,w), \; p \in R,$$

is nondecreasing for each $(x,y,w) \in E_{\text{imp}} \times C_{\text{imp}}[B^{(-)},R]$.

Theorem 3.1. *Suppose that Assumption H* $[f,g]$ *is satisfied and*

1) *the functions* $u,\, v \in C^{(1,2)}_{\text{imp}}(E^*,R)$ *satisfy the initial boundary inequality*

$$u(x,y) < v(x,y) \text{ on } E_0 \cup \partial_0 E, \tag{3.4}$$

2) *the functional differential inequality*

$$D_x u(x,u) - F[u](x,y) < D_x v(x,y) - F[v](x,y) \text{ on } E \setminus E_{\text{imp}} \tag{3.5}$$

and the inequality for impulses

$$\Delta u(x,y) - G[u](x,y) < \Delta v(x,y) - G[v](x,y) \text{ on } E_{\text{imp}}, \tag{3.6}$$

are satisfied,

3) *the function* f *is parabolic with respect to* u *on* $E \setminus E_{\text{imp}}$.

Then $u(x,y) < v(x,y)$ *on* E^*.

Proof. If the inequality is false then the set

$$J_+ = \{ x \in [0,a) : \text{ there is } y \in (-b,b) \text{ such that } u(x,y) \geq v(x,y) \}$$

is not empty. Defining $\tilde{x} = \inf J_+$ it follows from (3.4) that $\tilde{x} > 0$ and that there is $\tilde{y} \in (-b,b)$ such that

$$u(x,y) < v(x,y) \text{ on } E^* \cap ([-\tau_0,\tilde{x}) \times R^n), \quad u(\tilde{x},\tilde{y}) = v(\tilde{x},\tilde{y}). \tag{3.7}$$

There are two cases to be distinguished. If $(\tilde{x},\tilde{y}) \in E \setminus E_{\text{imp}}$ then $D_x(u - v)(\tilde{x},\tilde{y}) \geq 0$, $D_y(u-v)(\tilde{x},\tilde{y}) = 0$ and $D_{yy}u(\tilde{x},\tilde{y}) \leq D_{yy}v(\tilde{x},\tilde{y})$, which leads to a contradiction with (3.5).

Suppose now that $(\tilde{x},\tilde{y}) \in E_{\text{imp}}$. Then $\tilde{x} \in J_{\text{imp}}$ and $(u-v)(\tilde{x}^-,\tilde{y}) \leq 0$. It follows from (3.6) and from Assumption H $[f,g]$ that

$$(u-v)(\tilde{x},\tilde{y}) <$$
$$< (u-v)(\tilde{x}^-,\tilde{y}) + g(\tilde{x},\tilde{y},u(\tilde{x}^-,\tilde{y}),u_{(\tilde{x},\tilde{y})}) - g(\tilde{x},\tilde{y},v(\tilde{x}^-,\tilde{y}),v_{(\tilde{x},\tilde{y})}) \leq 0,$$

which contradicts (3.7). Hence J_+ is empty and the statement follows. □

Remark 3.2. In Theorem 3.1 we can assume instead of (3.5), (3.6) that

$$D_x u(x,u) - F[\,u\,](x,y) < D_x v(x,y) - F[\,v\,](x,y) \text{ for on } T_+ \setminus E_{\text{imp}}$$

and

$$\Delta u(x,y) - G[\,u\,](x,y) < \Delta v(x,y) - G[\,v\,](x,y) \text{ on } T_+ \cap \in E_{\text{imp}},$$

where

$$T_+ = \{(x,y) \in E : (u-v)(t,s) < 0 \text{ on } ([-\tau_0, x) \times R^n) \cap E \text{ and } (u-v)(x,y) = 0\}.$$

Now we consider weak impulsive functional differential inequalities.

Theorem 3.3. *Suppose that Assumption $H\,[\,f,g\,]$ is satisfied and*

1) *there are functions $\sigma : (J \setminus J_{\text{imp}}) \times R_+ \to R_+$ and $\sigma_0 : J_{\text{imp}} \times R_+ \to R_+$ satisfying Assumption $H\,[\,\sigma, \sigma_0\,]$ (Section 2.1) and such that*

$$f(x,y,p,w,q,r) - f(x,y,\bar{p},\bar{w},q,r) \ge -\sigma(\,x, \max\{\,\bar{p}-p,\ \|\bar{w}-w\|_B\}\,)$$

on Σ where $p \le \bar{p}$, $w \le \bar{w}$ and

$$g(x,y,p,w) - g(x,y,\bar{p},\bar{w}) \ge -\sigma_0(\,x, \max\{\,\bar{p}-p,\ \|\bar{w}-w\|_{B(-)}\}\,)$$

on Σ_{imp} where $p \le \bar{p}$, $w \le \bar{w}$,

2) *the function u, $v \in C^{(1,2)}_{\text{imp}}(E^*, R)$ satisfy the initial boundary inequality*

$$u(x,y) \le v(x,y) \text{ on } E_0 \cup \partial_0 E,$$

3) *f is parabolic with respect to u on $E \setminus E_{\text{imp}}$ and*

$$D_x u(x,u) - F[\,u\,](x,y) \le D_x v(x,y) - F[\,v\,](x,y) \quad \text{on } E \setminus E_{\text{imp}}$$

4) *the inequality for impulses*

$$\Delta u(x,y) - G[\,u\,](x,y) \le \Delta v(x,y) - G[\,v\,](x,y) \text{ on } E_{\text{imp}},$$

are satisfied.

Then $u(x,y) \le v(x,y)$ on E^*.

Proof. Suppose that $\tilde{a} \in (a_k, a)$. We prove that

$$u(x,y) \le v(x,y) \text{ on } ([-\tau_0, \tilde{a}) \times R^n) \cap E^*. \tag{3.8}$$

Consider problem (2.10), (2.11). There exists $\bar{\varepsilon} > 0$ such that for $0 < \varepsilon_i < \bar{\varepsilon}$, $i = 0,1,2$, the maximum solution $\omega(\,\cdot\,,\varepsilon)$, $\varepsilon = (\varepsilon_0, \varepsilon_1, \varepsilon_2)$ of (2.10), (2.11) is defined on $[0, \tilde{a})$. Let $\tilde{v}(x,y) = v(x,y) + \varepsilon_0$ on E_0 and $\tilde{v}(x,y) = v(x,y) + \omega(x,\varepsilon)$ on $([0,\tilde{a}) \times R^n) \cap (E \cup \partial_0 E)$. Then $u(x,y) < \tilde{v}(x,y)$ on $(E_0 \cup \partial_0 E) \cap ([-\tau_0, \tilde{a}) \times R^n)$ and

$$D_x u(x,u) - F[u](x,y) < D_x \tilde{v}(x,y) - F[\tilde{v}](x,y) \text{ on } (E \setminus E_{\text{imp}}) \cap ([0,\tilde{a}) \times R^n)$$
$$\Delta u(x,y) - G[\,u\,](x,y) < \Delta \tilde{v}(x,y) - G[\,\tilde{v}\,](x,y) \text{ on } E_{\text{imp}}.$$

It follows from Theorem 3.1 that

$$u(x,y) < v(x,y) \ \text{ on } \ E^* \cap ([-\tau_0, \tilde{a}) \times R^n).$$

Since $\lim_{\varepsilon \to 0} \omega(x,\varepsilon) = 0$ uniformly on $[0,\tilde{a})$, we obtain (3.8). The constant $\tilde{a} \in (a_k, a)$ is arbitrary and therefore the proof is completed. □

3.2. Comparison Results for Parabolic Problems

We prove estimates of functions satisfying impulsive parabolic inequalities by means of solutions of impulsive ordinary functional differential equations.

Let $C_{\text{imp}}[I,R]$ be the set define in Section 2.3. We define also in the case $\tau_0 > 0$

$$I^{(-)} = [-\tau_0, 0) \ \text{ and } \ C_{\text{imp}}[I^{(-)}, R] = \{ \eta \mid_{I^{(-)}} : \ \ \eta \in C_{\text{imp}}[I,R] \}.$$

Elements of the sets $C_{\text{imp}}[I,R]$ and $C_{\text{imp}}[I^{(-)},R]$ will be denoted by the same symbols. We denote by $\| \cdot \|_I$ and $\| \cdot \|_{I^{(-)}}$ the supremum norms in the space $C_{\text{imp}}[I,R]$ and $C_{\text{imp}}[I^{(-)},R]$ respectively.

Suppose that $\omega : I \cup J \to R$ and $x \in J$. Then $\omega_{(x)} : I \to R$ is the function defined by $\omega_{(x)}(t) = \omega(x+t), \ t \in I$. If $\tau_0 > 0$ then for the above ω and x we will consider also the function $\omega_{(x^-)} : I^{(-)} \to R$ given by $\omega_{(x^-)}(t) = \omega(x+t), \ t \in I^{(-)}$.

Assumption H $[\sigma, \tilde{\sigma}]$**.** Suppose that

1) the functions $\sigma : (J \setminus J_{\text{imp}}) \times R_+ \times C_{\text{imp}}[I, R_+] \to R_+$ and $\tilde{\sigma} : J_{\text{imp}} \times R_+ \times C_{\text{imp}}[I^{(-)}, R_+] \to R_+$ are continuous and nondecreasing with respect to the functional variables,

2) for each $(x,\eta) \in (J \setminus J_{\text{imp}}) \times C_{\text{imp}}[J_0^{(-)}, R]$ the function $\gamma(p) = p + \tilde{\sigma}(x,p,\eta)$ is nondecreasing on R_+

3) for each $\eta \in C(I,R)$ there exists the maximum solution $\omega(\,\cdot\,,\eta)$ of the problem

$$\omega'(x) = \sigma(x, \omega(x), \omega_{(x)}) \ \text{ on } \ J \setminus J_{\text{imp}}, \tag{3.9}$$

$$\Delta\omega(x) = \tilde{\sigma}(x, \omega(x^-), \omega_{(x^-)}) \text{ on } J_{\text{imp}}, \quad \omega(x) = \eta(x) \textit{ on } I. \tag{3.10}$$

We will need the following lemma on ordinary functional differential inequalities.

Lemma 3.4. *Suppose that Assumption H* $[\sigma, \tilde{\sigma}]$ *is satisfied and*

1) $\tilde{\eta} \in C(I, R_+)$ *and* $\omega(\,\cdot\,, \tilde{\eta}) : [-\tau_0, a) \to R_+$ *is the maximum solution of problem* (3.9)*,* (3.10) *with* $\eta = \tilde{\eta}$*,*

2) $\psi \in C_{\text{imp}}(I \cup J, R_+)$ *and*

$$\psi(x) \le \tilde{\eta}(x) \textit{ on } I, \quad \Delta\psi(x) \le \tilde{\sigma}(x, \psi(x^-), \psi_{(x^-)}) \textit{ on } J_{\text{imp}},$$

3) *for* $x \in J_+ = \{ x \in J \setminus J_{\text{imp}} : \ \psi(x) > \omega(x, \tilde{\eta}) \}$ *we have*

$$D_-\psi(x) \le \sigma(x, \psi(x), \psi_{(x)}).$$

Under these assumptions $\psi(x) \le \omega(x, \tilde{\eta})$ *for* $x \in J$.

We omit the proof of the Lemma.

Let $V : C_{\rm imp}[B, R] \to C_{\rm imp}[I, R]$ be the operator given by

$$(Vw)(t) = \max\{\, |w(t,s)| : \quad s \in [-\tau, \tau]\,\},$$

where $t \in [-\tau_0, 0]$. If $w \in C_{\rm imp}[B^{(-)}, R]$ then $Vw \in C_{\rm imp}[I^{(-)}, R]$ denotes a function given by the above formula for $t \in I^{(-)}$.

Theorem 3.5. *Suppose that Assumption* $H\,[\,\sigma, \tilde{\sigma}\,]$ *is satisfied and*

1) $f \in C(\,\Omega, R\,)$ *and for each* $(x, y, p, w) \in (E \setminus E_{\rm imp}) \times R \times C_{\rm imp}[B, R]$ *we have*

$$f(x, y, p, w, 0, 0)\ \text{sign}\ p \le \sigma(x, |p|, Vw) \tag{3.11}$$

where sign $p = 1$ *if* $p \ge 0$ *and* sign $p = -1$ *if* $p < 0$,

3) *the function* $u \in C^{(1,2)}_{\rm imp}(E^*, R)$ *is a solution of (*3.1*) and* f *is parabolic with respect to* u *on* $E \setminus E_{\rm imp}$,

4) $\tilde{\eta} \in C(I, R_+)$ *and* $|u(x, y)| \le \tilde{\eta}(x)$ *on* E_0,

5) *the boundary estimate* $|u(x, y)| \le \omega(x, \tilde{\eta})$ *on* $\partial_0 E$ *and inequality for impulses*

$$\Delta|u(x, y)| \le \tilde{\sigma}(\,x, |u(x^-, y)|, Vu_{(x^-)}\,) \ \ \text{on} \ \ E_{\rm imp} \cup \partial_0 E_{\rm imp}$$

are satisfied.

Under these assumptions we have $|u(x, y)| \le \omega(x, \tilde{\eta})$, $(x, y) \in E^*$.

Proof. We prove that the function $\psi = Tu$ satisfies all the assumption of Lemma 3.4. It follows from the initial estimate for u and from assumption 5) that condition 3) of Lemma 3.4 holds. Suppose that $x \in J \setminus J_{\rm imp}$ and $\psi(x) > \omega(x, \tilde{\eta})$. There exists $y \in [-c, c]$ such that $\psi(x) = |u(x, y)|$. It follows from the boundary estimate that $y \in (-b, b)$. There are two possibilities: either (i) $\psi(x) = u(x, y)$ or (ii) $\psi(x) = -u(x, y)$.

Consider the case (i). Then $D_y u(x, y) = 0$ and $D_{yy} u(x, y) \ge 0$. It follows that

$$\begin{aligned} D_-\psi(x) \le D_x u(x, y) = f(x, u, u(x, y), u_{(x,y)}, D_y u(x, y), D_{yy} u(x, y)\,) \le \\ \le \sigma(x, |u(x, y)|, V u_{(x,y)}) \le \sigma(x, \psi(x), \psi_{(x)}). \end{aligned}$$

Thus ψ satisfies condition 4) of Lemma 3.4. The case (ii) is analogous. Thus all conditions of Lemma 3.4 are satisfied and Theorem 3.5 follows.

Let us consider two problems: the problem (3.1)–(3.3) and the problem

$$D_x z(x, y) = \tilde{f}(x, y, z(x, y), z_{(x,y)}, D_y z(x, y), D_{yy} z(x, y)) \ \text{on}\ E \setminus E_{\rm imp}, \tag{3.12}$$

$$\Delta z(x, y) = \tilde{g}(\,x, y, z(x^-, y), z_{(x^-,y)}) \quad \text{on}\ E_{\rm imp} \cup \partial_0 E_{\rm imp}, \tag{3.13}$$

$$z(x, y) = \tilde{\varphi}(x, y) \ \ \text{on} \ \ E_0 \cup \partial_0 E, \tag{3.14}$$

where $\tilde{f} : \Sigma \to R$, $\tilde{g} : \Sigma_{\mathrm{imp}} \to R$, $\tilde{\varphi} : E_0 \cup \partial_0 E \to R$ are given functions. We prove a theorem on the estimate of the difference between solutions of (3.1)–(3.3) and (3.12)–(3.14). □

Theorem 3.6. *Suppose that Assumption* $H\,[\,\sigma, \tilde{\sigma}\,]$ *are satisfied and*

1) *the functions* f, $\tilde{f}$ *and* g, $\tilde{g}$ *satisfy the conditions*

$$\Big[f(x,y,p,w,q,r) - \tilde{f}(x,y,\bar{p},\bar{w},q,r) \Big] \operatorname{sign}(p-\bar{p}) \le$$
$$\le \sigma(x, |p-\bar{p}|, V(w-\bar{w})) \quad on \quad \Sigma$$

and

$$|g(x,y,p,w) - \tilde{g}(x,y,\bar{p},\bar{w})| \le \tilde{\sigma}(x, |p-\bar{p}|, V(w-\bar{w})) \quad on \ \ \Sigma_{\mathrm{imp}},$$

2) φ, $\tilde{\varphi} \in C_{\mathrm{imp}}(E_0 \cup \partial_0 E, R)$, *and* $|\varphi(x,y) - \tilde{\varphi}(x,y)| \le \tilde{\eta}(x)$ *on* E_0 *with* $\tilde{\eta} \in C(I, R_+)$,

3) u, $\tilde{u} \in C^{(1,2)}_{\mathrm{imp}}(E^*, R)$ *are solutions of* (3.1)–3.3) *and* (3.12)–(3.14) *respectively,*

4) *the boundary estimate* $|\varphi(x,y) - \tilde{\varphi}(x,y)| \le \omega(x, \tilde{\eta})$ *on* $\partial_0 E$ *is satisfied and* f *is parabolic with respect to* u *on* $E \setminus E_{\mathrm{imp}}$.

Under these assumptions we have $|u(x,y) - v(x,y)| \le \omega(x, \tilde{\eta})$ *on* E^*.

Proof. We prove that the function $\psi : [-\tau_0, a) \to R_+$ given by

$$\psi(x) = \max\{|u(x,y) - v(x,y)| : \ \ y \in [-c,c]\,\}$$

satisfies all the conditions of Lemma 3.4. Suppose that $x \in J_+$ where

$$J_+ = \{\, x \in J \setminus J_{\mathrm{imp}} : \ \ \psi(x) > \omega(x, \tilde{\eta})\,\}.$$

There is $y \in [-c,c]$ such that $\psi(x) = |u(x,y) - \tilde{u}(x,y)|$. From the boundary estimate it follows that $y \in (-b,b)$. There are two possibilities: either

$$(i)\ \psi(x) = u(x,y) - \tilde{u}(x,y) \ \text{ or } \ (ii)\ \psi(x) = -[\,u(x,y) - \tilde{u}(x,y)\,].$$

Consider the case (i). Then $D_y(u-\tilde{u})(x,y) = 0$ and $D_{yy}u(x,y) \le D_{yy}\tilde{u}(x,y)$. It follows that

$$D_-\psi(x) \le D_x(u-\tilde{u})(x,y) =$$
$$= f(\,x,y,u(x,y),u_{(x,y)},D_yu(x,y),D_{yy}u(x,y)\,) -$$
$$-\tilde{f}(\,x,y,\tilde{u}(x,y),\tilde{u}_{(x,y)},D_yu(x,y),D_{yy}\tilde{u}(x,y)\,)$$

and consequently

$$D_-\psi(x) \le \sigma(\,x,\psi(x),\psi_{(x)}\,)$$

The case (ii) is analogous. It is easy to see that ψ satisfies

$$\Delta\psi(x) \le \tilde{\sigma}(\,x,\psi(x^-),\psi_{(x^-)}\,)$$

for $x \in J_{\mathrm{imp}}$ such that $\psi(x) > \omega(x, \tilde{\eta})$ and $\psi(x) \le \tilde{\eta}(x)$ on I. Thus all the conditions of Lemma 3.4 are satisfied and the statement of the Theorem 3.6 follows. □

Theorem 3.7. *Suppose that Assumption $H\,[\,\sigma,\tilde{\sigma}\,]$ is satisfied and*
1) *the function $f \in C(\Sigma, R)$ and $g \in C(\Sigma_{\rm imp}, R)$ satisfy the conditions*

$$[\,f(x,y,p,w,q,r) - f(x,y,\bar{p},\bar{w},q,r)\,]\ \mathrm{sign}\,(p-\bar{p}) \leq \\ \leq \sigma(\,x,|p-\bar{p}|,\|w-\bar{w}\|_B\,)$$

and

$$|g(\,x,y,p,w\,) - g(\,x,y,\bar{p},\bar{w}\,)| \leq \tilde{\sigma}(\,x,|p-\bar{p}|,\|w-\bar{w}\|_{B^{(-)}}\,),$$

2) *the maximum solution of the problem with impulses*

$$\omega'(x) = \sigma(\,x,\omega(x),\omega_{(x)}\,) \text{ on } J \setminus J_{\rm imp}, \\ \Delta\omega(x) = \tilde{\sigma}(\,x,\omega(x^-),\omega_{(x^-)}\,) \text{ on } J_{\rm imp}, \quad \omega(x) = 0 \text{ on } I,$$

is $\bar{\omega}(x) = 0$, $x \in I \cup J$.

Then there is at most one solution of problem (3.1)–(3.3) *of class* $C^{(1,2)}_{\rm imp}(E^*, R)$.

Proof. The above Theorem follows from Theorem 3.6 for $\tilde{f}=f$ and $\tilde{g}=g$. □

Remark 3.8. Suppose that $\varrho : (J \setminus J_{\rm imp}) \times R_+ \times R_+ \to R_+$ and $\tilde{\varrho} : J_{\rm imp} \times R_+ \times R_+ \to R_+$ are given functions and σ, $\tilde{\sigma}$ are defined by

$$\sigma(\,x,p,\eta\,) = \varrho(\,x,p,\|\eta\|_I\,), \quad \tilde{\sigma}(\,x,p,\eta\,) = \tilde{\varrho}(\,x,p,\|\eta\|_{I^{(-)}}\,).$$

Then

(i) Estimates given in assumption 1) of Theorem 3.7 are equivalent to

$$\Big[\,f(x,y,p,w,q,r) - \tilde{f}(x,y,\bar{p},\bar{w},q,r)\Big]\ \mathrm{sign}\,(p-\bar{p}) \leq \\ \leq \varrho(\,x,|p-\bar{p}|,\|w-\bar{w}\|_B\,)$$

and

$$|\,g(\,x,y,p,w\,) - \tilde{g}(\,x,y,\bar{p},\bar{w}\,)\,| \leq \tilde{\varrho}(x,|p-\bar{p}|,\|w-\bar{w}\|_{B^{(-)}}\,).$$

(ii) If we assume that $\tilde{\eta} \in C(I, R_+)$ is nondecreasing on I then problem (3.9), (3.10) is equivalent to

$$\omega'(x) = \varrho(\,x,\omega(x),\omega(x)\,) \text{ on } J \setminus J_{\rm imp}, \\ \Delta\omega(x) = \tilde{\varrho}(\,x,\omega(x^-),\omega(x^-)\,) \text{ on } J_{\rm imp}, \ \ \omega(0) = \tilde{\eta}(0).$$

3.3. Discretization of Parabolic Problems

Suppose that $B,\ B^{(-)},\ I,\ J,\ E,\ E_0,\ \partial_0 E,\ E_{\text{imp}},\ \partial_0 E_{\text{imp}},\ E^*$ are the sets defined in Section 3.1. Write

$$\Sigma = (E \setminus E_{\text{imp}}) \times C_{\text{imp}}[B, R] \times R^n \times M_{n\times n}, \quad \Sigma_{\text{imp}} = E_{imp} \times C_{\text{imp}}[B^{(-)}, R]$$

and suppose that

$$f : \Sigma \to R, \ \ g : \Sigma_{\text{imp}} \to R, \quad \phi \in C_{\text{imp}}(E_0 \cup \partial_0 E, R)$$

are given functions. In this Section we consider the parabolic functional differential equation with impulses

$$D_x z(x,y) = f(x, y, z_{(x,y)}, D_y z(x,y), D_{yy} z(x,y)) \ \text{ on } \ E \setminus E_{\text{imp}}, \quad (3.15)$$

$$\Delta z(x,y) = g(x, y, z_{(x-,y)}) \quad \text{on } \ E_{\text{imp}}, \quad (3.16)$$

and the initial boundary condition

$$z(x,y) = \phi(x,y) \ \text{ on } \ E_0 \cup \partial_0 E. \quad (3.17)$$

We formulate a difference problem corresponding to (3.15)–(3.17). We start with a definition of a mesh in E^* and B. Suppose that $h = (h_0, h')$ where $h' = (h_1, \ldots, h_n)$ stand for steps of the mesh. For $(i, m) \in Z^{1+n}$ where $m = (m_1, \ldots, m_n)$ we define $y^{(m)}$ by $y^{(m)} = (\, y^{(m_1)}, \ldots, y^{(m_n)}\,) = m * h'$ and $x^{(i)} = ih_0$. Denote by Θ the set of all h such that there are $M = (M_1, \ldots, M_n) \in Z^n$ and $M_0 \in Z$ such that $M * h' = \tau$ and $M_0 h_0 = \tau_0$. We assume that $\Theta \neq \emptyset$ and that there is a sequence $\{h^{(j)}\}$, $h^{(j)} \in \Theta$, such that $\lim_{j\to\infty} h^{(j)} = 0$. Let

$$R_h^{1+n} = \Big\{ (x^{(i)}, y^{(m)}) : \quad (i,m) \in Z^{1+n} \Big\}.$$

We define the sets

$$E_{0.h} = E \cap R_h^{1+n}, \ \ B_h = B \cap R_h^{1+n}, \ \ E_h = E \cap R_h^{1+n},$$

and

$$\partial_0 E_h = \partial_0 E \cap R_h^{1+n}, \quad E_h^* = E_{0.h} \cup E_h \cup \partial_0 E_h.$$

Elements of the set E_h^* will be denoted by $(x^{(i)}, y^{(m)})$ or (x, y). For a function $z : E_h^* \to R$ and a point $(x^{(i)}, y^{(m)}) \in E_h^*$ we write $z^{(i,m)} = z(x^{(i)}, y^{(m)})$ and

$$\|z\|_{i.h} = \max\{\, |z^{(j,m)}| : \ \ (x^{(j)}, y^{(m)}) \in E_h^*, \ \ j \le i\,\}.$$

For the above z and for a point $(x^{(i)}, y^{(m)}) \in E_h$ we define the function $z_{(i,m)} : B_h \to R$ by

$$z_{(i,m)}(t,s) = z(x^{(i)} + t, y^{(m)} + s), \ \ (t,s) \in B_h.$$

The function $z_{(i,m)}$ is the restriction of z to the set

$$\Big([x^{(i)} - \tau_0, x^{(i)}] \times [y^{(m)} - \tau, y^{(m)} + \tau] \Big) \cap R_h^{1+n}$$

and this restriction is shifted to the set B_h. Let $\{ n_1, \dots, n_k \}$, $n_i \in N$, be defined by $n_i h_0 < a_i \le (n_i+1)h_0,\ i = 1, \dots, k$. Write

$$E_h^{\text{imp}} = \{ (x^{(i)}, y^{(m)}) \in E_h : \ \ i \in \{ n_1, \dots, n_k \} \},$$
$$E'_h = \{ (x^{(i)}, y^{(m)}) \in E_h : \ \ 0 \le i \le N_0 - 1 \}.$$

Let

$$J_h^{\text{imp}} = \{ x^{(i)} : \ i \in \{ n_1, \dots, n_k \} \},$$

and

$$I_h = \{ x^{(i)} : \ -M_0 \le i \le 0 \}, \ \ J_h = \{ x^{(i)} : 0 \le i \le N_0 \}, \ \ J'_h = J_h \setminus \{ x^{(N_0)} \}$$

where $N_0 \in \mathbf{N}$ is defined by $N_0\, h_0 < a \le (N_0 + 1)\, h_0$. For a function $\eta : I_h \cup J_h \to R$ and $x^{(i)} \in J_h$ we define a function $\eta_{(i)} : I_h \to R$ by $\eta_i(t) = \eta(x^{(i)} + t),\ t \in I_h$.

Put $X = \{ (i,j) : \ \ 1 \le i, j \le n \ \ \ i \ne j \}$ and assume that we have defined the sets $X_+,\ X_- \subset X$ such that $X_+ \cup X_- = X,\ X_+ \cap X_- = \emptyset$ (in particular, it may be $X_+ = \emptyset$ or $X_- = \emptyset$). We assume also that $(j,k) \in X_+$ if $(k,j) \in X_+$. We define for $1 \le j \le n$

$$\delta_j^+ z^{(i,m)} = \frac{1}{h_j} \left(z^{(i,m+e_j)} - z^{(i,m)} \right), \qquad \delta_i^- z^{(i,m)} = \frac{1}{h_i} \left(z^{(i,m)} - z^{(i,m-e_j)} \right),$$

where $e_j = (0, \dots, 0, 1, 0, \dots, 0) \in R^n$, 1 standing on j-th place. We consider difference operators

$$\delta_0, \ \delta = (\delta_1, \dots, \delta_n), \ \ \delta^{(2)} = \left[\, \delta_{jk}^{(2)} \, \right]_{j,k=1,\dots,n}$$

given by

$$\delta_0 z^{(m)} = \frac{1}{h_0} \left(z^{(i+1,m)} - z^{(i,m)} \right), \tag{3.18}$$

$$\delta_j z^{(i,m)} = \frac{1}{2} \left(\delta_j^+ z^{(i,m)} + \delta_j^- z^{(i,m)} \right), \ \ j = 1, \dots, n, \tag{3.19}$$

and

$$\delta_{jj}^{(2)} z^{(i,m)} = \delta_j^- \delta_j^+ z^{(i,m)}, \ \ \ j = 1, \dots, n, \tag{3.20}$$

$$\delta_{jk}^{(2)} z^{(i,m)} = \frac{1}{2} \left(\delta_j^+ \delta_k^+ z^{(i,m)} + \delta_j^- \delta_k^- z^{(i,m)} \right) \ \ \textit{for } (j,k) \in X_+, \tag{3.21}$$

$$\delta_{jk}^{(2)} z^{(i,m)} = \frac{1}{2} \left(\delta_j^+ \delta_k^- z^{(i,m)} + \delta_j^- \delta_k^+ z^{(i,m)} \right) \ \ \textit{for } (j,k) \in X_-. \tag{3.22}$$

Let

$$\Sigma_h = (E'_h \setminus E_h^{\text{imp}}) \times F(B_h, R) \times R^n \times M_{n \times n}, \quad \Sigma_h^{\text{imp}} = E_h^{\text{imp}} \times F(B_h, R).$$

Suppose that for each $h \in \Theta$ we have

$$f_h : \Sigma_h \to R, \ \ g_h : \Sigma_h^{\text{imp}} \to R, \ \ \phi_h : E_{0.h} \cup \partial_0 E_h \to R.$$

Write

$$F_h[\,z\,]^{(i,m)} = f_h(\,x^{(i)}, y^{(m)}, z_{(i,m)}, \delta z^{(i,m)}, \delta^{(2)} z^{(i,m)}\,) \;\text{ on }\; E'_h \setminus E_h^{\mathrm{imp}}$$

and

$$G_h[\,z\,]^{(i,m)} = g_h(\,x^{(i)}, y^{(m)}, z_{(i,m)}\,) \;\text{ on }\; E_h^{\mathrm{imp}}$$

where $z \in F(E_h^*, R)$.

We will approximate solutions of problem (3.15)–(3.17) by means of solutions of the functional difference equations

$$\delta_0 z^{(i,m)} = F_h[\,z\,]^{(i,m)} \quad \text{for} \quad (x^{(i)}, y^{(m)}) \in E'_h \setminus E_h^{\mathrm{imp}}, \tag{3.23}$$

$$\Delta z^{(i,m)} = G_h[\,z\,]^{(i,m)} \quad \text{for} \quad (x^{(i)}, y^{(m)}) \in E_h^{\mathrm{imp}}, \tag{3.24}$$

with the initial boundary condition

$$z^{(i,m)} = \phi_h^{(i,m)} \;\text{ on } E_{0.h} \cup \partial_0 E_h. \tag{3.25}$$

It is clear that there exists exactly one solution $v_h : E_h^* \to R$ of (3.23)–(3.25).

3.4. Convergence of the Difference Method for Parabolic Problems

We will consider two comparison functions σ_h and $\tilde{\sigma}_h$ corresponding to f_h and g_h respectively. Suppose that we have

$$\sigma : (J'_h \setminus J_h^{\mathrm{imp}}) \times F(I_h, R_+) \to R_+, \quad \tilde{\sigma}_h : J_h^{\mathrm{imp}} \times F(I_h, R_+) \to R_+.$$

Our main assumptions are the following.

Assumption H $[\,\sigma_h, \tilde{\sigma}_h\,]$**.** Suppose that

1) the functions $\sigma_h(x,\,\cdot\,) : F(I_h, R_+) \to R_+$ where $x \in J_h \setminus J_h^{\mathrm{imp}}$ and $\tilde{\sigma}_h(x,\,\cdot\,) : F(I_h, R_+) \to R_+$ where $x \in J_h^{\mathrm{imp}}$ are nondecreasing,

2) $\sigma_h(x, \theta_h) = 0$ for $x \in J_h \setminus J_h^{\mathrm{imp}}$ where $\theta_h(t) = 0$ for $t \in I_h$ and $\tilde{\sigma}_h(x, \theta_h) = 0$ for $x \in J_h^{\mathrm{imp}}$,

3) the functional difference problem with impulses

$$\eta^{(i+1)} = \eta^{(i)} + h_0 \sigma_h(x^{(i)}, \eta_{(i)}) \;\text{ for } x^{(i)} \in J'_h \setminus J_h^{\mathrm{imp}}, \tag{3.26}$$

$$\eta^{(i+1)} = \eta^{(i)} + \tilde{\sigma}_h(x^{(i)}, \eta^{(i)}, \eta_{(i)}) \;\text{ for } x^{(i)} \in J_h^{\mathrm{imp}}, \tag{3.27}$$

$$\eta^{(i)} = 0 \text{ for } x^{(i)} \in I_h, \tag{3.28}$$

is stable in the following sense: if $\eta_h : I_h \cup J_h \to R_+$ is the solution of the problem

$$\eta^{(i+1)} = \eta^{(i)} + h_0 \sigma_h(x^{(i)}, \eta_{(i)}) + h_0 \gamma(h) \;\text{ for }\; x^{(i)} \in J'_h \setminus J_h^{\mathrm{imp}},$$

$$\eta^{(i+1)} = \eta^{(i)} + \tilde{\sigma}_h(x^{(i)}, \eta^{(i)}, \eta_{(i)}) + \tilde{\gamma}(h) \;\text{ for } x^{(i)} \in J_h^{\mathrm{imp}},$$

$$\eta^{(i)} = \alpha_0(h) \text{ for } x^{(i)} \in I_h,$$

where $\gamma,\ \tilde{\gamma},\ \alpha_0 : \Theta \to R_+$ and

$$\lim_{h\to 0} \gamma(h) = \lim_{h\to 0} \tilde{\gamma}(h) = \lim_{h\to 0} \alpha_0(h) = 0,$$

then there exists $\beta : \Theta \to R_+$ such that $\eta_h^{(i)} \le \beta(h)$ for $x^{(i)} \in J_h$ and $\lim_{h\to 0} \beta(h) = 0$.

In a convergence theorem we will estimate a function of several variables by means of a function of one variable. Therefore we will need the following operator $V_h : F(B_h, R) \to F(I_h, R_+)$. If $w : B_h \to R$ then

$$(V_h w)(x^{(i)}) = \max\{\ |w(x^{(i)}, y^{(m)})| :\ \ -M \le m \le M\ \},\ \ -M_0 \le i \le 0.$$

Assumption H $[\, f_h, g_h \,]$**.** Suppose that σ_h and $\tilde{\sigma}_h$ satisfy Assumption H $[\, \sigma_h, \tilde{\sigma}_h \,]$ and

$$|f_h(x, y, w, q, r) - f_h(x, y, \bar{w}, q, r)| \le \sigma(\, x, V_h(w - \bar{w})\,) \ \text{ on } \Sigma_h, \quad (3.29)$$

$$|g_h(x, y, w) - g_h(x, y, \bar{w})| \le \tilde{\sigma}_h(\, x, V_h(w - \bar{w})\,) \ \text{ on } \Sigma_h^{\text{imp}}. \quad (3.30)$$

Assumption H $[\, D_q f_h, D_r f_h \,]$**.** Suppose that the function $f_h : \Sigma_h \to R$ of the variables (x, y, w, q, r), $q = (q_1, \ldots, q_n)$, $r = [r_{ij}]_{i,j=1,\ldots,n}$, satisfies the conditions:

1) for each $P = (x, y, w, q, r) \in \Sigma_h$ there exist the derivatives

$$D_q f_h(P) = (D_{q_1} f_h(P), \ldots, D_{q_n} f_h(P))\,,\ \ D_r f_h(P) = \big[\, D_{r_{ij}} f_h(P)\,\big]_{i,j=1,\ldots,n}$$

and

$$D_q f_h(x, y, w, \cdot) \in C(R^n \times M_{n\times n}, R^n),\ D_r f_h(x, y, w, \cdot) \in C(R^n \times M_{n\times n}, M_{n\times n}),$$

2) the matrix $D_r f_h$ is symmetric and for $P = (x, y, w, q, r) \in \Sigma_h$ we have

$$D_{r_{ij}} f_h(P) \ge 0\ \ for\ (i, j) \in X_+,\ \ D_{r_{ij}} f_h(P) \le 0\ \text{ for } (i, j) \in X_- \quad (3.31)$$

$$1 - 2h_0 \sum_{i=1}^{n} \frac{1}{h_i^2} D_{r_{ii}} f_h(P) + h_0 \sum_{(i,j)\in X} \frac{1}{h_i h_j} |D_{r_{ij}} f_h(P)| \ge 0, \quad (3.32)$$

$$-\frac{1}{2}|D_{q_j} f_h(P)| + \frac{1}{h_j} D_{r_{jj}} f_h(P) - \sum_{i\in\pi[j]} \frac{1}{h_i} |D_{r_{ij}} f_h(P)| \ge 0,\ 1 \le j \le n, \quad (3.33)$$

where $\pi[j] = \{1, \ldots, j-1, j+1, \ldots, n\}$.

Theorem 3.9. *Suppose that Assumptions* $H[\, \sigma_h, \tilde{\sigma}_h \,]$*,* $H[\, f_h, g_h \,]$*,* $H[\, D_q f_h, D_r f_h \,]$ *are satisfied and*

1) *there is* $\alpha_0 : \Theta \to R_+$ *such that*

$$|\phi_h^{(i,m)} - \phi^{(i,m)}| \le \alpha_0(h)\ \ on\ E_{0.h} \cup \partial_0 E_h\ \ and\ \ \lim_{h\to 0} \alpha_0(h) = 0, \quad (3.34)$$

and $v_h : E_h^* \to R$ *is a solution of* (3.23)–(3.25)*,*

2) *$u \in C_{\rm imp}(E^*, R)$ is a solution of* (3.15)–(3.17), *u is of class C^3 on $E \setminus E_{\rm imp}$ and the partial derivatives of the third order of u are bounded on $\bar{E} \setminus E_{\rm imp}$,*

3) *there exists $C > 0$ such that $h_i h_j^{-1} \le C$ for $i, j = 1, \ldots, n$, $h \in \Theta$,*

4) *there exist $\tilde{\beta}_1$, $\tilde{\beta}_2 : \Theta \to R_+$ such that the compatibility conditions are satisfied*

$$\Big| F_h[\,u_h\,]^{(i,m)} - f(\,x^{(i)}, y^{(m)}, u_{(x^{(i)},y^{(m)})}, \delta u_h^{(i,m)}, \delta^{(2)} u_h^{(i,m)}\,) \Big| \le \\ \le \tilde{\beta}_1(h) \ \text{ on } E'_h \setminus E_h^{(\rm imp}, \quad \lim_{h\to 0} \tilde{\beta}_1(h) = 0, \tag{3.35}$$

and

$$\Big| G_h[\,u_h\,]^{(n_i,m)} - g(a_i, y^{(m)}, u_{(x^{(i)-},y^{(m)})}) \Big| \le \\ \le \tilde{\beta}_2(h) \ \text{ on } E_h^{(\rm imp}, \quad \lim_{h\to 0} \tilde{\beta}_2(h) = 0, \tag{3.36}$$

where u_h is the restriction of u to the set E_h^*. Then there exists $\gamma : \Theta \to R_+$ such that

$$|u_h^{(i,m)} - v_h^{(i,m)}| \le \gamma(h) \ \text{ on } \ E_h, \quad \lim_{h\to 0} \gamma(h) = 0. \tag{3.37}$$

Proof. Let $\Gamma_h : E'_h \to R$ be defined by

$$\delta_0 u_h^{(i,m)} = F_h[\,u_h\,]^{(i,m)} + \Gamma_h^{(i,m)} \quad \text{on } \ E'_h \setminus E_h^{\rm imp}, \tag{3.38}$$

$$\Delta u_h^{(i,m)} = G_h[\,u_h\,]^{(i,m)} + \Gamma_h^{(i,m)} \ \text{ on } \ E_h^{\rm imp}. \tag{3.39}$$

It follows that there is $\beta : \Theta \to R_+$ such that $|\Gamma_h^{(i,m)}| \le \beta(h)$ on E'_h and $\lim_{h\to 0} \beta(h) = 0$.

Let $\eta_h : I_h \cup X_h \to R_+$ be given by

$$\eta_h^{(i)} = \max\{\, |\, u_h^{(i,m)} - v_h^{(i,m)} \,| : \ (x^{(i)}, y^{(m)}) \in E_h^* \,\}, \ \ -M_0 \le i \le N_0, \tag{3.40}$$

We prove that η_h satisfies the difference inequalities

$$\eta_h^{(i+1)} \le \max\{\, \alpha_0(h), \ \eta_h^{(i)} + h_0 \sigma_h(x^{(i)}, (\eta_h)_{(i)}) + h_0 \beta(h) \,\}, \tag{3.41}$$
$$x^{(i)} \in J'_h \setminus J_h^{\rm imp},$$

$$\eta_h^{(i+1)} \le \max\{\, \alpha_0(h), \ \eta_h^{(i)} + \tilde{\sigma}_h(x^{(i)}, \eta_h^{(i)}, (\eta_h)_{(i)}) + \beta(h) \,\}, \tag{3.42}$$
$$x^{(i)} \in J_h^{\rm imp}.$$

We prove (3.41). Suppose that $(x^{(i)}, y^{(m)}) \in E'_h \setminus E_h^{\mathrm{imp}}$. It follows from Assumptions H $[\, f_h, g_h \,]$, H $[\, D_q f_h, D_r f_h \,]$ that

$$
\begin{aligned}
&|(u_h - v_h)^{(i+1,m)}| \leq \\
&\leq \Big| (u_h - v_h)^{(i,m)} + h_0 \Big[f_h(x^{(i)}, y^{(m)}, (u_h)_{(i,m)}, \delta u_h^{(i,m)}, \delta^{(2)} u_h^{(i,m)}) - \\
&\qquad - f_h(x^{(i)}, y^{(m)}, (u_h)_{(i,m)}, \delta v_h^{(i,m)}, \delta^{(2)} v_h^{(i,m)}) \Big] \Big| + \\
&\qquad + h_0\, \sigma_h \Big(x^{(i)}, V_h(\, (u_h)_{(i,m)} - (v_h)_{(i,m)} \,) \Big) + h_0 \beta(h)
\end{aligned}
$$

and consequently

$$
\begin{aligned}
&\Big| (u_h - v_h)^{(i+1,m)} \Big| \leq \\
&\leq \Big| (u_h - v_h)^{(i,m)} \Big[1 - 2h_0 \sum_{j=1}^{n} \frac{1}{h_j^2} D_{r_{jj}} f_h(Q) + h_0 \sum_{(j,k) \in X} \frac{1}{h_j h_k} |D_{r_{jk}} f_h(Q)| \Big] \Big| + \\
&\quad + \Big| \sum_{j=1}^{n} (u_h - v_h)^{(i,m+e_j)} \Big[\frac{h_0}{2h_j} D_{q_j} f_h(Q) + \frac{h_0}{h_j^2} D_{r_{jj}} f_h(Q) - \\
&\qquad - \sum_{k \in \pi[j]} \frac{h_0}{h_j h_k} |D_{r_{jk}} f_h(Q)| \Big] \Big| + \\
&\quad + \Big| \sum_{j=1}^{n} (u_h - v_h)^{(i,m-e_j)} \Big[- \frac{h_0}{2h_j} D_{q_j} f_h(Q) + \frac{h_0}{h_j^2} D_{r_{jj}} f_h(Q) - \\
&\qquad - \sum_{k \in \pi[j]} \frac{h_0}{h_j h_k} |D_{r_{jk}} f_h(Q)| \Big] \Big| + \\
&+ \sum_{(j,k) \in X_+} \frac{h_0}{2h_j h_k} D_{r_{jk}} f_h(Q) \Big[\Big| (u_h - v_h)^{(i,m+e_j+e_k)} + (u_h - v_h)^{(i,m-e_j-e_k)} \Big| \Big] - \\
&- \sum_{(j,k) \in X_-} \frac{h_0}{2h_j h_k} D_{r_{jk}} f_h(Q) \Big[\Big| (u_h - v_h)^{(i,m+e_j-e_k)} + (u_h - v_h)^{(i,m-e_j+e_k)} \Big| \Big] + \\
&\qquad + h_0\, \sigma_h(\, x^{(i)}, (\eta_h)_{(i)} \,) + h_0 \beta(h),
\end{aligned}
\tag{3.43}
$$

where $Q = (x^{(i)}, y^{(m)}, (u_h)_{(i,m)}, q^{(i,m)}, r^{(i,m)}) \in \Sigma_h$ is an intermediate point. The above estimates and (3.31)–(3.33) imply

$$
|(u_h - v_h)^{(i+1,m)}| \leq \eta_h^{(i)} + h_0\, \sigma_h(\, x^{(i)}, (\eta_h)_{(i)} \,) + h_0\, \beta(h) \tag{3.44}
$$

on $E'_h \setminus E_h^{\mathrm{imp}}$. If $(x^{(i)}, y^{(m)}) \in \partial_0 E_h \setminus E_h^{\mathrm{imp}}$ then we have

$$
|(u_h - v_h)^{(i+1,m)}| \leq \alpha_0(h).
$$

The above inequalities imply (3.41).

Suppose now that $(x^{(i)}, y^{(m)}) \in E_h^{\text{imp}}$. Then we have

$$|(u_h - v_h)^{(i+1,m)}| \le \eta_h^{(i)} + \tilde{\sigma}_h(x^{(i}, \eta_h^{(i)}, (\eta_h)_{(i)}) + \beta(h).$$

If $(x^{(i)}, y^{(m)}) \in \partial_0 E_h$ and $i \in \{n_1, \ldots, n_k\}$ then

$$|(u_h - v_h)^{(i+1,m)}| \le \alpha_0(h).$$

The above inequalities imply (3.42).

Denote by $\tilde{\eta}_h : I_h \cup J_h \to R_+$ the solution of the problem

$$\eta^{(i+1)} = \eta^{(i)} + h_0 \sigma_h(x^{(i)}, \eta_{(i)}) + h_0 \beta(h) \;\text{ for }\; x^{(i)} \in J'_h \setminus J_h^{\text{imp}}, \quad (3.45)$$

$$\eta^{(i+1)} = \eta^{(i)} + \tilde{\sigma}_h(x^{(i)}, \eta^{(i)}, \eta_{(i)}) + \beta(h) \;\; for \;\; x^{(i)} \in J_h^{\text{imp}}, \quad (3.46)$$

$$\eta^{(i)} = \alpha_0(h) \;\text{ for } x^{(i)} \in I_h. \quad (3.47)$$

Since the function η_h satisfies (3.41), (3.42) it follows from the monotonicity of σ and $\tilde{\sigma}$ that $\eta_h^{(i)} \le \tilde{\eta}_h^{(i)}$ for $x^{(i)} \in J_h$ and consequently $|u_h^{(i,m)} - v_h^{(i,m)}| \le \tilde{\eta}_h^{(i)}$ *on* E_h. The assertion of our theorem follows from the stability of problem (3.26)–(3.28). □

Remark 3.10. The condition 2) of Assumption H $[\, D_q f_h, D_r f_h \,]$ is very complicated because we consider the functional differential problem with all the derivatives $[\, D_{y_i y_j} z \,]_{i,j=1,\ldots,n}$. We have obtained estimate (3.44) from (3.43) because the appropriate coefficients in (3.43) are nonnegative. Consider the simple equation

$$D_x z(x, y) = \tilde{f}(\, x, y, z_{(x,y)} \,) + \sum_{j=1}^{n} D_{y_j y_j} z(x, y) \;\text{ on }\; E \setminus E_{\text{imp}}$$

where $\tilde{f} : (E \setminus E_{\text{imp}}) \times C_{\text{imp}}[B, R] \to R$ is a given function. Then the corresponding difference equation has the form

$$\delta_0 z^{(i,m)} = \tilde{f}_h(\, x^{(i)}, y^{(m)}, z_{(i,m)} \,) + \sum_{j=1}^{n} \delta_{jj}^{(2)} z^{(i,m)}$$

where $(x^{(i)}, y^{(m)}) \in E'_h \setminus E_h^{\text{imp}}$. Then condition 2) of Assumption H $[\, D_q f_h,$ $D_r f_h \,]$ is equivalent to

$$1 - 2h_0 \sum_{j=1}^{n} \frac{1}{h_j^2} \ge 0$$

which is known in literature.

3.5. Difference Methods for Almost Linear Problems

Write $\Sigma_0 = (E \setminus E_{\rm imp}) \times C(B,R) \times R^n$ and suppose that

$$F_0 \in C(\,\Sigma_0, R\,),\; g \in C(\,\Sigma_{\rm imp}, R\,),\; \phi \in C_{\rm imp}(E_0 \cup \partial_0 E, R)$$

and

$$F \in C(\,E \setminus E_{\rm imp}, M_{n\times n}\,),\quad F = [\;F_{ij}\;]_{i,j=1,\dots,n}$$

are given functions. In this Section we consider the almost linear equation with impulses

$$D_x z(x,y) = F_0(\,x,y,z_{(x,y)}, D_y z(x,y)\,) + \\ + \sum_{i,j=1}^{n} F_{i,j}(\,x,y\,)\, D_{y_i y_j} z(x,y) \;\text{ on }\; E \setminus E_{\rm imp}, \tag{3.48}$$

$$\Delta z(x,y) = g(x,y,z_{(x-,y)} \;\text{ on }\; E_{\rm imp} \tag{3.49}$$

and the initial boundary condition (3.17).

Let

$$\Sigma_{0.h} = (E'_h \setminus E_h^{\rm imp}) \times F(B_h, R) \times R^n$$

and suppose that

$$F_{0.h} : \Sigma_{0.h} \to R,\;\; g_h : \Sigma_h^{\rm imp} \to R,\;\; \phi_h : E_{0.h} \cup \partial_0 E_h \to R,$$

are given functions. Consider the difference equations

$$\delta_0 z^{(i,m)} = F_{0.h}(x^{(i)}, y^{(m)}, z_{(i,m)}, \delta z^{(i,m)}) + \\ + \sum_{i,j=1}^{n} F_{ij}(x^{(i)}, y^{(m)})\, \delta_{ij}^{(2)} z^{(i,m)},\;\; (x^{(i)}, y^{(m)}) \in E'_h \setminus E_h^{\rm imp}, \tag{3.50}$$

$$z^{(i+1,m)} = z^{(i,m)} + g_h(x^{(i)}, y^{(m)}, z_{(i,m)}),\;\; (x^{(i)}, y^{(m)}) \in E_h^{\rm imp} \tag{3.51}$$

with initial boundary condition (3.25).

If we apply Theorem 3.9 to problems (3.48), (3.49) and (3.50), (3.51), (3.25) then we need the following assumption on F : for each $(j,k) \in X$ the function

$$\tilde F_{jk}(x,y) = \text{sign}\;\, F_{jk}(x,y),\;\; (x,y) \in E \setminus E_{\rm imp},$$

is constant (see condition (3.31)).

Now we prove that this condition can be omitted in the case of almost linear problems. We define for $(i,j) \in X$:

$$\delta_{jk}^{(2)} z^{(i,m)} = \frac{1}{2}\left(\delta_j^+ \delta_k^- z^{(i,m)} + \delta_j^- \delta_k^+ z^{(i,m)}\right) \;\text{ if }\; \\ F_{jk}^{(i,m)} = F_{jk}(x^{(i)}, y^{(m)}) \le 0, \tag{3.52}$$

$$\delta_{jk}^{(2)} z^{(i,m)} = \frac{1}{2}\left(\delta_j^+ \delta_k^+ z^{(i,m)} + \delta_j^- \delta_k^- z^{(i,m)}\right) \;\text{ if }\; F_{jk}^{(i,m)} > 0. \tag{3.53}$$

We consider the difference problem (3.50), (3.51) with $\delta_0,\ \delta$ given by (3.18), (3.19) and $\delta^{(2)}$ given by (3.20), (3.52), (3.53).

Assumption H $[\,F_{0.h}, F\,]$ Suppose that the functions $F_{0.h}$ and F satisfy the conditions:

1) for each $Q=(x,y,w,q)\in\Sigma_{0.h}$ there exist the derivatives

$$(\,D_{q_1}F_{0.h}(Q),\dots,D_{q_n}F_{0.h}(Q)\,)=D_qF_{0.h}(Q)$$

and $D_qF_{0.h}(x,y,w,\,\cdot\,)\in C(R^n,R^n)$ for $(x,y,w)\in(E'_h\setminus E_h^{\mathrm{imp}})\times F(B_h,R)$,

2) the matrix F is symmetric and

$$1-2h_0\sum_{i=1}^n\frac{1}{h_i^2}F_{ii}(x,y)+h_0\sum_{(i,j)\in X}\frac{1}{h_ih_j}|F_{ij}(x,y)|\ge 0,$$

$$-\frac{1}{2}|D_{q_j}F_{0.h}(Q)|+\frac{1}{h_j}F_{jj}(x,y)-\sum_{i\in\pi[j]}\frac{1}{h_i}|F_{ij}(x,y)|\ge 0,\ \ 1\le j\le n,$$

where $(x,y)\in E'_h\setminus E_h^{\mathrm{imp}},\ Q=(x,y,w,q)\in\Sigma_{0.h}$.

Assumption H $[\,F_{0.h}, g_h\,]$**.** Suppose that

1) σ_h and $\tilde\sigma_h$ satisfy Assumption H $[\,\sigma_h,\tilde\sigma_h\,]$ and

2) for each $(x,y,w,q)\in\Sigma_{0.h},\ \bar w\in F(B_h,R)$ we have

$$|F_{0.h}(x,y,w,q)-F_{0.h}(x,y,\bar w,q)|\le\sigma_h(x,V_h(w-\bar w)\,),$$

3) the function g_h satisfies the estimate (3.30).

Theorem 3.11. *Suppose that Assumptions H* $[\,\sigma_h,\tilde\sigma_h\,]$*, H* $[\,F_{0.h},F\,]$*, H* $[\,F_{0.h},g_h\,]$ *are satisfied and*

1) *there is* $\alpha_0:\Theta\to R_+$ *such that the condition (*3.34*) is satisfied and* $v_h:E_h^*\to R$ *is a solutions of problem* (3.50)–(3.53), (3.18)–(3.20),

2) $u\in C_{\mathrm{imp}}(E^*,R)$ *is a solution of* (3.48), (3.49), u *is of class* C^3 *on* $E\setminus E_{\mathrm{imp}}$ *and the derivatives of the third order of* u *are bounded on* $E\setminus E_{\mathrm{imp}}$,

3) *there exists* $C>0$ *such that* $h_ih_j^{-1}\le C,\ i,j=1,\dots,n,\ h\in\Theta$,

4) *there exist* $\tilde\beta_1,\ \tilde\beta_2:\Theta\to R_+$ *such that*

$$|F_0(x^{(i)},y^{(m)},u_{(x^{(i)},y^{(m)})},\delta u_h^{(i,m)})-F_{0.h}(x^{(i)},y^{(m)},(u_h)_{(i,m)},\delta u_h^{(i,m)})|\le$$
$$\le\tilde\beta_1(h),\ (x^{(i)},y^{(m)})\in E_h\setminus E_h^{\mathrm{imp}},\quad\lim_{h\to 0}\tilde\beta_1(h)=0,$$

and condition (3.36) *is satisfied.*

Then there is $\gamma:\Theta\to R_+$ *such that* $|u_h^{(i,m)}-v_h^{(i,m)}|\le\gamma(h)$ *on* E_h *and* $\lim_{h\to 0}\gamma(h)=0$.

Proof. Let $\Gamma_h:E'_h\to R$ be defined by (3.38), (3.39) for

$$f_h(x,y,w,q,r)=F_{0.h}(x,y,w,q)+\sum_{i,j=1}^nF_{ij}\,r_{ij},\ \ (x,y,w,q,r)\in\Sigma_h,$$

and for $\delta_0,\ \delta,\ \delta^{(2)}$ given by (3.18)–(3.20), (3.52), (3.53). It follows from the compatibility conditions and from assumption 1) that there is $\beta : \Theta \to R_+$ such that $|\,\Gamma_h^{(i,m)}| \le \beta(h)$ on E'_h and $\lim_{h\to 0}\beta(h) = 0$. Let $\eta_h : I_h \cup J_h \to R_+$ be defined by (3.40). We prove that η_h satisfies (3.41), (3.42).

Suppose that $(x^{(i)}, y^{(m)}) \in E'_h \setminus E_h^{\rm imp}$. Let

$$X_+[i,m] = \{(j,k) \in X :\ F_{jk}(x^{(i)}, y^{(m)}) \ge 0\},\ X_-[i,m] = X \setminus X_+[i,m]$$

and $z_h = u_h - v_h$. It follows from Assumptions H $[F_{0.h}, F]$, H $[F_{0.h}, g_h]$ that

$$\begin{aligned}
|z_h^{(i+1,m)}| &\le h_0\,\sigma_h(x^{(i)}, V_h(z_h)_{(i,m)}) + h_0\beta(h) + \\
&+\Big|\, z_h^{(i,m)} + h_0 \sum_{j,k=1}^{n} F_{jk} z^{(i,m)}\,\delta z_h^{(i,m)} + \\
&+h_0\Big[F_{0.h}(x^{(i)}, y^{(m)}, (u_h)_{(i,m)}, \delta u_h^{(m)}) - F_{0.h}(x^{(i)}, y^{(m)}, (u_h)_{(i,m)}, \delta v_h^{(m)})\Big]\,\Big| \le \\
&\le h_0\,\sigma_h(x^{(i)}, V_h(z_h)_{(i,m)}) + h_0\beta(h) + \\
&+\Big|\, z_h^{(i,m)} + h_0 \sum_{j=1}^{n} D_{q_j} F_{0.h}(\tilde{Q}) \frac{1}{2h_j}\Big(z_h^{(i,m+e_j)} - z_h^{(i,m-e_j)}\Big) + \\
&+h_0 \sum_{j=1}^{n} F_{jj}^{(i,m)} \frac{1}{h_j^2}\Big(z_h^{(i,m+e_j)} - 2z_h^{(i,m)} + z^{(i,m-e_j)}\Big) + \\
&+h_0 \sum_{(j,k)\in X_+[i,m]} \frac{1}{2h_j h_k} F_{jk}^{(i,m)}\Big(-z_h^{(i,m+e_j)} - z_h^{(i,m+e_k)} - z_h^{(i,m-e_j)} - \\
&\quad -z_h^{(i,m-e_k)} + 2z_h^{(i,m)} + z_h^{(i,m+e_j+e_k)} + z_h^{(i,m-e_j-e_k)}\Big) - \\
&-h_0 \sum_{(j,k)\in X_-[i,m]} F_{jk}^{(i,m)} \frac{1}{2h_j h_k}\Big(z_h^{(i,m+e_j)} + z_h^{(i,m+e_k)} + z_h^{(i,m-e_j)} + \\
&\quad +z_h^{(i,m-e_k)} - 2z_h^{(i,m)} - z_h^{(i,m+e_j-e_k)} - z_h^{(i,m-e_j+e_k)}\Big)\Big|,
\end{aligned}$$

where $\tilde{Q} = (x^{(i)}, y^{(m)}, (u_h)_{(i,m)}, q^{(i,m)}) \in \Sigma_{0.h}$ is an intermediate point. The above estimate implies

$$\begin{aligned}
|z_h^{(i+1,m)}| &\le h_0\,\sigma_h(x^{(i)}, V_h(z_h)_{(i,m)}) + h_0\beta(h) + \\
&+\Big|\, z_h^{(i,m)}\,\Big| \left[1 - 2h_0 \sum_{j=1}^{n} \frac{1}{h_j^2} F_{ii}^{(i,m)} + h_0 \sum_{(j,k)\in X} \frac{1}{h_j h_k} |F_{jk}^{(i,m)}|\right] + \\
&+\sum_{j=1}^{n} \Big|\, z_h^{(i,m+e_j)}\,\Big| \left[\frac{h_0}{2h_j} D_{q_j} F_{0.h}(\tilde{Q}) + \frac{h_0}{h_j^2} F_{jj}^{(i,m)} + h_0 \sum_{k\in\pi[j]} \frac{1}{h_j h_k} |F_{jk}^{(i,m)}|\right] +
\end{aligned}$$

$$+\sum_{j=1}^{n}\left|z_h^{((i,m-e_j)}\right|\left[-\frac{h_0}{2h_j}D_{q_j}F_{0.h}(\tilde{Q})+\frac{h_0}{h_j^2}F_{jj}^{(i,m)}+h_0\sum_{k\in\pi[j]}\frac{1}{h_jh_k}|F_{jk}^{(i,m)}|\right]+$$
$$+h_0\sum_{(j,k)\in X_+[i,m]}|F_{jk}^{(i,m)}|\frac{1}{2h_jh_k}|z_h^{(i,m+e_j+e_k)}+z_h^{(i,m-e_j-e_k)}|+$$
$$+h_0\sum_{(j,k)\in X_-[i,m]}|F_{jk}^{(i,m)}|\frac{1}{2h_jh_k}|z_h^{(i,m-e_j+e_k)}+z_h^{(i,m+e_j-e_k)}|.$$

It follows from the assumption 2) that

$$|(u_h-v_h)^{(i+1,m)}|\le\eta_h^{(i)}+h_0\,\sigma_h(x^{(i)},(\eta_h)_{(i)})+h_0\beta(h)\ \ on\ E'_h\setminus E_h^{\rm imp}.$$

The above inequality and (3.40) imply (3.41). The proof of (3.42) is the same as in the proof of Theorem 3.9 Therefore we omit the details. From (3.41), (3.42) we see that $\eta_h^{(i)}\le\tilde{\eta}_h^{(i)},\ x^{(i)}\in J_h$, where $\tilde{\eta}_h$ is the solution of (3.45)–(3.47). Then we have $|u_h^{(i,m)}-v_h^{(i,m)}|\le\tilde{\eta}_h^{(i)}$ on E_h, and the assertion of our theorem follows from the stability of problem (3.26) (3.28). □

3.6. Remarks on Nonlinear Estimates for Increment Functions

In this Section we investigate the condition 3) of Assumption H $[\,\sigma_h,\tilde{\sigma}_h\,]$ on the stability of difference problem (3.26) (3.28). We consider a class of difference equations (3.23), (3.24) where f_h and g_h are superpositions of f and g and suitable interpolation operators. We will consider the operators

$$T_{h.i}:F(B_h,R)\to F(B,R),\quad 0\le i\le N_0,$$

defined in Section 2.5 with $B,\ B_h$ considered in Section 3.1. The interpolating operators $T_{h.i}$ were first introduced in [45] and were adopted in [14] for equations with impulses.

Consider the initial boundary value problem (3.15)–(3.17) and the difference equations

$$\delta_0 z^{(i,m)}=f\left(x^{(i)},y^{(m)},T_{h.i}z_{(i,m)},\delta z^{(i,m)},\delta^{(2)}z^{(i,m)}\right),$$
$$(x^{(i)},y^{(m)})\in E'_h\setminus E_h^{\rm imp},\tag{3.54}$$
$$z^{(i+1,m)}=z^{(i,m)}+g(x^{(i)},y^{(m)},T_{h.i}z^-_{(i,m)}),\ \ (x^{(i)},y^{(m)})\in E_h^{\rm imp},\tag{3.55}$$

with initial boundary condition (3.25) where $T_{h,i}z_{(i,m)^-}$ is the restriction of the function $T_{h,i}z_{(i,m)}$ to the set $B^{(-)}$. Suppose that $\sigma:(J\backslash I_{\rm imp})\times R_+\to R_+$ and $\tilde{\sigma}:I_{\rm imp}\times R_+\to R_+$ are given functions. We consider the comparison problem with impulses

$$\omega'(x)=\sigma(x,\omega(x))\ \ for\ x\in J\setminus J_{\rm imp},\tag{3.56}$$
$$\omega(x)=\omega(x^-)+\tilde{\sigma}(x,\omega(x^-))\ \ for\ x\in J_{\rm imp},\qquad\omega(0)=0.\tag{3.57}$$

For $w \in C_{\text{imp}}[B, R]$ we write

$$\|w\|_B = \sup\{\, |w(t,s)| : \;\; (t,s) \in B\,\}.$$

We will denote by $\|\cdot\|_{B^{(-)}}$ the supremum norm in the space $C_{\text{imp}}[B^{(-)}, R]$. Suppose that the following conditions are satisfied:

(I) σ and $\tilde{\sigma}$ are continuous, $\sigma(x, 0) = 0$ for $x \in J \setminus J_{\text{imp}}$, $\tilde{\sigma}(x, 0) = 0$ for $x \in J_{\text{imp}}$.

(II) The function σ is nondecreasing with respect to $(x, p) \in (J \setminus J_{\text{imp}}) \times R_+$ and the function $\tilde{\sigma}$ is nondecreasing with respect to the second variable.

(III) The maximum solution of problem (3.56), (3.57) is $\omega(x) = 0$ for $x \in J$.

(IV) We have the estimates

$$|f(x, y, w, q, r) - f(x, y, \bar{w}, q, r)| \le \sigma(x, \|w - \bar{w}\|_B) \;\text{ on }\; \Sigma$$

and

$$|g(x, y, w) - g(x, y, \bar{w})| \le \tilde{\sigma}(x, \|w - \bar{w}\|_{B^{(-)}}) \;\text{ on }\; \Sigma_{\text{imp}}.$$

The method (3.54), (3.55) is a particular case of (3.23), (3.24) for

$$f_h(x^{(i)}, y^{(m)}, w, q, r) = f(x^{(i)}, y^{(m)}, T_{h.i}w, q, r) \quad \text{on }\; \Sigma_h,$$
$$g_h(x^{(i)}, y^{(m)}, w) = g(x^{(i)}, y^{(m)}, (T_{h,i}w)_{(-)}) \quad \text{on }\; \Sigma_h^{\text{imp}},$$

where $(T_{h,i}w)_{(-)}$ is the restriction of $T_{h.i}w$ to the set $B^{(-)}$.

It is easy to see that the above functions f_h and g_h satisfy the compatibility conditions (3.35), (3.36) and that

$$|f_h(x^{(i)}, y^{(m)}, w, q, r) - f_h(x^{(i)}, y^{(m)}, \bar{w}, q, r)| \le \sigma(x^{(i)}, \|w - \bar{w}\|_h) \;\text{ on } \Sigma_h,$$
$$|g_h(x^{(i)}, y^{(m)}, w) - g_h(x^{(i)}, y^{(m)}, \bar{w})| \le \tilde{\sigma}(x^{(i)}, \|w - \bar{w}\|_h) \;\text{ on } \Sigma_h^{\text{imp}}.$$

Now we prove that the difference problem

$$\eta^{(i+1)} = \eta^{(i)} + h_0\, \sigma(x^{(i)}, \eta^{(i)}) \;\text{ for } x^{(i)} \in J'_h \setminus J_h^{\text{imp}}, \tag{3.58}$$
$$\eta^{(i+1)} = \eta^{(i)} + \tilde{\sigma}(x^{(i)}, \eta^{(i)}) \;\text{ for } x^{(i)} \in J_h^{\text{imp}}, \qquad \eta(0) = 0, \tag{3.59}$$

is stable in the sense of Assumption H $[\,\sigma_h, \tilde{\sigma}_h\,]$.

Let $\eta_h : J_h \to R_+$ be a solution of the problem

$$\eta^{(i+1)} = \eta^{(i)} + h_0\, \sigma(x^{(i)}, \eta^{(i)}) + h_0\gamma(h) \;\text{ for } x^{(i)} \in J'_h \setminus J_h^{\text{imp}},$$
$$\eta^{(i+1)} = \eta^{(i)} + \tilde{\sigma}(x^{(i)}, \eta^{(i)}) + \tilde{\gamma}(h) \;\text{ for } x^{(i)} \in J_h^{\text{imp}},$$
$$\eta(0) = \alpha_0(h),$$

with $\gamma,\ \tilde{\gamma},\ \alpha_0 : \Theta \to R_+$.

Denote by $\omega_h : J \to R_+$ the solution of the initial problem with impulses

$$\omega'(x) = \sigma(x, \omega(x)) \;+\; \gamma(h) \;\text{ for } x \in J \setminus J_{\text{imp}},$$
$$\omega(x) = \eta(x^-) + \tilde{\sigma}(x, \omega(x^-) + \tilde{\gamma}(h) \;\text{ for } x \in J_{\text{imp}},$$
$$\omega(0) = \alpha_0(h).$$

There is $\varepsilon_0 > 0$ such that for $|h| < \varepsilon_0$ the solution ω_h is defined on J and

$$\lim_{h\to 0} \omega_h(x) = 0 \text{ uniformly with respect to } x \in J.$$

It follows from condition (II) that ω_h satisfies the difference inequalities

$$\omega_h^{(i+1)} \geq \omega_h^{(i)} + h_0\, \sigma(x^{(i)}, \omega_h^{(i)}) + h_0\gamma(h) \text{ for } x^{(i)} \in J_h \setminus J_h^{\text{imp}},$$
$$\omega_h^{(i+1)} \geq \omega_h^{(i)} + \tilde{\sigma}(a_i, \omega_h^{(i)}) + \tilde{\gamma}(h), \quad x^{(i)} \in J_h^{\text{imp}}.$$

From the above estimates we have $\eta_h^{(i)} \leq \omega_h^{(i)} \leq \omega_h(a)$ for $x^{(i)} \in J_h$. Then the problem (3.58), (3.59) is stable in the sense of Assumption H $[\,\sigma_h, \tilde{\sigma}_h\,]$.

Remark 3.12. The results of this Chapter can be extended for weakly coupled parabolic systems with impulses

$$D_x z_i(x,y) = f_i(x, y, z_{(x,y)}, D_y z_i(x,y), D_{yy} z_i(x,y)),$$
$$(x,y) \in E \setminus E_{\text{imp}}, \quad i = 1, \ldots, k,$$
$$\Delta z(x,y) = g(x, y, z_{(x^-,y)}), \quad (x,y) \in E_{\text{imp}},$$

where

$$f = (f_1, \ldots, f_k) : (\Sigma \setminus \Sigma_{\text{imp}}) \times C_{\text{imp}}[B, R^k] \times R^n \times M_{n\times n} \ \to R^k,$$
$$g = (g_1, \ldots, g_k) : \Sigma_{\text{imp}} \times R^k \times C_{\text{imp}}[B^{(-)}, R^k]$$

and $z = (z_1, \ldots, z_k)$.

3.7. Numerical Example

For $n = 2$ we put

$$E = (0,\ 1.5] \times (-1,\ 1) \times (-1,\ 1),$$
$$B = \{\,0\,\} \times [-0.5,\ 0.5] \times [-0.5,\ 0.5], \quad J_{\text{imp}} = \{0.5,\ 1\}$$

and

$$E_0 = \{0\} \times [-1.5, 1.5] \times [-1.5, 1.5],$$
$$\partial_0 E = (0,\ 1.5] \times ([-1.5, 1.5] \times [-1.5, 1.5] \setminus (-1,\ 1) \times (-1,\ 1))\,.$$

Consider the differential - integral equation

$$D_x z(x,y) = \frac{1}{4}\,[D_{y_1y_1} z(x,y) + D_{y_2y_2} z(x,y)] + \frac{1}{8}(x + y_1 + y_2) D_{y_1y_2} z(x,y) +$$
$$+6x \left[\int_B z(x, y+s)ds - z(x,y)\right] + f(x,y), \quad (x,y) \in E \setminus E_{\text{imp}}, \quad (3.60)$$

with impulses

$$\Delta z(x,y) = \int_B z(x-0.25, y+s)ds - z(x-0.25,y) + g(x,y), \quad (3.61)$$

$$(x,y) \in E_{\text{imp}},$$

and with initial boundary conditions

$$\begin{gathered} z(0,y) = 0 \ \ for\ y \in [-1.5,\ 1.5] \times [-1.5,\ 1.5], \\ z(x,y) = \psi(x,y) \ \text{ for } (x,y) \in \partial_0 E, \end{gathered} \quad (3.62)$$

where

$$f(x,y) = 1 + y_1^2 + y_2^2 + y_1 y_2 - x - x^2 - \frac{x}{8}(x + y_1 + y_2),$$
$$\psi(x,y) = x(1 + y_1^2 + y_2^2 + y_1 y_2) \ \text{ for } 0 < x < 0.5, \ \ (x,y) \in \partial_0 E,$$
$$\psi(x,y) = x(1 + y_1^2 + y_2^2 + y_1 y_2) + y_1 + y_2 \ \text{ for } 0.5 \le x < 1, \ \ (x,y) \in \partial_0 E,$$
$$\psi(x,y) = x(1 + y_1^2 + y_2^2 + y_1 y_2) - \frac{1}{2}(y_1 + y_2)$$
$$\text{for } \ 1 \le x \le 1.5, \ \ (x,y) \in \partial_0 E,$$

and

$$g(0.5,\, y) = y_1 + y_2 - \frac{1}{24}, \quad g(1,y) = -\frac{3}{2}(y_1+y_2) - \frac{1}{8}, \quad y \in (-1,\, 1)\times(-1,\, 1).$$

Let

$$B_h = \{(x^{(i)}, y_1^{(j)}, y_2^{(k)}) : \ i = 0, \ \ -K_1 \le j \le K_1, \ -K_2 \le k \le K_2\},$$

where $K_1, K_2 \in N$ and $K_1 h_1 = K_2 h_2 = 0.5$. We define the operator $T_h : F(B_h, R) \to F(B, R)$ in the following way. Suppose that $w \in F(B_h, R)$ and $y = (y_1, y_2), \ -0.5 \le y_i \le 0.5, \ i = 1, 2$. There exists $m = (m_1, m_2) \in Z^2$ such that $-K_i \le m_i < K_i, \ i = 1, 2$ and $y_i^{(m_i)} \le y_i \le y_i^{m_i+1)}, \ i = 1, 2$. We put

$$(T_h w)(0,y) = w^{(0,m_1+1,m_2+1)} \ \frac{y_1 - y_1^{(m_1)}}{h_1} \ \frac{y_2 - y_2^{(m_2)}}{h_2} +$$
$$+ w^{(0,m_1,m_2+1)} \ \left(1 - \frac{y_1 - y_1^{(m_1)}}{h_1}\right) \ \frac{y_2 - y_2^{(m_2)}}{h_2} +$$
$$+ \ w^{(0,m_1+1,m_2)} \ \frac{y_1 - y_1^{(m_1)}}{h_1} \ \left(1 - \frac{y_2 - y_2^{(m_2)}}{h_2}\right) +$$
$$+ \ w^{(0,m_1,m_2)} \left(1 - \frac{y_1 - y_1^{(m_1)}}{h_1}\right) \ \left(1 - \frac{y_2 - y_2^{(m_2)}}{h_2}\right).$$

The difference method for the above problem has the form

$$\begin{aligned}
z^{(i+1,j,k)} &= z^{(i,j,k)} + \frac{h_0}{4}\left(\delta_{11}^{(2)} z^{(i,j,k)} + \delta_{22}^{(2)} z^{(i,j,k)}\right) + \\
&+ h_0 \frac{1}{8}\left(x^{(i)} + y_1^{(j)} + y_2^{(k)}\right) \delta_{12}^{(2)} z^{(i,j,k)} + \\
&+ 6x^{(i)} h_0 \left(\int_B T_h z_{(i,j,k)}(0,s)ds - z^{(i,j,k)}\right) + \\
&+ h_0 f(x^{(i)}, y_1^{(j)}, y_2^{(k)}), \quad (x^{(i)}, y_1^{(j)}, y_2^{(k)}) \in E_h' \setminus \Sigma_h^{\text{imp}}, \qquad (3.63)\\
z^{(i+1,j,k)} &= z^{(i,j,k)} + \int_B T_h z_{(i',j,k)}(0,s)ds - z^{(i',j,k)} + \\
&+ g(x^{(i)}, y_1^{(j)}, y_2^{(k)}), \quad (x^{(i)}, y_1^{(j)}, y_2^{(k)}) \in \Sigma_h^{\text{imp}}, \qquad (3.64)
\end{aligned}$$

and

$$\begin{aligned}
z^{(0,j,k)} &= 0 \ \text{ for } (0, y_1^{(j)}, y_2^{(k)}) \in E_{0.h}, \\
z^{(i,j,k)} &= \psi^{(i,j,k)} \ \text{ for } (x^{(i)}, y_1^{(j)}, y_2^{(k)}) \in \partial_0 E_h,
\end{aligned} \qquad (3.65)$$

where $\delta_{11}^{(2)},\ \delta_{22}^{(2)}$ are given by (3.20) for $n = 2$ and $\delta_{12}^{(2)}$ is defined by

$$\begin{aligned}
\delta_{12}^{(2)} z^{(i,j,k)} &= \frac{1}{2h_1h_2}\left(z^{(i,j+1,k)} + z^{(i,j-1,k)} + z^{(i,j,k+1)} + z^{(i,j,k-1)} - \right.\\
&\left. -2z^{(i,j,k)} - z^{(i,j+1,k-1)} - z^{(i,j+1,k-1)}\right) \ \text{ if } x^{(i)} + y_1^{(j)} + y_2^{(k)} < 0,\\
\delta_{12}^{(2)} z^{(i,j,k)} &= \frac{1}{2h_1h_2}\left(-z^{(i,j+1,k)} - z^{(i,j-1,k)} - z^{(i,j,k+1)} - z^{(i,j,k-1)} + \right.\\
&\left. +2z^{(i,j,k)} + z^{(i,j+1,k+1)} + z^{(i,j-1,k-1)}\right) \ \text{ if } x^{(i)} + y_1^{(j)} + y_2^{(k)} \geq 0.
\end{aligned}$$

The above formulas are identical with (3.52), (3.53) for $n = 2$. The index i' in (3.64) is defined by $i'h_0 = x^{(i)} - 0.25$.

If $h_1 = h_2 = \tilde{h}$ and $1 - h_0\,\tilde{h}^{-2} \geq 0$ then the difference method (3.63)–(3.65) is convergent.

We take $h_0 = 10^{-4},\ h_1 = h_2 = 10^{-2}$ Let $\Omega = \bar{E} \cup \partial_0 E$. The function

$$\begin{aligned}
v(x,y) &= x(1 + y_1^2 + y_2^2 + y_1y_2) \ \text{ for } 0 \leq x < 0.5,\ (x,y) \in \Omega,\\
v(x,y) &= x(1 + y_1^2 + y_2^2 + y_1y_2) + y_1 + y_2 \ \text{ for } 0.5 \leq x < 1,\ (x,y) \in \Omega,\\
v(x,y) &= x(1 + y_1^2 + y_2^2 + y_1y_2) - \frac{1}{2}(y_1 + y_2) \ \text{ for } 1 \leq x \leq 1.5,\ (x,y) \in \Omega,
\end{aligned}$$

is the solution of (3.60)–(3.62). Let $u_h : E_h \to R$ be a solution of (3.63)–(3.65) and $\varepsilon_h = u_h - v_h$. Some values of $\varepsilon_h^{(i,j,k)}$ are listed in the table for $x = 0.75,\ x = 1.25$.

TABLE OF ERRORS, x = 0.75

(y_1, y_2)	-0.3	0	0.3
-0.3	$-6.33\ 10^{-2}$	$-5.19\ 10^{-2}$	$-4.01\ 10^{-2}$
0	$-5.43\ 10^{-2}$	$-5.43\ 10^{-2}$	$-3.11\ 10^{-2}$
0.3	$-5.21\ 10^{-2}$	$-4.99\ 10^{-2}$	$-6.01\ 10^{-2}$

TABLE OF ERRORS, x = 1.25

(y_1, y_2)	-0.3	0	0.3
-0.3	$4.54\ 10^{-2}$	$-4.10\ 10^{-2}$	$-3.98\ 10^{-2}$
0	$-3.34\ 10^{-2}$	$-2.98\ 10^{-2}$	$2.87\ 10^{-2}$
0.3	$3.71\ 10^{-2}$	$3.27\ 10^{-2}$	$4.56\ 10^{-2}$

The computation was performed by the computer IBM AT.

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