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ON THE COMPARISON THEOREMS FOR STABILITY OF LINEAR SYSTEMS OF IMPULSIVE EQUATIONS

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In the present note, we consider a linear system of impulsive equations of the form given in [1]

$$\frac{dx}{dt} = Q(t)x + q(t) \quad \text{for } t \in \mathbb{R}_+, \tag{1}$$

$$x(t_j+) - x(t_j-) = G_j x(t_j-) + g_j \quad (j = 1, 2, \dots), \tag{2}$$

where $Q : \mathbb{R}_+ \rightarrow \mathbb{R}^{n \times n}$ and $q : \mathbb{R}_+ \rightarrow \mathbb{R}^n$ are, respectively, a matrix and the vector functions such that each of their components is integrable in the Lebesgue sense on every closed segment from \mathbb{R}_+ ; G_j and g_j ($j = 1, 2, \dots$) are, respectively, constant matrices and vectors; $t_j \in \mathbb{R}_+$ ($j = 1, 2, \dots$), $0 < t_1 < t_2 < \dots$, $\lim_{j \rightarrow +\infty} t_j = +\infty$.

We give some sufficient conditions guaranteeing the stability of the system (1), (2) in the Lyapunov sense with respect to small perturbations. Analogous results are given in [2] for linear systems of ordinary differential equations and in [3-5] for linear systems of generalized ordinary differential equations.

We use the following notation and definitions:

$\mathbb{R} =] - \infty, +\infty[$ is the set of all real numbers, $\mathbb{R}_+ = [0, +\infty[$;

$\mathbb{R}^{n \times m}$ is the space of all real $n \times m$ -matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm

$$\|X\| = \sum_{j=1}^m \sum_{i=1}^n |x_{ij}|;$$

$\mathbb{R}^n = \mathbb{R}^{n \times 1}$ is the space of all real column n -vectors $x = (x_i)_{i=1}^n$;

If $X \in \mathbb{R}^{n \times n}$, then X^{-1} and $\det(X)$ are, respectively, the matrix inverse to X and the determinant of X ; I_n is the identity $n \times n$ -matrix;

$X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of the matrix-function $X : \mathbb{R}_+ \rightarrow \mathbb{R}^{n \times m}$ at the point t ;

$L_{\text{loc}}(\mathbb{R}_+; \mathbb{R}^{n \times m})$ is the set of all matrix-functions $X : \mathbb{R}_+ \rightarrow \mathbb{R}^{n \times m}$ such that each of their components is a measurable and integrable functions in Lebesgue sense on every closed segment from \mathbb{R}_+ ;

If I is an arbitrary interval from \mathbb{R}_+ , then $\tilde{C}_{\text{loc}}(I; \mathbb{R}^{n \times m})$ is the set of all matrix-functions $X : I \rightarrow \mathbb{R}^{n \times m}$ such that each of their components is absolutely continuous on every closed segment from I ; $\tilde{C}_{\text{loc}}(\mathbb{R}_+ \setminus T; \mathbb{R}^{n \times m})$, where $T = \{t_1, t_2, \dots\}$, is the set of all matrix-functions $X : \mathbb{R}_+ \rightarrow \mathbb{R}^{n \times m}$ with restrictions on $]t_j, t_{j+1}[$ belonging to $\tilde{C}_{\text{loc}}(]t_j, t_{j+1}[; \mathbb{R}^{n \times m})$ for every $j \in \{1, 2, \dots\}$.

Under a solution of the system (1), (2) we understand a continuous from the left vector-function $x \in \tilde{C}_{\text{loc}}(\mathbb{R}_+ \setminus T; \mathbb{R}^n)$ satisfying the system (1) almost everywhere on $]t_j, t_{j+1}[$ and the relation (2) at the point t_j for every $j \in \{1, 2, \dots\}$.

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We will assume that $Q = (q_{ik})_{i,k=1}^n \in L_{\text{loc}}(\mathbb{R}_+; \mathbb{R}^{n \times n})$, $G_j = (g_{jik})_{i,k=1}^n \in \mathbb{R}^{n \times n}$ ($j = 1, 2, \dots$) and

$$\det(I_n + G_j) \neq 0 \quad (j = 1, 2, \dots).$$

Definition 1. Let $\xi \in \tilde{C}_{\text{loc}}(\mathbb{R}_+ \setminus T; \mathbb{R}_+)$ be a continuous from the left nondecreasing function such that

$$\lim_{t \rightarrow +\infty} \xi(t) = +\infty. \quad (3)$$

A solution x_0 of the system (1), (2) is called ξ -exponentially asymptotically stable if there exists a positive number η such that for every $\varepsilon > 0$ there exists a positive number $\delta = \delta(\varepsilon)$ such that an arbitrary solution x of the system (1), (2) satisfying the inequality

$$\|x(t_0) - x_0(t_0)\| < \delta$$

for some $t_0 \in \mathbb{R}_+$, admits the estimate

$$\|x(t) - x_0(t)\| < \varepsilon \exp(-\eta(\xi(t) - \xi(t_0))) \quad \text{for } t \geq t_0.$$

It is evident that the exponential asymptotical stability is a particular case of the ξ -exponential asymptotical stability if we assume $\xi(t) \equiv t$.

Stability, uniform stability and asymptotical stability are defined just in the same way as for systems of ordinary differential equations (see, e.g., [2] or [6]).

Definition 2. The system (1), (2) is called stable in one or another sense if every solution of this system is stable in the same sense.

As in the case of ordinary differential equations, the system (1), (2) is stable in one or another sense if and only if its corresponding homogeneous system

$$\frac{dx}{dt} = Q(t)x \quad \text{for } t \in \mathbb{R}_+, \quad (1_0)$$

$$x(t_j+) - x(t_j-) = G_j x(t_j-) \quad (j = 1, 2, \dots) \quad (2_0)$$

is stable in the same sense.

Definition 3. The pair $(Q, \{G_j\}_{j=1}^{+\infty})$ is called stable in one or another sense if the system (1₀), (2₀) is stable in the same sense.

Theorem 1. Let the matrix functions $Q = (q_{ik})_{i,k=1}^n \in L_{\text{loc}}(\mathbb{R}_+; \mathbb{R}^{n \times n})$ and $Q_0 = (q_{oik})_{i,k=1}^n \in L_{\text{loc}}(\mathbb{R}_+; \mathbb{R}^{n \times n})$, the constant matrices $G_j = (g_{jik})_{i,k=1}^n \in \mathbb{R}^{n \times n}$ and $G_{oj} = (g_{ojik})_{i,k=1}^n \in \mathbb{R}^{n \times n}$ ($j = 1, 2, \dots$) be such that

$$\begin{aligned} \|G_{oj}\| &< 1 \quad (j = 1, 2, \dots), \\ q_{ii}(t) &\leq q_{oii}(t) \quad (i = 1, \dots, n), \\ |q_{ik}(t)| &\leq q_{oik}(t) \quad (i \neq k; \quad i, k = 1, \dots, n) \end{aligned}$$

and

$$|g_{jik}| \leq g_{ojik} \quad (i, k = 1, \dots, n; \quad j = 1, 2, \dots)$$

almost everywhere on $[t^*, +\infty[$ for some $t^* \in \mathbb{R}_+$. Let, moreover, the pair $(Q_0, \{G_{oj}\}_{j=1}^{+\infty})$ be stable (uniformly stable, asymptotically stable, ξ -exponentially asymptotically stable). Then the pair $(Q, \{G_j\}_{j=1}^{+\infty})$ is stable (uniformly stable, asymptotically stable, ξ -exponentially asymptotically stable) as well.

Theorem 2. Let the pair $(Q_0, \{G_{oj}\}_{j=1}^{+\infty})$ be uniformly stable, where $Q_0 \in L_{\text{loc}}(\mathbb{R}_+; \mathbb{R}^{n \times n})$, and $G_{oj} \in \mathbb{R}^{n \times n}$ ($j = 1, 2, \dots$) are constant matrices such that

$$\det(I_n + G_{oj}) \neq 0 \quad (j = 1, 2, \dots). \quad (4)$$

Let, moreover, the matrix function $Q \in L_{\text{loc}}(\mathbb{R}_+; \mathbb{R}^{n \times n})$ and the constant matrices $G_j \in \mathbb{R}^{n \times n}$ ($j = 1, 2, \dots$) satisfy the conditions

$$\int_0^{+\infty} \|Q(t) - Q_0(t)\| dt < +\infty$$

and

$$\sum_{j=1}^{+\infty} \|(I_n + G_{0j})^{-1}(G_j - G_{0j})\| < +\infty.$$

Then the pair $(Q, \{G_j\}_{j=1}^{+\infty})$ is uniformly stable.

Theorem 3. Let the pair $(Q_0, \{G_{0j}\}_{j=1}^{+\infty})$ be ξ -exponentially asymptotically stable and let the condition (4) hold, where $Q_0 \in L_{\text{loc}}(\mathbb{R}_+; \mathbb{R}^{n \times n})$, $G_{0j} \in \mathbb{R}^{n \times n}$ ($j = 1, 2, \dots$). Let, moreover, $Q \in L_{\text{loc}}(\mathbb{R}_+; \mathbb{R}^{n \times n})$ and $G_j \in \mathbb{R}^{n \times n}$ ($j = 1, 2, \dots$) be such that

$$\lim_{t \rightarrow +\infty} \int_t^{\nu(\xi)(t)} \|Q(\tau) - Q_0(\tau)\| d\tau = 0$$

and

$$\lim_{t \rightarrow +\infty} \sum_{t \leq t_j < \nu(\xi)(t)} \|(I_n + G_{0j})^{-1}(G_j - G_{0j})\| = 0,$$

where $\nu(\xi)(t) = \sup\{\tau \geq t : \xi(\tau) \leq \xi(t+) + 1\}$. Then the pair $(Q, \{G_j\}_{j=1}^{+\infty})$ is ξ -exponentially asymptotically stable as well.

Corollary 1. Let the components q_{ik} ($i, k = 1, \dots, n$) and g_{jik} ($i, k = 1, \dots, n$; $j = 1, 2, \dots$) of the matrix-function $Q \in L_{\text{loc}}(\mathbb{R}_+; \mathbb{R}^{n \times n})$ and of the constant matrices $G_j \in \mathbb{R}^{n \times n}$ ($j = 1, 2, \dots$), respectively, satisfy the conditions

$$\begin{aligned} 1 + g_{jii} &\neq 0 \quad (i = 1, \dots, n; j = 1, 2, \dots), \\ \lim_{t \rightarrow +\infty} \int_t^{\nu(\xi)(t)} |q_{ik}(\tau)| d\tau &= 0 \quad (i, k = 1, \dots, n), \\ \lim_{t \rightarrow +\infty} \sum_{t \leq t_j < \nu(\xi)(t)} |(1 + g_{jii})^{-1} g_{jik}| &= 0, \\ q_{ii}(t) &\leq -\eta \xi'(t) \quad \text{for } t \in [t^*, +\infty[\setminus T \quad (i = 1, \dots, n) \end{aligned}$$

and

$$g_{jii} \leq -\eta(\xi(t_j+) - \xi(t_j)) \quad (i = 1, \dots, n; j = 1, 2, \dots),$$

where $\eta > 0$, $t^* \in \mathbb{R}_+$, $\xi \in \tilde{C}_{\text{loc}}(\mathbb{R}_+ \setminus T; \mathbb{R}_+)$ is the continuous from left nondecreasing function satisfying the condition (3), and $\nu(\xi) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is the function defined as in Theorem 3. Then the pair $(Q, \{G_j\}_{j=1}^{+\infty})$ is ξ -exponentially asymptotically stable.

Note that the system (1), (2) can be rewritten in the form of the system of so called generalized linear ordinary differential equations

$$dx(t) = dA(t) \cdot x(t) + df(t) \quad \text{for } t \in \mathbb{R}_+, \quad (5)$$

where $A : \mathbb{R}_+ \rightarrow \mathbb{R}^{n \times n}$ and $f : \mathbb{R}_+ \rightarrow \mathbb{R}^n$ are, respectively, matrix and vector functions with the components having bounded variation on every closed interval from \mathbb{R}_+ (see [7]).

Under a solution of the system (5) we understand a vector-function $x : \mathbb{R}_+ \rightarrow \mathbb{R}^{n \times n}$ with the components having bounded variation on every closed interval from \mathbb{R}_+ and

such that

$$x(t) = x(s) + \int_s^t dA(\tau) \cdot x(\tau) + f(t) - f(s) \quad \text{for } 0 \leq t \leq s,$$

where the integral is understood in Lebesgue–Stieltjes sense.

The vector-function x is a solution of (1), (2) iff it is a solution of (5), where

$$A(t) \equiv \int_0^t Q(\tau) d\tau + \sum_{0 \leq t_j < t} G_j, \quad f(t) \equiv \int_0^t q(\tau) d\tau + \sum_{0 \leq t_j < t} g_j.$$

Hence the results given above immediately follow from analogous results contained in [3-5] for the system (5).

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