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Haishen Lü, Donal O'Regan and Ravi P. Agarwal

NONUNIFORM NONRESONANCE AT THE FIRST EIGENVALUE OF THE ONE-DIMENSIONAL SINGULAR p-LAPLACIAN

Abstract. In this paper, general existence theorems are presented for the singular equation

$$\begin{cases} -\left(\varphi_p\left(u'\right)\right)' = f\left(t, u, u'\right), \quad 0 < t < 1, \\ u\left(0\right) = u\left(1\right) = 0. \end{cases}$$

Throughout, our nonlinearity is allowed to change sign. The singularity may occur at $u = 0$, $t = 0$, $t = 1$ and f may be nonuniform nonresonant at the first eigenvalue.

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$$\begin{cases} -\left(\varphi_p\left(u'\right)\right)' = f\left(t, u, u'\right), \quad 0 < t < 1, \\ u\left(0\right) = u\left(1\right) = 0 \end{cases}$$

სინგულარული ამოცანისთვის, ამასთან არაწრფივი წევრი შეიძლება ნიშანცვლადი იყოს. $u = 0$, $t = 0$, $t = 1$ წერტილებში შეიძლება ადგილი ჰქონდეს სინგულარობას, ხოლო f შეიძლება არათანაბრად არარეზონანსული იყოს პირველ საკუთრივ მნიშვნელობაზე.

1. Introduction

In this paper, we study the singular boundary value problem

$$\begin{cases} -\left(\varphi_p\left(u'\right)\right)' = f\left(t, u, u'\right), & 0 < t < 1, \\ u\left(0\right) = u\left(1\right) = 0, \end{cases} \tag{1.1}$$

where $\varphi_p\left(s\right) = \left|s\right|^{p-2} s,\ p > 1$. The singularity may occur at $u = 0$, $t = 0$ and $t = 1$, and the function f is allowed to change sign and is nonuniform nonresonant at the first eigenvalue. Note that f may not be a Carathéodory function because of the singular behavior of the u variable. In the literature $[8, 9, 12]$, (1.1) has been discussed extensively when $f\left(t, u, v\right) \equiv f\left(t, u\right)$ and f is positive, i.e., $f : \left(0, 1\right) \times \left(0, \infty\right) \to \left(0, \infty\right)$. Recently [1], [13] (1.1) was discussed when $f\left(t, u, v\right) \equiv f\left(t, u\right)$ and $f : \left(0, 1\right) \times \left(0, \infty\right) \to R$. The case when f depends on the u' variable has received very little attention in the literature, see [2], [3], [7] and references therein. In [14], the author studied nonuniform nonresonance at the first eigenvalue of the $p-$Laplacian when the function f is not singular. This paper presents a new and very general result for (1.1) when $f : \left(0, 1\right) \times \left(0, \infty\right) \times R \to R$ and f is nonuniform nonresonant at the first eigenvalue.

The nonlinear eigenvalue problem associated with the problem (1.1) is

$$\begin{cases} -\left(\varphi_p\left(u'\right)\right)' = \lambda \varphi_p\left(u\right), & 0 < t < 1, \\ u\left(0\right) = u\left(1\right) = 0. \end{cases} \tag{1.2}$$

It is well-known (see [14]) that (1.2) has eigenvalues

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \quad \text{as} \quad n \to \infty.$$

In what follows, we will use $\left\|\cdot\right\|_p$ to denote the L^p-norm defined by

$$\left\|u\right\|_p = \left(\int_0^1 \left|u\left(t\right)\right|^p dt \right)^{\frac{1}{p}}.$$

The $C\left[0, 1\right]$-norm is

$$\left\|u\right\|_\infty = \sup_{0 \le t \le 1} \left|u\left(t\right)\right|.$$

We present some results from literature which will be needed in Section 2. Let $W = W_0^{1,p}\left(\left[0, 1\right], R\right)$ be the Sobolev space. The following lemma is a result of embedding inequalities.

Lemma 1.1 ([14])**.** (1) *We have*

$$\left\|u\right\|_p \le \lambda_1^{-\frac{1}{p}} \left\|u'\right\|_p \quad \textit{for} \quad \forall u \in W. \tag{1.3}$$

Moreover, the equality in (1.3) *holds if and only if u is an eigenfunction corresponding to the eigenvalue λ_1.*

(2)

$$\left\|u\right\|_\infty \le \left(\frac{1}{2}\right)^{1/q} \left\|u'\right\|_p \quad \textit{for} \quad \forall u \in W, \tag{1.4}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Lemma 1.2 ([14])**.** *Suppose that* $a\in C[0,1]$ *satisfies the condition:*

$$a(t)<\|a\|_\infty$$

on a subset of $[0,1]$ *of positive measure. Then there exists* $\varepsilon>0$ *such that*

$$\int_0^1 a(t)\,|u(t)|^p\,dt\le\left(\|a\|_\infty\lambda_1^{-1}-\varepsilon\right)\|u'\|_p^p\quad\text{for all}\quad u\in W. \tag{1.5}$$

Lemma 1.3 ([7])**.** *Let* $e_n=\left[\frac{1}{2^{n+1}},1\right]\ (n\ge1)$, $e_0=\varnothing$. *If there exists a sequence* $\{\varepsilon_n\}\downarrow0$ *and* $\varepsilon_n>0$ *for* $n\ge1$, *then there exists a function* $\lambda\in C^1[0,1]$ *such that*

(1) $\varphi_p(\lambda')\in C^1[0,1]$ *and* $\max\limits_{0\le t\le1}\left|(\varphi_p(\lambda'(t)))'\right|>0$, *and*

(2) $\lambda(0)=\lambda(1)=0$ *and* $0<\lambda(t)\le\varepsilon_n$, $t\in e_n\backslash e_{n-1}$, $n\ge1$.

2. Main Existence Theorem

We present a general existence theorem for the BVP (1.1).

Theorem 2.1. *Let* $n_0\in\{1,2,\dots\}$ *be fixed and suppose the following conditions are satisfied:*

$$f:(0,1)\times(0,\infty)\times R\to R\ \ \text{is continuous,} \tag{2.1}$$

$$\begin{cases}\text{let } n\in\{n_0,n_0+1,\cdots\}\equiv N_0 \text{ and associated with each } n\in N_0\\ \text{we have a constant } \rho_n \text{ such that } \{\rho_n\} \text{ is a nondecreasing}\\ \text{sequence with } \lim_{n\to\infty}\rho_n=0 \text{ and}\\ \text{for } \frac{1}{2^{n+1}}\le t\le1 \text{ we have } f(t,\rho_n,0)\ge0,\end{cases} \tag{2.2}$$

$$\begin{cases}\text{there exists } \alpha\in C[0,1],\ \alpha(0)=0=\alpha(1),\ \ \alpha>0 \text{ on } (0,1),\\ \text{such that if } h:(0,1)\times(0,\infty)\times R\to R\\ \text{is any continuous function with}\\ h(t,u,v)\ge f(t,u,v),\ \forall(t,u,v)\in(0,1]\times(0,\infty)\times R\\ \text{and if } u\in C^1[0,1],\ \varphi(u')\in C^1(0,1),\ u(t)>0 \text{ for } t\in[0,1],\\ \text{is any solution of}\\ -(\varphi_p(u'))'=h(t,u,u'),\ \text{then } u(t)\ge\alpha(t) \text{ for } t\in[0,1],\end{cases} \tag{2.3}$$

$$\begin{cases}\text{for any } \varepsilon>0 \text{ there exist } \gamma,\tau \text{ with } 1\le\gamma<p,\ 0\le\tau<p-1,\\ \text{functions } a,b\in C[0,1] \text{ with } a\ge0,\ b\ge0,\ \text{on } [0,1],\\ \text{functions } c\in L^1[0,1],\ d\in L^{\frac{p}{p-\tau}}[0,1],\ h_\varepsilon\in L^1[0,1]\\ \text{with } c\ge0,\ d\ge0,\ h_\varepsilon\ge0 \text{ a.e. on } [0,1],\ \text{such that}\\ uf(t,u,v)\le a(t)u^p+b(t)u|v|^{p-1}+c(t)u^\gamma+\\ +d(t)u|v|^\tau+uh_\varepsilon(t)\ \text{for } t\in(0,1),\ u\ge\varepsilon \text{ and } v\in R,\end{cases} \tag{2.4}$$

$$\begin{cases} \textit{either} \\ \text{(i) } a(t) < |a|_\infty \ \textit{on a subset of } [0,1] \textit{ of positive measure} \\ \ \textit{and } a\left(\frac{1}{2^{n_0+1}}\right) < \|a\|_\infty \\ \textit{or} \\ \text{(ii) } b(t) < |b|_\infty \ \textit{on a subset of } [0,1] \textit{ of positive measure} \\ \textit{and } b\left(\frac{1}{2^{n_0+1}}\right) < \|b\|_\infty, \end{cases} \tag{2.5}$$

$$\lambda_1^{-1} \|a\|_\infty + \lambda_1^{-\frac{1}{p}} \|b\|_\infty \leq 1 \tag{2.6}$$

and

$$\begin{cases} \textit{for any } \varepsilon > 0, \textit{ there exist } \delta, \beta, \textit{ with } 1 \leq \delta < p, \ 0 \leq \beta < p, \\ \textit{functions } a_0 \in L^1[0,1], \ b_0 \in L^{\frac{p}{p-\beta}} \textit{ and } \eta_\varepsilon \in L^1[0,1] \textit{ with} \\ a_0 \geq 0, \ b_0 \geq 0, \ \eta_\varepsilon \geq 0 \textit{ a.e. on } [0,1], \textit{ such that} \\ |f(t,u,v)| \leq a_0(t) u^\delta + b_0(t) |v|^\beta + \eta_\varepsilon(t) \\ \textit{for } t \in (0,1), \ u \geq \varepsilon \textit{ and } v \in R. \end{cases} \tag{2.7}$$

Then (1.1) *has a solution* $u \in C[0,1]$ *with* $u(t) \geq \alpha(t)$ *for* $t \in [0,1]$ (*here* α *is given in* (2.3)).

Proof. For $n = n_0, n_0+1, \ldots$ let

$$e_n = \left[\frac{1}{2^{n+1}}, 1\right] \quad \text{and} \quad \theta_n(t) = \max\left\{\frac{1}{2^{n+1}}, t\right\}, \quad 0 \leq t \leq 1,$$

and

$$f_n(t,x,y) = \max\{f(\theta_n(t), x, y), f(t,x,y)\}.$$

Next we define inductively

$$g_{n_0}(t,x,y) = f_{n_0}(t,x,y)$$

and

$$g_n(t,x,y) = \min\{f_{n_0}(t,x,y), \ldots, f_n(t,x,y)\}, \quad n = n_0+1, \ n_0+2, \ldots .$$

Notice

$$f(t,x,y) \leq \cdots \leq g_{n+1}(t,x,y) \leq g_n(t,x,y) \leq \cdots \leq g_{n_0}(t,x,y)$$

for $(t,x,y) \in (0,1] \times (0,\infty) \times R$ and

$$g_n(t,x,y) = f(t,x,y) \quad \text{for} \quad (t,x,y) \in e_n \times (0,\infty) \times R.$$

We begin with the boundary value problem

$$\begin{cases} -(\varphi_p(u'))' = g_{n_0}^*(t,u,u'), \ \ 0 < t < 1, \\ u(0) = u(1) = \rho_{n_0}, \end{cases} \tag{2.8}$$

where

$$g_{n_0}^*(t,u,v) = \begin{cases} g_{n_0}(t, \rho_{n_0}, v) + r(\rho_{n_0} - u), & u \leq \rho_{n_0}, \\ g_{n_0}(t,u,v), & \rho_{n_0} \leq u, \end{cases}$$

with $r : R \to [-1, 1]$ the radial retraction defined by

$$r(u) = \begin{cases} u, & |u| \le 1, \\ \frac{u}{|u|}, & |u| > 1. \end{cases}$$

To show that (2.8) has a solution, we consider [7, 11] the family of problems

$$\begin{cases} -(\varphi_p(u'))' = \lambda g^*_{n_0}(t, u, u'), \quad 0 < t < 1, \\ u(0) = u(1) = \rho_{n_0}, \end{cases} \qquad (2.9)_\lambda$$

where $0 < \lambda < 1$. Let u be any solution of $(2.9)_\lambda$ for some $0 < \lambda \le 1$. We first show

$$u(t) \ge \rho_{n_0}, \quad t \in [0, 1]. \qquad (2.10)$$

Suppose (2.10) is not true. Then there exists a $t_0 \in (0, 1)$ with $u(t_0) < \rho_{n_0}$, $u'(t_0) = 0$ and

$$(\varphi_p(u'))'(t_0) \ge 0.$$

However note

$$\begin{aligned} (\varphi_p(u'))'(t_0) &= -\lambda [g_{n_0}(t_0, \rho_{n_0}, u'(t_0)) + r(\rho_{n_0} - u(t_0))] = \\ &= -\lambda [g_{n_0}(t_0, \rho_{n_0}, 0) + r(\rho_{n_0} - u(t_0))]. \end{aligned}$$

We need to discuss two cases, namely $t_0 \in [\frac{1}{2^{n_0+1}}, 1)$ and $t_0 \in \left(0, \frac{1}{2^{n_0+1}}\right)$.

Case 1. $t_0 \in \left[\frac{1}{2^{n_0+1}}, 1\right)$.

Then since $g_{n_0}(t_0, u, v) = f(t_0, u, v)$ for $(u, v) \in (0, \infty) \times R$ (note $t_0 \in e_{n_0}$), we have

$$\left(\varphi_p\left(u'_{n_0}\right)\right)'(t_0) = -\lambda f(t_0, \rho_{n_0}, 0) - r(\rho_{n_0} - u(t_0)) < 0,$$

a contradiction.

Case 2. $t_0 \in \left(0, \frac{1}{2^{n_0+1}}\right)$.

Then since

$$g_{n_0}(t_0, u, v) = \max\left\{ f\left(\frac{1}{2^{n_0+1}}, u, v\right),\ f(t_0, u, v)\right\},$$

we have

$$g_{n_0}(t_0, u, v) \ge f(t_0, u, v) \text{ and } g_{n_0}(t_0, u, v) \ge f\left(\frac{1}{2^{n_0+1}}, u, v\right)$$

for $(u, v) \in (0, \infty) \times R$. Thus

$$\begin{aligned} \left(\varphi_p\left(u'_{n_0}\right)\right)'(t_0) &= -\lambda [g_{n_0}(t_0, \rho_{n_0}, 0) + r(\rho_{n_0} - u(t_0))] \le \\ &\le -\lambda \left[f\left(\frac{1}{2^{n_0+1}}, \rho_{n_0}, 0\right) + r(\rho_{n_0} - u(t_0))\right] < 0, \end{aligned}$$

a contradiction.

Consequently (2.10) is true. Next we show

$$u_{n_0}(t) \le M_{n_0} \quad \text{for} \quad t \in [0, 1], \qquad (2.11)$$

where $M_{n_0}\ (\geq \rho_{n_0})$ is a predetermined constant (see (2.15)). Notice that (2.7) (with $\varepsilon = \rho_{n_0}$) guarantees the existence of a_0, b_0, η_ε, δ and β (as described in (2.7)) with

$$\left|g^*_{n_0}(t, u(t), u'(t))\right| \leq \phi_1(t)\left|u(t)\right|^\delta + \phi_2(t)\left|u'(t)\right|^\beta + \phi_3(t) \tag{2.12}$$

for $t \in (0,1)$; here

$$\phi_1(t) = \max\left\{a_0(t), a_0(\theta_{n_0}(t))\right\}, \quad \phi_2(t) = \max\left\{b_0(t), b_0(\theta_{n_0}(t))\right\}$$

and

$$\phi_3(t) = \max\left\{\eta_\varepsilon(t), \eta_\varepsilon(\theta_{n_0}(t))\right\};$$

notice that (2.12) is immediate since for $t \in (0,1)$ we have

$$g_{n_0}(t, u(t), u'(t)) = \max\left\{f(\theta_{n_0}(t)), u(t), u'(t), f(t, u(t), u'(t))\right\}.$$

Next notice that (2.4) (with $\varepsilon = \rho_{n_0}$) guarantees the existence of a, b, c, d, h_ε, γ and τ (as described in (2.4)) with

$$\begin{aligned} u(t)\, g^*_{n_0}(t, u(t), u'(t)) \leq \phi_4(t)\left|u(t)\right|^p + \phi_5(t)\left|u(t)\right|\left|u'(t)\right| + \\ + \phi_6(t)\left|u\right|^\gamma + \phi_7(t)\left|u\right|\left|u'\right|^\tau + u\phi_8(t) \end{aligned}$$

for $t \in (0,1)$; here

$$\begin{aligned} \phi_4(t) &= \max\left\{a(t), a(\theta_{n_0}(t))\right\},\ \phi_5(t) = \max\left\{b(t), b(\theta_{n_0}(t))\right\}, \\ \phi_6(t) &= \max\left\{c(t), c(\theta_{n_0}(t))\right\},\ \phi_7(t) = \max\left\{d(t), d(\theta_{n_0}(t))\right\} \end{aligned}$$

and

$$\phi_8(t) = \max\left\{h_\varepsilon(t), h_\varepsilon(\theta_{n_0}(t))\right\}.$$

Let $v = u - \rho_{n_0}$, so $v(0) = v(1) = 0$ and

$$-v\left(\left|v'\right|^{p-2} v'\right)' = \lambda u g^*_{n_0}(t, u, u') - \lambda \rho_{n_0} g^*_{n_0}(t, u, u') \quad \text{for} \quad t \in (0,1).$$

As a result, we have

$$\begin{aligned} \left\|v'\right\|_p^p \leq & \int_0^1 \phi_4(t)\left[v(t) + \rho_{n_0}\right]^p dt + \int_0^1 \phi_5(t)\left[v(t) + \rho_{n_0}\right]\left|v'(t)\right|^{p-1} dt + \\ & + \int_0^1 \phi_6(t)\left[v(t) + \rho_{n_0}\right]^\gamma dt + \int_0^1 \phi_7(t)\left[v(t) + \rho_{n_0}\right]\left|v'(t)\right|^\tau dt + \\ & + \int_0^1 \phi_8(t)\left[v(t) + \rho_{n_0}\right] dt + \rho_{n_0}\int_0^1 \phi_1(t)\left[v(t) + \rho_{n_0}\right]^\delta dt + \\ & + \rho_{n_0}\int_0^1 \phi_2(t)\left|v'(t)\right|^\beta dt + \rho_{n_0}\int_0^1 \phi_3(t)\, dt \leq \\ \leq & \int_0^1 \phi_4(t)\left[v(t) + \rho_{n_0}\right]^p dt + \int_0^1 \phi_5(t)\left|v(t)\right|\left|v'(t)\right|^{p-1} dt. \end{aligned}$$

$$+ \text{ lower order terms.}$$

Note that

$$\int_0^1 \phi_4(t)\left[v(t)+\rho_{n_0}\right]^p dt \le \int_0^1 \phi_4(t)(v(t))^p dt + \text{lower order terms,}$$

and so (note also (1.4) and Hölder inequality)

$$\left\|v'\right\|_p^p \le \int_0^1 \phi_4(t)(v(t))^p dt + \int_0^1 \phi_5(t)\left|v(t)\right|\left|v'(t)\right|^{p-1} dt + \\ + \text{ lower order terms.} \tag{2.13}$$

Case A. Suppose $a(t) < |a|_\infty$ on a subset of $[0,1]$ of positive measure and $a\left(\frac{1}{2^{n_0+1}}\right) < |a|_\infty$.

This of course implies $\phi_4(t) < \|\phi_4\|_\infty = \|a\|_\infty$ on a subset of $[0,1]$ of positive measure. From (1.5), there exists $\varepsilon > 0$ with

$$\int_0^1 \phi_4(t)(v(t))^p dt \le \left(\lambda_1^{-1}\|\phi_4\|_\infty - \varepsilon\right)\left\|v'\right\|_p^p = \left(\lambda_1^{-1}\|a\|_\infty - \varepsilon\right)\left\|v'\right\|_p^p,$$

where λ_1 is defined as in Lemma 1.1. Also

$$\int_0^1 \phi_5(t)\left|v(t)\right|\left|v'(t)\right|^{p-1} dt \le \|\phi_5\|_\infty \|v\|_p \left\|v'\right\|_p^{p-1} \le \lambda_1^{-\frac{1}{p}}\|b\|_\infty \left\|v'\right\|_p^p.$$

Thus, we have

$$\left\|v'\right\|_p^p \le \left(\lambda_1^{-1}\|a\|_\infty - \varepsilon\right)\left\|v'\right\|_p^p + \lambda_1^{-\frac{1}{p}}\|b\|_\infty \left\|v'\right\|_p^p + \\ + \text{ lower order terms,}$$

so

$$\left(1 - \lambda_1^{-1}\|a\|_\infty - \lambda_1^{-\frac{1}{p}}\|b\|_\infty\right)\left\|v'\right\|_p^p + \varepsilon\left\|v'\right\|_p^p \le \text{lower order terms.}$$

As a result (see (2.6)),

$$\varepsilon\left\|v'\right\|_p^p \le \text{lower order terms.}$$

Thus there exists K_{n_0} (independent of λ) such that $K_{n_0} \ge \rho_{n_0}$ and

$$\left\|u'\right\|_p = \left\|v'\right\|_p \le K_{n_0}. \tag{2.14}$$

Case B. Suppose that $b(t) < |b|_\infty$ on a subset of $[0,1]$ of positive measure and $b\left(\frac{1}{2^{n_0+1}}\right) < |b|_\infty$.

This of course implies $\phi_5(t) < \|\phi_5\|_\infty = \|b\|_\infty$ on a subset of $[0,1]$ of positive measure. From (1.5), there exists $\varepsilon > 0$ with

$$\int_0^1 [\phi_5(t)]^p v^p(t)\,dt \le \left(\lambda_1^{-1}\|\phi_5\|_\infty^p - \varepsilon\right)\|v'\|_p^p = \left(\lambda_1^{-1}\|b\|_\infty^p - \varepsilon\right)\|v'\|_p^p .$$

Also there exists a $\delta > 0$ with

$$\left(\lambda^{-1}\|b\|_\infty^p - \varepsilon\right)^{\frac{1}{p}} \le \lambda^{-\frac{1}{p}}\|b\|_\infty - \delta,$$

so

$$\int_0^1 \phi_5(t)\,|v(t)|\,|v'(t)|^{p-1}\,dt \le \left(\lambda_1^{-1}\|b\|_\infty^p - \varepsilon\right)^{\frac{1}{p}}\|v'\|_p^p \le$$

$$\le \left(\lambda^{-\frac{1}{p}}\|b\|_\infty - \delta\right)\|v'\|_p^p .$$

Also

$$\int_0^1 \phi_4(t)\,(v(t))^p\,dt \le \|\phi_4\|_\infty \|v\|_p^p \le \lambda_1^{-1}\|a\|_\infty \|v'\|_p^p .$$

Now (2.13) yields

$$\left(1 - \lambda_1^{-\frac{1}{p}}\|b\|_\infty - \lambda_1^{-1}\|a\|_\infty\right)\|v'\|_p^p + \delta\|v'\|_p^p \le \text{lower order terms.}$$

As a result (see (2.6)),

$$\delta\|v'\|_p^p \le \text{lower order terms.}$$

Thus there exists K_{n_0} (independent of λ) such that $K_{n_0} \ge \rho_{n_0}$ and

$$\|u'\|_p = \|v'\|_p \le K_{n_0}.$$

In both cases (2.14) holds, and now since $\|v\|_\infty \le \frac{1}{2^{1/q}}\|v'\|_p$, we have $\|v\|_\infty \le \frac{1}{2^{1/q}}K_{n_0}$ and as a result we have

$$\|u\|_\infty \le \frac{1}{2^{1/q}}K_{n_0} + \rho_{n_0} \equiv M_{n_0} \text{ and } \|u'\|_p \le K_{n_0} \tag{2.15}$$

for any solution u to $(2.9)_\lambda$. Also (2.7) (with $\varepsilon = \rho_{n_0}$) implies

$$\int_0^1 \left(|u'|^{p-2}u'\right)' dt \le$$

$$\le M_{n_0}^\delta \int_0^1 \phi_1(t)\,dt + \left[\int_0^1 \phi_2^{\frac{p}{p-\beta}}(t)\right]^{\frac{p-\beta}{p}} \|u'\|_p^\beta + \int_0^1 \phi_3(t)\,dt \le$$

$$\le M_{n_0}^\delta \int_0^1 \phi_1(t)\,dt + \left[\int_0^1 \phi_2^{\frac{p}{p-\beta}}(t)\right]^{\frac{p-\beta}{p}} K_{n_0}^\beta + \int_0^1 \phi_3(t)\,dt \equiv L_{n_0},$$

and so since $u(0) = u(1) = \rho_{n_0}$, we have

$$\|u'\|_\infty \le \varphi_p^{-1}\left(\int_0^1 \left(|u'|^{p-2}u'\right)' dt\right) \le \varphi_p^{-1}(L_{n_0}) \equiv R_{n_0}.$$

Now a standard existence principle from the literature [7, 11] guarantees that $(2.9)_1$ has a solution u_{n_0} with $\rho_{n_0} \le u_{n_0}(t) \le M_{n_0}$ for $t \in [0,1]$ and $\left\|u'_{n_0}\right\|_\infty \le R_{n_0}$. □

Remark 2.1. In [11] we assumed that φ_p^{-1} is continuously differentiable on $(-\infty, \infty)$, so $1 < p \le 2$. However, this assumption is only needed in [11] to show that $N_\lambda \Omega$ is equicontinuous on $[0,1]$ (here N_λ and Ω are defined in [11]). It is well known that this assumption can be removed once one notices that $\varphi_p N_\lambda \Omega$ is equicontinuous on $[0,1]$ and uses also the fact that φ_p^{-1} is continuous.

Also notice that if we take $h(t,u,v) = g_{n_0}(t,u,v)$ in (2.3), then since $g_{n_0} \ge f$ and u_{n_0} satisfies $-(\varphi_p(u'))' = g_{n_0}(t,u,u')$ on $(0,1)$ with $u_{n_0}(t) \ge \rho_{n_0}$ for $t \in [0,1]$, we have

$$u_{n_0}(t) \ge \alpha(t) \quad \text{for} \quad t \in [0,1].$$

Next we consider the boundary value problem

$$\begin{cases} -(\varphi_p(u'))' = g^*_{n_0+1}(t,u,u'), & 0 < t < 1, \\ u(0) = u(1) = \rho_{n_0+1}, \end{cases} \tag{2.16}$$

where

$$g^*_{n_0+1}(t,u,v) = \begin{cases} g_{n_0+1}(t,\rho_{n_0+1},v^*) + r(\alpha_{n_0+1}(t) - u), & u \le \rho_{n_0+1}, \\ g_{n_0+1}(t,u,v^*), & \rho_{n_0+1} \le u \le u_{n_0}(t), \\ g_{n_0+1}(t,u_{n_0}(t),v^*) + r(u_{n_0}(t) - u), & u \ge u_{n_0}(t), \end{cases}$$

with

$$v^* = \begin{cases} R_{n_0+1}, & v > R_{n_0+1}, \\ v, & -R_{n_0+1} \le v \le R_{n_0+1}, \\ -R_{n_0+1}, & v < -R_{n_0+1}; \end{cases}$$

here $R_{n_0+1} \ge R_{n_0}$ is a predetermined constant (see (2.20)). Now Schauder's fixed point theorem guarantees that there exists a solution $u_{n_0+1} \in C^1[0,1]$ with $\varphi_p\left(u'_{n_0+1}\right) \in C^1(0,1)$ to (2.16). We first show

$$u_{n_0+1}(t) \ge \rho_{n_0+1}, \quad t \in [0,1]. \tag{2.17}$$

Suppose (2.17) is not true. Then there exists a $t_1 \in (0,1)$ with $u_{n_0+1}(t_1) < \rho_{n_0+1}$, $u'_{n_0+1}(t_1) = 0$ and

$$\left(\varphi_p\left(u'_{n_0+1}\right)\right)'(t_1) \ge 0.$$

We need to discuss two cases, namely $t_1 \in \left[\frac{1}{2^{n_0+2}}, 1\right)$ and $t_1 \in \left(0, \frac{1}{2^{n_0+2}}\right)$.

Case (1). $t_1 \in \left[\frac{1}{2^{n_0+2}}, 1\right)$.

Then since $g_{n_0+1}(t_1,u,v) = f(t_1,u,v)$ for $(u,v) \in (0,\infty)\times R$ (note $t_1 \in e_{n_0+1}$), we have

$$\left(\varphi_p\left(u'_{n_0+1}\right)\right)'(t_1) =$$
$$= -\left[g_{n_0+1}\left(t_1,\rho_{n_0+1},\left(u'_{n_0+1}(t_1)\right)^*\right) + r\left(\rho_{n_0+1} - u_{n_0+1}(t_1)\right)\right]$$
$$= -\left[f\left(t_1,\rho_{n_0+1},0\right) + r\left(\rho_{n_0+1} - u_{n_0+1}(t_1)\right)\right] < 0$$

from (2.2), a contradiction.

Case (2). $t_1 \in \left(0, \frac{1}{2^{n_0+2}}\right)$.

Then since $g_{n_0+1}(t_1,u,v)$ equals

$$\min\left\{\max\left\{f\left(\frac{1}{2^{n_0+1}},u,v\right), f(t_1,u,v)\right\},\right.$$
$$\left.\max\left\{f\left(\frac{1}{2^{n_0+2}},u,v\right), f(t_1,u,v)\right\}\right\},$$

we have

$$g_{n_0+1}(t_1,u,v) \geq f(t_1,u,v)$$

and

$$g_{n_0+1}(t_1,u,v) \geq \min\left\{f\left(\frac{1}{2^{n_0+1}},u,v\right),\ f\left(\frac{1}{2^{n_0+2}},u,v\right)\right\}$$

for $(u,v) \in (0,\infty)\times R$. Thus we have

$$\left(\varphi_p\left(u'_{n_0+1}\right)\right)'(t_1) =$$
$$= -\left[g_{n_0+1}\left(t_1,\rho_{n_0+1},\left(u'_{n_0+1}(t_1)\right)^*\right) + r\left(\rho_{n_0+1} - u_{n_0+1}(t_1)\right)\right] \leq$$
$$\leq -\left\{\min\left\{f\left(\frac{1}{2^{n_0+1}},\rho_{n_0+1},0\right), f\left(\frac{1}{2^{n_0+2}},\rho_{n_0+1},0\right)\right\} +\right.$$
$$\left. + r\left(\rho_{n_0+1} - u_{n_0+1}(t_1)\right)\right\} < 0,$$

since

$$f\left(\frac{1}{2^{n_0+1}},\rho_{n_0+1},0\right) \geq 0 \quad \text{and} \quad f\left(\frac{1}{2^{n_0+2}},\rho_{n_0+1},0\right) \geq 0$$

because

$$f\left(t,\rho_{n_0+1},0\right) \geq 0 \quad \text{for} \quad t \in \left[\frac{1}{2^{n_0+2}},1\right]$$

and $\frac{1}{2^{n_0+1}} \in \left[\frac{1}{2^{n_0+2}},1\right]$.

Consequently (2.18) is true. Next we show

$$u_{n_0+1}(t) \leq u_{n_0}(t) \quad \text{for} \quad t \in [0,1]. \tag{2.18}$$

If (2.18) is not true, then $u_{n_0+1} - u_{n_0}$ would have a positive absolute maximum at say $\tau_0 \in (0,1)$, in which case $\left(u_{n_0+1} - u_{n_0}\right)'(\tau_0) = 0$ and

$$\left(\varphi_p\left(u'_{n_0+1}\right)\right)'(\tau_0) - \left(\varphi_p\left(u'_{n_0}\right)\right)'(\tau_0) \leq 0; \tag{2.19}$$

the proof is contained in [7].

Then $u_{n_0+1}(\tau_0) > u_{n_0}(\tau_0)$ together with $g_{n_0}(\tau_0,u,v) \geq g_{n_0+1}(\tau_0,u,v)$ for $(u,v) \in (0,\infty)\times R$ gives (note $\left(u'_{n_0+1}(\tau_0)\right)^* = \left(u'_{n_0}(\tau_0)\right)^* = u'_{n_0}(\tau_0)$ since $R_{n_0+1} \geq R_{n_0}$ and $\left\|u'_{n_0}\right\|_\infty \leq R_{n_0}$)

$$\begin{aligned}&\left(\varphi_p\left(u'_{n_0+1}\right)\right)'(\tau_0) - \left(\varphi_p\left(u'_{n_0}\right)\right)'(\tau_0) =\\&= -\left[g_{n_0+1}\left(\tau_0, u_{n_0}(\tau_0), \left(u'_{n_0+1}(\tau_0)\right)^*\right) + r\left(u_{n_0}(\tau_0) - u_{n_0+1}(\tau_0)\right)\right] -\\&\quad - \left(\varphi_p\left(u'_{n_0}\right)\right)'(\tau_0) \geq -\left[\left(\varphi_p\left(u'_{n_0}\right)\right)'(\tau_0) + g_{n_0}\left(\tau_0, u_{n_0}(\tau_0), u'_{n_0}(\tau_0)\right)\right] -\\&\quad - r\left(u_{n_0}(\tau_0) - u_{n_0+1}(\tau_0)\right)\\&= -r\left(u_{n_0}(\tau_0) - u_{n_0+1}(\tau_0)\right) > 0,\end{aligned}$$

a contradiction. Thus (2.18) holds. In addition, since $\|u_{n_0+1}\|_\infty \leq \|u_{n_0}\|_\infty \leq M_{n_0}$, then (2.7) (with $\varepsilon = \rho_{n_0+1}$) guarantees the existence of a_0, b_0, η_ε, δ and β (as described in (2.7)) with (we only need to note that $g^*_{n_0+1}(t, u_{n_0+1}(t), u'_{n_0+1}(t)) = g_{n_0+1}(t, u_{n_0+1}(t), (u'_{n_0+1}(t))^*)$

$$\begin{aligned}\left|g^*_{n_0+1}\left(t, u_{n_0+1}, u'_{n_0+1}\right)\right| &\leq \phi_9(t)\left[u_{n_0+1}(t)\right]^\delta +\\&\quad + \phi_{10}(t)\left|\left(u'_{n_0+1}(t)\right)^*\right|^\beta + \phi_{11}(t) \leq\\&\leq \phi_9(t) M^\delta_{n_0} + \phi_{10}(t)\left|u'_{n_0+1}(t)\right|^\beta + \phi_{11}(t)\end{aligned}$$

for $t \in (0,1)$ (note that $|v^*| \leq |v|$); here

$$\phi_9(t) = \max\left\{a_0(t),\ a_0(\theta_{n_0}(t)),\ a_0(\theta_{n_0+1}(t))\right\},$$
$$\phi_{10}(t) = \max\left\{b_0(t),\ b_0(\theta_{n_0}(t)),\ b_0(\theta_{n_0+1}(t))\right\}$$

and

$$\phi_{11}(t) = \max\left\{\eta_\varepsilon(t),\ \eta_\varepsilon(\theta_{n_0}(t)),\ \eta_\varepsilon(\theta_{n_0+1}(t))\right\}.$$

As a result,

$$\begin{aligned}\left\|u'_{n_0+1}\right\|_p^p &= \left|\int_0^1 \left(u_{n_0+1}(t) - \rho_{n_0+1}\right)\left(\left|u'_{n_0+1}(t)\right|^{p-2} u'_{n_0+1}(t)\right)'\right| \leq\\&\leq M^\delta_{n_0}\left(M_{n_0} + \rho_{n_0+1}\right)\int_0^1 \phi_9(t)\,dt +\\&\quad + \left(M_{n_0} + \rho_{n_0+1}\right)\left\|u'_{n_0+1}\right\|_p^\beta \left(\int_0^1 \phi_{10}^{\frac{p-\beta}{p}}(t)dt\right)^{\frac{p}{p-\beta}} +\\&\quad + \left(M_{n_0} + \rho_{n_0+1}\right)\int_0^1 \phi_{11}\,dt,\end{aligned}$$

so there exists a constant $K_{n_0+1} \geq \rho_{n_0+1}$ with

$$\left\|u'_{n_0+1}\right\|_p \leq K_{n_0+1}.$$

Also since $u_{n_0+1}(0) = u_{n_0+1}(1) = \rho_{n_0+1}$, we have

$$\left\|u'_{n_0+1}\right\|_\infty \leq \varphi_p^{-1}\left(\int_0^1 \left(\left|u'_{n_0+1}(t)\right|^{p-2} u'_{n_0+1}(t)\right)' dt\right) \leq$$
$$\leq M_{n_0}^{\delta}\int_0^1 \phi_9(t)\,dt + K_{n_0+1}^{\beta}\left(\int_0^1 [\phi_{10}(t)]^{\frac{p}{p-\beta}}\,dt\right)^{\frac{p-\beta}{p}} +$$
$$+\int_0^1 \phi_{11}(t)\,dt,$$

so there exists a constant $R_{n_0+1} \geq R_{n_0}$ with

$$\left\|u'_{n_0+1}\right\|_\infty \leq R_{n_0+1}. \tag{2.20}$$

As a result, if we take $h(t,u,v) = g_{n_0+1}(t,u,v)$ in (2.3), then since $g_{n_0+1} \geq f$ and u_{n_0+1} satisfies $-(\varphi_p(u'))' = g_{n_0+1}(t,u,u')$ on $(0,1)$ with $u_{n_0+1}(t) \geq \rho_{n_0+1}$ for $t \in [0,1]$, we have

$$u_{n_0}(t) \geq \alpha(t) \quad \text{for} \quad t \in [0,1].$$

Now proceed inductively to construct $u_{n_0+2}, u_{n_0+3}, \ldots$ as follows. Suppose we have u_k for some $k \in \{n_0+1, n_0+2, \}$ with $\alpha(t) \leq u_k(t) \leq u_{k-1}(t)$ for $t \in [0,1]$.

Then consider the boundary value problem

$$\begin{cases} -(\varphi_p(u'))' = g_{k+1}^*(t,u,u'), & 0 < t < 1, \\ u(0) = u(1) = \rho_{k+1}, \end{cases} \tag{2.21}$$

where

$$g_{k+1}^*(t,u,v) = \begin{cases} g_{k+1}(t,\rho_{k+1},v^*) + r(\rho_{k+1} - u), & u \leq \rho_{k+1}, \\ g_{k+1}(t,u,v^*), & \rho_{k+1} \leq u \leq u_k, \\ g_{k+1}(t,u_k,v^*) + r(u_k - u), & u \geq u_k, \end{cases}$$

with

$$v^* = \begin{cases} M_{k+1}, & v > M_{k+1}, \\ v, & -M_{k+1} \leq v \leq M_{k+1}, \\ -M_{k+1}, & v < -M_{k+1}; \end{cases}$$

here $M_{k+1} \geq M_k$ is a predetermined constant. Now Schauder's fixed point theorem guarantees that (2.21) has a solution $u_{k+1} \in C^1[0,1]$ with $\varphi_p(u'_k) \in C^1(0,1)$ and essentially the same reasoning as above yields

$$\rho_{k+1} \leq u_{k+1}(t) \leq u_k(t), \quad \left|u'_{k+1}(t)\right| \leq M_{k+1} \quad \text{for} \quad t \in [0,1] \tag{2.22}$$

with

$$u_{k+1}(t) \geq \alpha(t) \quad \text{for} \quad t \in [0,1]$$

and

$$-\left(\varphi_p\left(u'_{k+1}\right)\right)' = g_{k+1}\left(t, u_{k+1}, u'_{k+1}\right) \quad \text{for} \quad 0 < t < 1.$$

Now let us look at the interval $\left[\frac{1}{2^{n_0+1}}, 1 - \frac{1}{2^{n_0+1}}\right]$. We claim

$$\begin{cases} \left\{u_n^{(j)}\right\}_{n=n_0+1}^{\infty}, \ j = 0, 1, \text{ is a bounded, equicontinuous} \\ \text{family on } \left[\frac{1}{2^{n_0+1}}, 1 - \frac{1}{2^{n_0+1}}\right]. \end{cases} \tag{2.23}$$

Firstly note

$$\|u_n\|_\infty \le \|u_{n_0}\|_\infty \le M_{n_0} \quad \text{for} \quad t \in [0,1] \quad \text{and} \quad n \ge n_0 + 1. \tag{2.24}$$

Let

$$\varepsilon = \min_{t \in \left[\frac{1}{2^{n_0+1}}, 1 - \frac{1}{2^{n_0+1}}\right]} \alpha(t).$$

Then (2.7) guarantees the existence of a_0, b_0, η_ε, δ and β (as described in (2.7)) with

$$|g_n(t, u_n(t), u'_n(t))| = |f(t, u_n(t), u'_n(t))| \le$$
$$\le a_0(t) M_{n_0}^\delta + b_0(t) |u'_n(t)|^\beta + \eta_\varepsilon(t)$$

for $t \in [a,b] \equiv \left[\frac{1}{2^{n_0+1}}, 1 - \frac{1}{2^{n_0+1}}\right] \subseteq e_{n_0}$ and $n \ge n_0 + 1$. Let

$$r_n(t) = u_n(t) - \left\{u_n(a) + \frac{[u_n(b) - u_n(a)]}{b-a}(t-a)\right\},$$

so for $n \ge n_0 + 1$ we have

$$\left|\int_a^b r_n(t)\left(\varphi_p(u')\right)' dt\right| = -\int_a^b |u'_n|^p dt + \frac{u_n(b) - u_n(a)}{b-a}\int_a^b \varphi_p(u'_n)\, dt.$$

Now since $r_n(t) \le 2M_{n_0}$ for $t \in [a,b]$, we have for any $n \ge n_0 + 1$ that

$$\int_a^b |u'_n(t)|^p dt \le \frac{2M_{n_0}}{b-a}\int_a^b |u_n|^{p-1} dt + 2M_{n_0}\int_a^b \left(\varphi_p(u')\right)' dt \le$$
$$\le \frac{2M_{n_0}}{(b-a)^{\frac{p+1}{p}}}\|u_n\|_p^{p-1} + 2M_{n_0}\left[M_{n_0}^\delta \int_a^b a_0(t)\, dt + \right.$$
$$\left. + \left(\int_a^b |b_0(t)|^{\frac{p}{p-\beta}} dt\right)^{\frac{p-\beta}{p}} \|u'_n\|_p^\beta + \int_a^b \eta_\varepsilon(t)\, dt\right],$$

so there exists Q_{n_0} with

$$\|u'_n\|_p^p \le Q_{n_0} \quad \text{for} \quad n \ge n_0 + 1. \tag{2.25}$$

Also there exists $t_n \in (a,b)$ with $u_n'(t_n) = \frac{u_n(b)-u_n(a)}{b-a}$, so for $n \geq n_0+1$ we have (using (2.25))

$$\sup_{t\in[a,b]} |u_n'(t)|^{p-1} \leq |\varphi_p(u_n')(t_n)| + \int_a^b (\varphi_p(u_n'))' \, dt \leq$$

$$\leq \left[\frac{2M_{n_0}}{b-a}\right]^{p-1} + M_{n_0}^{\delta} \int_a^b a_0(t)\, dt +$$

$$+ Q_{n_0}^{\frac{\beta}{p}} \left(\int_a^b [b_0(t)]^{\frac{p}{p-\beta}} \right)^{\frac{p-\beta}{p}} + \int_a^b \eta_\varepsilon(t)\, dt \equiv L_{n_0},$$

i.e.,

$$\sup_{t\in[a,b]} |u_n'(t)| \leq L_{n_0}^{\frac{1}{p-1}} \quad \text{for} \quad n \geq n_0+1. \tag{2.26}$$

Now (2.24), (2.25) and (2.26) guarantee that (2.23) holds. The Arzela–Ascoli theorem guarantees the existence of a subsequence N_{n_0} of integers and a function $z_{n_0} \in C^1\left[\frac{1}{2^{n_0+1}}, 1-\frac{1}{2^{n_0+1}}\right]$ with $u_n^{(j)}$, $j=0,1$, converging uniformly to $z_{n_0}^{(j)}$ on $\left[\frac{1}{2^{n_0+1}}, 1-\frac{1}{2^{n_0+1}}\right]$ as $n\to\infty$ through N_{n_0}. Similarly

$$\begin{cases} \left\{u_n^{(j)}\right\}_{n=n_0+2}^{\infty}, \ j=0,1, \ \text{is a bounded, equicontinuous} \\ \text{family on } \left[\frac{1}{2^{n_0+2}}, 1-\frac{1}{2^{n_0+2}}\right], \end{cases}$$

so there is a subsequence N_{n_0+1} of N_{n_0} and a function

$$z_{n_0+1} \in C^1\left[\frac{1}{2^{n_0+2}}, 1-\frac{1}{2^{n_0+2}}\right]$$

with $u_n^{(j)}$, $j=0,1$, converging uniformly to $z_{n_0+1}^{(j)}$ on $\left[\frac{1}{2^{n_0+2}}, 1-\frac{1}{2^{n_0+2}}\right]$ as $n\to\infty$ through N_{n_0+1}. Note $z_{n_0+1} = z_{n_0}$ on $\left[\frac{1}{2^{n_0+1}}, 1-\frac{1}{2^{n_0+1}}\right]$ since $N_{n_0+1} \subseteq N_{n_0}$. Proceed inductively to obtain subsequences of integers

$$N_{n_0} \supseteq N_{n_0+1} \supseteq \cdots \supseteq N_k \supseteq \cdots$$

and the function

$$z_k \in C^1\left[\frac{1}{2^{k+1}}, 1-\frac{1}{2^{k+1}}\right]$$

with

$$u_n^{(j)}, \ j=0,1, \ \text{converging uniformly to } z_k^{(j)} \ \text{on} \ \left[\frac{1}{2^{k+1}}, 1-\frac{1}{2^{k+1}}\right]$$

as $n\to\infty$ through N_k, and

$$z_k = z_{k-1} \quad \text{on} \quad \left[\frac{1}{2^k}, 1-\frac{1}{2^k}\right].$$

Define a function $u : [0,1] \to [0,\infty)$ by $u(t) = z_k(t)$ on $\left[\frac{1}{2^{k+1}}, 1 - \frac{1}{2^{k+1}}\right]$ and $u(0) = u(1) = 0$. Notice that u is well defined and

$$\alpha(t) \le u(t) \le u_{n_0}(t) \quad \text{for} \quad t \in (0,1).$$

Now let $[a,b] \subset (0,1)$ be a compact interval. There is an index n^* such that $[a,b] \subset \left[\frac{1}{2^{n+1}}, 1 - \frac{1}{2^{n+1}}\right]$ for all $n > n^*$ and therefore, for all $n > n^*$

$$-\left(\varphi_p(u_n')\right)' = f(t, u_n, u_n') \quad \text{for} \quad a \le t \le b.$$

A standard argument [7, 11] guarantees that

$$-\left(\varphi_p(u')\right)' = f(t, u, u') \quad \text{for} \quad a \le t \le b.$$

Since $[a,b] \subset (0,1)$ is arbitrary, we find that

$$(\varphi(u')' \in C(0,1) \quad \text{and} \quad -\left(\varphi_p(u')\right)' = f(t, u, u') \quad \text{for} \quad 0 < t < 1.$$

It remains to show that u is continuous at 0 and 1. Let $\varepsilon > 0$ be given. Now since $\lim_{n\to\infty} u_n(0) = 0$, there exists $n_1 \in \{n_0, n_0+1, \ldots\}$ with $u_{n_1}(0) < \frac{\varepsilon}{2}$. Next since $u_{n_1} \in C[0,1]$, there exists $\delta_{n_1} > 0$ with

$$u_{n_1}(t) < \frac{\varepsilon}{2} \quad \text{for} \quad t \in [0, \delta_{n_1}].$$

Now for $n \ge n_1$ we have, since $\{u_n(t)\}_{n \in N_0}$ is nonincreasing for each $t \in [0,1]$,

$$\alpha(t) \le u_n(t) \le u_{n_1}(t) < \frac{\varepsilon}{2} \quad \text{for} \quad t \in [0, \delta_{n_1}].$$

Consequently,

$$\alpha(t) \le u(t) \le \frac{\varepsilon}{2} < \varepsilon \quad \text{for} \quad t \in (0, \delta_{n_1}]$$

and so u is continuous at 0. Similarly u is continuous at 1. As a result, $u \in C[0,1]$.

Remark 2.2. In (2.2) it is possible to replace $\frac{1}{2^{n+1}} \le t \le 1$ with either (i) $0 \le t \le 1 - \frac{1}{2^{n+1}}$, (ii) $\frac{1}{2^{n+1}} \le t \le 1 - \frac{1}{2^{n+1}}$, or (iii) $0 \le t \le 1$. This is clear once one changes the definition of e_n and θ_n. For example, in case (ii) take

$$e_n = \left[\frac{1}{2^{n+1}}, 1 - \frac{1}{2^{n+1}}\right] \quad \text{and} \quad \theta_n(t) = \max\left\{\frac{1}{2^{n+1}}, \min\left\{t, 1 - \frac{1}{2^{n+1}}\right\}\right\}.$$

Finally we discuss the condition (2.3). Suppose the following condition is satisfied:

$$\begin{cases} \text{let } n \in \{n_0, n_0+1, \ldots\} \text{ and associated with each } n \text{ we} \\ \text{have a constant } \rho_n \text{ such that } \{\rho_n\} \text{ is a decreasing} \\ \text{sequence with } \lim_{n\to\infty} \rho_n = 0 \text{ and for any } r > 0 \\ \text{there exists a constant } k_r > 0 \text{ such that for } \frac{1}{2^{n+1}} \le t \le 1, \\ 0 < u \le \rho_n \text{ and } v \in [-r, r] \text{ we have } f(t,u,v) > k_r. \end{cases} \tag{2.27}$$

A slight modification of the argument in [7, Proposition 4] guarantees that (2.3) is true.

Remark 2.3. In (2.27) if $\frac{1}{2^{n+1}} \le t \le 1$ is replaced by (i), (ii), or (iii) in Remark 2.2, then (2.3) is also true.

Theorem 2.2. *Let* $n_0 \in \{1, 2, \ldots\}$ *be fixed and suppose* (2.1), (2.4) − (2.7) *and* (2.27) *hold. Then* (1.1) *has a solution* $u \in C[0,1]$ *with* $u(t) > 0$ *for* $t \in (0,1)$.

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Authors' addresses:

Haishen Lü
Department of Applied Mathematics
Hohai University
Nanjing, 210098
China
E-mail: Haishen2001@yahoo.com.cn

Donal O'Regan
Department of Mathematics
University of Ireland, Galway
Ireland
E-mail: donal.oregan@nuigalway.ie

R. P. Agarwal
Department Mathematical Sciences
Florida Institute of Technology
Florida 32901-6975
USA
E-mail: agarwal@fit.edu