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BOUNDARY-VALUE PROBLEMS OF PIEZOELECTRICITY IN DOMAINS WITH CUTS

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In a domain with cut static boundary-value problems of piezoelectricity are investigated. The existence and uniqueness of solutions of the considered boundary value problems are proved and an asymptotic expansion of solutions near the edges of the cut are obtained.

Let Ω and Ω_1 ($\overline{\Omega}_1 \subset \Omega$) be a bounded domains in the three-dimensional Euclidean space $\mathbb{R}^3$ with infinitely smooth boundaries $\partial\Omega$ and $\partial\Omega_1$, respectively. We assume that the boundary $\partial\Omega_1$ of the domain Ω_1 is the union of two surfaces: $\partial\Omega_1 = \overline{S} \cup S_0$, and that the boundary $\partial S = \partial S_0$ is an infinitely smooth curve. Denote $\Omega_2 = \Omega \backslash \overline{\Omega}_1$.

We suppose that the domain $\Omega \backslash \overline{S}$ is filled with an anisotropic homogeneous piezoelectric material having a cut at $\overline{S}$.

In the domain $\Omega \backslash \overline{S}$, let us consider the system of static equations of piezoelectricity for a homogeneous anisotropic medium [1]:

$$A(D_x)u + F = 0, \tag{1}$$

where $u = (u_1, u_2, u_3, u_4)$; u_1, u_2, u_3 are the components of the displacement vector, u_4 is the electric potential, F_i, $i = 1, 2, 3$, are the components of the mass force, F_4 is the electric charge density, $A(D_x)$ is the differential operator of the form

$$\begin{aligned} A(D_x) &= \|A_{jk}(D_x)\|_{5\times 5}, \\ A_{jk}(D_x) &= c_{ijlk}\partial_i\partial_l, \quad A_{j4}(D_x) = e_{kjl}\partial_k\partial_l, \\ A_{4k}(D_x) &= -e_{ikl}\partial_i\partial_l, \quad A_{44}(D_x) = \varepsilon_{il}\partial_i\partial_l, \quad j,k = 1,2,3, \end{aligned} \tag{2}$$

where c_{ijlk}, e_{ikl}, ε_{ik} are respectively the elastic, piezoelectric and dielectric constants.

In the equalities (1) it is assumed that summation is carried out over the repeated indices. We will follow this rule in the sequel.

The constants in (1) satisfy the symmetry conditions

$$\begin{aligned} & c_{ijlk} = c_{jilk} = c_{lkij}, \quad e_{kjl} = e_{klj}, \quad \varepsilon_{ik} = \varepsilon_{ki}, \\ & i,j,k,l = 1,2,3, \end{aligned} \tag{3}$$

and the condition of positiveness of the internal energy:

$$\forall(\xi_{ij}),\ \ (\eta_i): \ \ \xi_{ij} = \xi_{ji}, \ \ \exists c_0 > 0 \ \ c_{ijkl}\xi_{ij}\xi_{kl} \geq c_0\xi_{ij}\xi_{ij}, \quad \varepsilon_{ij}\eta_i\eta_j \geq c_0\eta_i\eta_i. \tag{4}$$

Note that the operator $A(D_x)$ is a strongly elliptic operator, but is not formally self-adjoint or positive definite.

We introduce the following electromechanical stress operator $T(D_y, n)$:

$$\begin{aligned} T(D_y,n) &= \big\|T_{jk}(D_y,n)\big\|_{4\times 4}, \\ T_{jk}(D_y,n) &= c_{ijlk}n_l\partial_i, \quad T_{j4}(D_y,n) = e_{kjl}n_l\partial_k, \\ T_{4k}(D_y,n) &= -e_{ikl}n_i\partial_l, \quad T_{44}(D_y,n) = \varepsilon_{ij}n_j\partial_i, \quad j,k = 1,2,3, \end{aligned}$$

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where $n(y) = (n_1(y), n_2(y), n_3(y))$ is the unit normal at the point $y \in \partial\Omega_2$, directed outward from Ω_2.

Let us suppose that the considered piezoelectric body is grounded and mechanically clamped over the part S_1 of the boundary $\partial\Omega$, whereas the remaining part S_2 of the boundary $\partial\Omega$ is mechanically free and electrically isolated. A metal foil is located at the cut at S and connected to the electric source with a known potential. Then the corresponding boundary value problem can be formulated as follows.

We look for a vector-function

$$u = (u_1, u_2, u_3, u_4)^\top : \Omega \to \mathbb{R}^4$$

belonging to the space $\left[W^1{}_p(\Omega \backslash \overline{S}\,)\right]^4$ and satisfying the conditions

$$\begin{cases} A(D_x)u = 0 & \text{in} \ \ \Omega\backslash S, \\ r_S\{[T(D_y,n)u]_j\}^{\pm} = 0, \quad j = 1,2,3; & \text{on} \ \ S, \\ r_S\{u_4\}^{\pm} = \varphi, \quad i = 1,2, & \text{on} \ \ S, \\ r_{S_1}\{u\}^{+} = 0 & \text{on} \ \ S_1, \\ r_{S_2}\{[T(D_y,n)u]\}^{+} = 0 & \text{on} \ \ S_2, \end{cases} \tag{5}$$

where $\{f\}^+$, denotes the trace of f on $\partial\Omega_2$ from Ω_2 and $\{f\}^-$ denotes the trace of the function f on $\partial\Omega_1$ from Ω_1. r_S, r_{S_1}, r_{S_2} are restriction operators on S, S_1 and S_2 respectively; φ belongs to the Besov space $B^{1/p'}_{p,p}(S)$, $1 < p < \infty$, $p' = \frac{p}{p-1}$.

Using the potential theory and the general theory of pseudodifferential equations on manifolds with boundary we have proved the existence and uniqueness of a solution of the considered problem.

Employing an asymptotic expansion of solutions of strongly elliptic pseudodifferential equations and an asymptotic expansion of potential-type functions obtained in [2, 3], for sufficiently smooth data of the problem we have obtained a complete asymptotic expansion of the solution near cut's edge ∂S as well as near the curve ∂S_1, where the type of the boundary conditions changes. It turned out that the exponent of the first term of an asymptotic expansion of solution near the cut's edge ∂S is $\frac{1}{2}$.

We have found an important class of anisotropic piezoelectric media for which the oscillation of solutions vanishes near the curve ∂S_1, at least the first three terms of the asymptotic do not contain logarithms and the singularity of solutions is calculated by a simple formula.

Denote by $V_0(h)$ the single layer potential:

$$V_0(h)(x) = \int_{\partial\Omega} T(\partial_x, n(x)) H(x-y) h(y) d_y S, \quad x \in \Omega,$$

where $H(x)$ is the fundamental matrix-function of the differential operator $A(\partial_x)$, and let β_k, $k = \overline{1,4}$, be the eigenvalues of the matrix $\sigma_{V_0}(x_1, 0, +1)$, where $\sigma_{V_0}(x', \xi')$ is the principal homogeneous symbol of the operator V_0.

In the case where the conditions

$$\beta_{1,2} = \pm a(x_1), \quad \beta_{3,4} = \pm i b(x_1) \quad (b(x_1) > 0, \ x_1 \in \partial S_1) \tag{6}$$

are fulfilled, solutions in the neighborhood of the curve ∂S_1 possess the following properties:

1) The solutions near the ∂S admit the asymptotic expansion

$$c_1\,\rho^{\gamma_1} + c_2\,\rho^{\frac{1}{2}\pm i\delta} + c_3\,\rho^{\gamma_2} + \cdots,$$

where

$$\gamma_1 = \frac{1}{2} - \frac{1}{\pi}\,\text{arctg}\ 2b, \quad \gamma_2 = \frac{1}{2} + \frac{1}{\pi}\,\text{arctg}\ 2b.$$

The singularity of the solutions is characterized by the quantity

$$\gamma_1 = \frac{1}{2} - \frac{1}{\pi} \sup_{\partial S_1} \operatorname{arctg}\, 2b,$$

which depends on the elastic, piezoelectric and dielectric constants and on the geometry of ∂S_1 and can take any value from the interval $(0; 1/2)$.

2) Since $\gamma_1 < \frac{1}{2}$, the oscillation vanishes in some neighborhood of ∂S_1.

The class of piezoelectric media for which the conditions (6) are fulfilled is not empty (for example TeO_2 belongs to this class). On the other hand, for media that do not possess piezoelectric properties (i.e. are pure elastic) we have $\gamma_1 = \frac{1}{2}$ and the solutions oscillate. Thus, in the case of piezoelectricity we reveal effects which are not observed in the case of classical elasticity.

For some class of piezoelectric media the singularity of solution in the neighborhood of ∂S_1 can also reach the value $\frac{1}{2}$. For example, such property is possessed by piezoelectric media with cubic anisotropy.

Finally note that the boundary value problems of piezoelectricity of different type in domains with cuts were considered in [4].

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