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GENERALIZED EULER METHOD FOR FIRST ORDER PARTIAL DIFFERENTIAL FUNCTIONAL EQUATIONS

Abstract. We present a new class of numerical methods for nonlinear first order partial differential equations. Classical solutions of mixed problems are approximated in the paper by solutions of suitable quasilinear systems of difference equations. We give a complete convergence analysis for the methods and we show by an example that the new methods are considerably better than the classical schemes. The proof of the stability is based on a comparison technique with nonlinear estimates of the Perron type.

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1. Introduction

For any metric spaces X and Y we denote by $C(X,Y)$ the class of all continuous functions defined on X and taking values in Y. We will use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components.

Let $a > 0, d_0 \in R_+, d = (d_1, \dots, d_n) \in R_+^n$ with $R_+ = [0, +\infty)$ and $b = (b_1, \dots, b_n) \in R_+^n$ with $b_i > 0$, $1 \le i \le n$, be given. Write $c = b + d, c = (c_1, \dots, c_n)$. Let us define the sets

$$E = [0,a] \times (-b,b)\,,\ E_0 = [-d_0, 0] \times [-c,c],$$
$$\partial_0 E = (0,a] \times ([-c,c] \backslash (-b,b))\,,\ B = E_0 \cup E \cup \partial_0 E,$$
$$D = [-d_0, 0] \times [-d,d].$$

Suppose that $z\colon B \to R$ and $(t,x) \in \bar{E}$, where $\bar{E}$ is the closure of E. We define the function $z_{(t,x)}\colon D \to R$ as follows

$$z_{(t,x)}(\tau,\xi) = z(t+\tau, x+\xi), \quad (\tau,\xi) \in D.$$

Let us denote by $\|\cdot\|_0$ the supremum norm in the space $C(D,R)$. Put $\Omega = \bar{E} \times C(D,R) \times R^n$ and suppose that $f : \Omega \to R$ is a given function of the variables (t,x,w,q) with $x = (x_1, \dots, x_n)$ and $q = (q_1, \dots, q_n)$. Let $\varphi\colon E_0 \cup \partial_0 E \to R$ and $\alpha_0\colon [0,a] \to R$, $\alpha' : \bar{E} \to R^n$, $\alpha' = (\alpha_1, \dots, \alpha_n)$ be given functions.
The requirements on α_0 and α' are that $(\alpha_0(t), \alpha'(t,x)) \in \bar{E}$ and $\alpha_0(t) \le t$ for $t \in [0,a]$. Write $\alpha(t,x) = (\alpha_0(t), \alpha'(t,x))$ for $(t,x) \in \bar{E}$. For a function $z\colon B \to R$ and for a point $(t,x) \in E$ we write

$$\partial_x z = (\partial_{x_1} z, \dots, \partial_{x_n} z).$$

We consider the nonlinear functional differential equation

$$\partial_t z(t,x) = f(t,x,z_{\alpha(t,x)}, \partial_x z(t,x)) \tag{1}$$

with the initial boundary condition

$$z(t,x) = \varphi(t,x) \text{ for } (t,x) \in E_0 \cup \partial_0 E. \tag{2}$$

We consider classical solutions of (1),(2).

In recent years a number of papers concerning numerical methods for functional partial differential equations have been published.

Difference methods for nonlinear parabolic functional differential equations with initial boundary conditions of the Dirichlet type were considered in [11], [18]. Difference schemes for the Cauchy problem were investigated in [12], [19].

The main question in these investigations is to find a difference functional equation which satisfies consistency conditions with respect to the original problem and is stable. The method of difference inequalities or theorems on recurrent inequalities are used in the investigation of the stability.

The numerical method of lines for nonlinear partial functional equations was considered in [10], [20]. By using a disretization in spatial variables, parabolic equations or first order partial equations with functional variables are replaced by sequences of initial problems for ordinary functional differential equations. The question of under what conditions the solutions of ordinary equations tend to solutions of original problems are investigated in these papers. The method of differential inequalities is used.

Difference methods and monotone iterative techniques for nonlinear problems were investigated in [13], [14]. Numerical methods of functional differential equations with impulses are investigated in [9].

For further bibliographical information see the references in [5], [14] and in the monograph [17].

Finite difference approximations relative to initial or initial boundary value problems for first order partial functional differential equations were investigated in [1], [6], [15], [16].

Error estimates implying the convergence of difference schemes are obtained in these papers by a comparison technique. Theorems on functional difference inequalities are used. The proofs of the convergence were also based on a general theorem on error estimates of approximate solutions to functional difference equations of the Volterra type with initial-boundary conditions and with unknown function of several variables. The monograph [8] contains an exposition of the theory of numerical methods for hyperbolic functional differential problems.

In the paper we present another approach to the numerical solving of the problem (1),(2). We transform the nonlinear equation (1) into a quasilinear system of difference functional equations.

The following two difference methods for problem (1),(2) are known in literature: the Euler method and the Lax scheme. They have the following properties. Assumptions on the regularity of f in convergence theorems are the same for both methods. It is required that the function f of the variables (t, x, w, q) satisfies the Lipschitz condition with respect to the functional variable and it is of class C^1 with respect to $q = (q_1, \ldots, q_n)$. Some nonlinear estimates for f with respect to w are also considered. For the analysis of the stability of the Euler method we need the assumption that the function

$$\operatorname{sign} \partial_q f = (\operatorname{sign} \partial_{q_1} f, \ldots, \operatorname{sign} \partial_{q_n} f) \tag{3}$$

is constant. We do not need this assumption if we use the Lax scheme for (1),(2).

There are equations for which both methods can be used. It follows that in this case the Euler method is more suitable than the Lax scheme. The aim of the paper is to show that for each equation (1) with sufficiently regular function f the Euler difference method can be constructed. It is important in our considerations that the assumption that the function (3) is constant is omitted. In other words, we show that the Lax scheme is superfluous in the numerical approximation of classical solutions of (1),(2).

Our main ideas are based on a quasilinearization of the equation (1) with respect to the last variable and on the theory of bicharacteristics for

nonlinear hyperbolic problems. The method was introduced and studied by S. Cinquini and M. Cinquini-Cibrario [2], [3].
The authors have used the method of quasilinearization and the theory of bicharacteristics in the existence and uniqueness theory for generalized solutions of hyperbolic differential systems. This method is also adapted to nonlinear functional differential problems in [8], Chapter V.

The paper is organized as follows. In Section 2 we propose a quasilinear system of difference equations of the Euler type to the problem (1),(2). A convergence result and an error estimate of approximate solutions are presented in Section 3. Numerical examples are given in the last part of the paper.

We use in the paper general ideas concerning numerical methods for partial differential equations which were introduced in [4], [8].

Below, we give examples of equations which can be derived from (1) by specifying the operator f.

Example 1. A general class of equations with deviated variables can be obtained in the following way. Suppose that $F : E \times R \times R^n \to R$, $\beta_0 : [0, a] \to R$, $\beta' : E \to R^n$, $\beta' = (\beta_1, \ldots, \beta_n)$ are given functions and

$$-d_0 \le \beta_0(t) - \alpha_0(t) \le 0, \;\; -d \le \beta'(t,x) - \alpha'(t,x) \le d, \;\; (t,x) \in E. \quad (4)$$

We define the operator f as follows:

$$f(t,x,w,q) = F(t,x,w(\beta_0(t) - \alpha_0(t), \beta'(t,x) - \alpha'(t,x)), q), \quad (t,x,w,q) \in \Omega. \quad (5)$$

Then

$$f(t,x,z_{\alpha(t,x)},q) = F(t,x,z(\beta(t,x)),q),$$

where $\beta(t,x) = (\beta_0(t), \beta'(t,x))$ and the equation (1) is equivalent to

$$\partial_t z(t,x) = F(t,x,z(\beta(t,x)), \partial_x z(t,x)). \quad (6)$$

Example 2. Now we consider differential integral equations. Suppose that $\gamma_0 : [0, a] \to R$, $\gamma' : E \to R^n$, $\gamma' = (\gamma_1, \ldots, \gamma_n)$ are given functions and

$$-d_0 \le \gamma_0(t) - \alpha_0(t) \le 0, \;\; -d \le \gamma'(t,x) - \alpha'(t,x) \le d, \;\; (t,x) \in E. \quad (7)$$

For the above given functions β satisfying (4) and $F : E \times R \times R^n \to R$ we define the operator f in the following way:

$$f(t,x,w,q) = F(t,x, \int_{\beta_0(t)-\alpha_0(t)}^{\gamma_0(t)-\alpha_0(t)} \int_{\beta'(t,x)-\alpha'(t,x)}^{\gamma'(t,x)-\alpha'(t,x)} w(\tau,y) dy \, d\tau, q), \quad (8)$$

where $(t,x,w,q) \in \Omega$. Then

$$f(t,x,z_{\alpha(t,x)},q) = F(t,x, \int_{\beta(t,x)}^{\gamma(t,x)} z(\tau,y) dy \, d\tau, q)$$

and (1) reduces to the differential integral equation

$$\partial_t z(t,x) = F(t,x, \int_{\beta(t,x)}^{\gamma(t,x)} z(\tau,y)dy\,d\tau, \partial_x z(t,x)). \tag{9}$$

Existence results for mixed problems can be found in [8], Chapter V.

2. Discretizations of the Mixed Problem

Let us denote by $\mathbf{F}(X,Y)$ the class of all functions defined on X and taking values in Y, where X and Y are arbitrary sets. Let $\mathbf{N}$ and $\mathbf{Z}$ be the sets of natural numbers and integers, respectively. Let us fix our notation of vectors and matrices. We will denote by $M_{n\times n}$ the space of all $n\times n$ matrices with real elements. For $x\,, y \in R^n, U \in M_{n\times n}$, where

$$x = (x_1,\ldots,x_n), y = (y_1,\ldots,y_n), U = [u_{ij}]_{i,j=1,\ldots,n}$$

we put

$$\|x\| = \sum_{j=1}^{n} |x_j|\,,\ x \diamond y = (x_1y_1,\ldots,x_ny_n)\,,$$

$$\|U\| = \max\Big\{ \sum_{j=1}^{n} |u_{ij}| \colon 1 \le i \le n \Big\}.$$

The product of two matrices is denoted by " $\star$ ". If $U \in M_{n\times n}$, then U^T is the transpose matrix. We use the symbol " $\circ$ " to denote the scalar product in R^n. For a function $z \in C(B,R)$ and for a point $t \in [0,a]$, we write

$$\|z\|_t = \max\{\, |z(\tau,y)| \colon (\tau,y) \in B \cap ([-d_0,t] \times R^n)\}.$$

We define a mesh on the set B in the following way. Let (h_0,h'), $h' = (h_1,\ldots,h_n)$, stand for steps of the mesh. For $h = (h_0,h')$ and $(r,m) \in \mathbf{Z}^{1+n}$, where $m = (m_1,\ldots,m_n)$, we define nodal points as follows

$$t^{(r)} = rh_0\,,\ x^{(m)} = m \diamond h'\,,\ x^{(m)} = (x_1^{(m_1)},\ldots,x_n^{(m_n)}).$$

Let us denote by H the set of all h such that there are $N = (N_1,\ldots,N_n) \in \mathbf{N}^n$ and $N_0 \in \mathbf{N}$ such that $N \diamond h' = c$ and $N_0h_0 = d_0$. There is $K \in \mathbf{N}$ such that $Kh_0 \le a < (K+1)h_0$. For $h \in H$ we put $\|h\| = h_0 + h_1 + \ldots + h_n$. Write

$$R_h^{1+n} = \{\, (t^{(r)},x^{(m)}) \colon (r,m) \in \mathbf{Z}^{1+n}\},\ I_h = \{\, t^{(r)} \colon 0 \le r \le K\}$$

and

$$E_{0.h} = E_0 \cap R_h^{1+n},\ E_h = E \cap R_h^{1+n},\ \partial_0 E_h = \partial_0 E \cap R_h^{1+n},$$
$$B_h = E_{0.h} \cup E_h \cup \partial_0 E_h.$$

Set

$$B_{r.h} = B_h \cap \Big([-d_0,t^{(r)}]\ \times\ R^n \Big)\,,\quad 0 \le r \le K,$$

and

$$E'_h = \{\, (t^{(r)},x^{(m)}) \in E_h :\ 0 \le r \le K-1\}.$$

For functions $\omega : I_h \to R,\ z : B_h \to R\,, u : B_h \to R^n$ we write

$$\omega^{(r)} = \omega(t^{(r)}),\ z^{(r,m)} = z(t^{(r)}, x^{(m)}),\ u^{(r,m)} = u(t^{(r)}, x^{(m)}),$$

$$\|z\|_{r.h} = \max\{\, |z^{(i,m)}| \colon (t^{(i)}, x^{(m)}) \in B_{r.h}\}$$

and

$$\|u\|_{r.h} = \max\{\, \|u^{(i,m)}\| \colon (t^{(i)}, x^{(m)}) \in B_{r.h}\},$$

where $0 \le r \le K$. Let $e_j = (0, \dots, 0, 1, 0, \dots, 0) \in R^n$, 1 standing on the j-th place, $1 \le j \le n$.

Classical difference methods for the problem (1),(2) consist in replacing partial derivatives ∂_t and $(\partial_{x_1}, \dots, \partial_{x_n}) = \partial_x$ with difference operators δ_0 and $(\delta_1, \dots, \delta_n) = \delta$ respectively. Moreover, because the equation (1) contains the functional variable $z_{\alpha(t,x)}$ which is an element of the space $C(D, R)$, we need an interpolating operator $T_h \colon F(B_h, R) \to C(B, R)$. This leads to the difference functional equation

$$\delta_0 z^{(r,m)} = f(t^{(r)}, x^{(m)}, T_h[z]_{\alpha^{(r,m)}}, \delta z^{(r,m)}) \tag{10}$$

with the initial boundary condition

$$z^{(r,m)} = \varphi_h^{(r,m)} \ \text{ on } \ E_{0.h} \cup \partial_0 E_h, \tag{11}$$

where $\varphi_h \colon E_{0.h} \cup \partial_0 E_h \to R$ is a given function.

The following examples of the equation (10) are known in literature. The Euler difference method is obtained by putting

$$\delta_0 z^{(r,m)} = \frac{1}{h_0}[z^{(r+1,m)} - z^{(r,m)}], \tag{12}$$

$$\delta_j z^{(r,m)} = \frac{1}{h_j}[z^{(r,m+e_j)} - z^{(r,m)}],\ \ 1 \le j \le \kappa, \tag{13}$$

and

$$\delta_j z^{(r,m)} = \frac{1}{h_j}[z^{(r,m)} - z^{(r,m-e_j)}],\ \ \kappa + 1 \le j \le n, \tag{14}$$

where $0 \le \kappa \le n$ is fixed. The Lax scheme is the second important example. It is obtained by putting

$$\delta_0 z^{(r,m)} = \frac{1}{h_0}\Big[z^{(r+1,m)} - \Delta[z]^{(r,m)}\Big] \tag{15}$$

where

$$\Delta[z]^{(r,m)} = \frac{1}{2n}\sum_{j=1}^{n}\Big[z^{(r,m+e_j)} + z^{(r,m-e_j)}\Big], \tag{16}$$

and

$$\delta_j z^{(r,m)} = \frac{1}{2h_j}\Big[z^{(r,m+e_j)} - z^{(r,m-e_j)}\Big],\ \ 1 \le j \le n. \tag{17}$$

Suppose that the function f is continuous on Ω and that there exist the partial derivatives

$$(\partial_{q_1} f, \dots, \partial_{q_n} f) = \partial_q f.$$

The stability of difference equations generated by the equation (1) is strictly connected with the so-called Courant-Friedrichs-Levy (CFL) conditions (see

[4], Chapter 3). The (CFL) conditions for nonlinear equation (1) and for the Euler difference method have the form:

(i) for each $P = (t, x, w, q) \in \Omega$ we have

$$1 - h_0 \sum_{j=1}^{n} \frac{1}{h_j} |\partial_{q_j} f(P)| \geq 0; \tag{18}$$

(ii) the function (3) is constant on Ω.

The (CFL) condition for (1) and for the Lax scheme has the form

$$1 - \frac{n h_0}{h_j} |\partial_{q_j} f(P)| \geq 0, \ \ 1 \leq j \leq n, \tag{19}$$

where $P \in \Omega$.

Note that the assumptions (18) and (19) are quite similar. It follows from the above condition (ii) that we need more restrictive assumptions on f for the Euler difference method than for the Lax scheme.

There are difference problems (10), (11) such that both the above difference methods are convergent. It follows from the bicharacteristics theory for nonlinear functional differential equations that in this case the numerical results obtained by using the Euler difference method are better than those obtained by the Lax scheme. Then we are interested in proving convergence results of the Euler method for a large class of nonlinear problems.

We will show that there are difference methods of the Euler type for which the assumption (ii) can be omitted.

In the paper we present a new approach to the numerical solving of (1),(2) . We transform initial boundary value problem into a quasilinear system of difference equations. Our main ideas are based on a quasilinearization of (1) with respect to the last variable and on the theory of bicharacteristics for functional differential equations. Unknown functions in new systems are z and the partial derivatives of z with respect to spatial variables. We use the above general ideas for the construction of new numerical methods for (1),(2).

We will need the following assumptions on the interpolating operator $T_h \colon \mathbf{F}(B_h, R) \to \mathbf{F}(B, R)$.

Assumption $\mathbf{H}[T_h]$. Suppose that the operator T_h is such that

1) $T_h \colon \mathbf{F}(B_h, R) \to C(B, R)$;

2) for any functions $z, \bar{z} \colon \mathbf{F}(B_h, R)$ we have

$$\|T_h[z] - T_h[\bar{z}]\|_{t^{(r)}} \leq \|z - \bar{z}\|_{r.h}, \quad 0 \leq r \leq K;$$

3) if a function $z \colon E \to R$ is of class C^1 then there is a function $\tilde{\gamma} \colon H \to R$ such that

$$\|z - T_h[z_h]\|_t \leq \tilde{\gamma}(h), \quad t \in [0, a],$$

and $\lim_{h \to 0} \tilde{\gamma}(h) = 0$, where z_h is the restriction of z to the set B_h.

Remark. The above assumption 2) implies that T_h fulfills the following Volterra condition: if the functions $z, \bar{z}$ are such functions that $z\big|_{B_{r.h}} =$

$\bar{z}\big|_{B_{r.h}}$, then $T_h[z]^{(r,m)} = T_h[\bar{z}]^{(r,m)}$ for $-N \le m \le N$. Condition 2) states also that T_h satisfies the Lipcshitz condition with the constant $L = 1$. The meaning of the assumption 3) is that $T_h[z_h]$ is an approximation of z and the approximation error is bounded by $\tilde{\gamma}(h)$.

An example of the operator T_h satisfying Assumption H$[T_h]$ can be found in [8], Chapter V.

If $u \in \mathbf{F}(B_h, R^n)$ and $u = (u_1, \dots, u_n)$, then we put

$$T_h[u] = \Big(T_h[u_1], \dots, T_h[u_n]\Big).$$

We denote by $CL(D, R)$ the set of all continuous and real functions defined on $C(D, R)$ and by $\|\cdot\|_\star$ the norm in $CL(D, R)$.

Assumption H$_0$[f]. Suppose that $f \in C(\Omega, R)$, the partial derivatives

$$\partial_x f(P) = (\partial_{x_1} f(P), \dots, \partial_{x_n} f(P)),$$

$$\partial_w f(P), \ \partial_q f(P) = (\partial_{q_1} f(P), \dots, \partial_{q_n} f(P))$$

exist for $P = (t, x, w, q) \in \Omega$ and $\partial_x f$, $\partial_q f \in C(\Omega, R^n)$, $\partial_w f \in CL(D, R)$.

Assumption H$[\alpha_0, \alpha']$. Suppose that the functions $\alpha_0 \colon [0, a] \to [0, a]$, $\alpha' \colon \bar{E} \to [-b, b]$ are continuous, the partial derivatives

$$\partial_x \alpha'(t, x) = [\partial_{x_j} \alpha_i(t, x)]_{i,j=1,\dots,n}$$

exist on $\bar{E}$ and $\partial_x \alpha' \in C(\bar{E}, M_{n\times n})$.

Now we formulate a difference problem corresponding to (1),(2). We will denote by δ_0 the difference operator with respect to the variable t and by $\delta = (\delta_1, \dots, \delta_n)$ the difference operator for the spatial variables $(x_1, \dots, x_n) = x$. Write

$$\delta z^{(r,m)} = \Big(\delta_1 z^{(r,m)}, \dots, \delta_n z^{(r,m)}\Big),$$

$$\delta_0 u^{(r,m)} = \Big(\delta_0 u_1^{(r,m)}, \dots, \delta_0 u_n^{(r,m)}\Big), \ \delta u^{(r,m)} = \Big[\delta_j u_\tau^{(r,m)}\Big]_{\tau,j=1,\dots,n}.$$

Set

$$P^{(r,m)}[z, u] = \Big(t^{(r)}, x^{(m)}, (T_h[z])_{\alpha^{(r,m)}}, u^{(r,m)}\Big).$$

If $u \colon B \to R^n$ and $u = (u_1, \dots, u_n)$, then we put

$$u_{\alpha(t,x)} = ((u_1)_{\alpha(t,x)}, \dots, (u_n)_{\alpha(t,x)}) \ \text{ for } \ (t, x) \in \bar{E}.$$

We consider the following system of quasilinear difference equations with unknown functions z and $(u_1, \dots, u_n) = u$.

$$\delta_0 z^{(r,m)} = f(P^{(r,m)}[z, u]) + \partial_q f(P^{(r,m)}[z, u]) \circ (\delta z^{(r,m)} - u^{(r,m)}), \tag{20}$$

$$\delta_0 u^{(r,m)} = \partial_x f(P^{(r,m)}[z, u]) + \partial_w f(P^{(r,m)}[z, u])\,(T_h[u])_{\alpha^{(r,m)}} \star \partial_x {\alpha'}^{(r,m)} + \\ + \partial_q f(P^{(r,m)}[z, u]) \star \Big[\delta u^{(r,m)}\Big]^T, \tag{21}$$

with the initial-boundary condition

$$z^{(r,m)} = \varphi_h^{(r,m)},\ u^{(r,m)} = \psi_h^{(r,m)} \text{ on } E_{0.h} \cup \partial_0 E_h, \tag{22}$$

where

$$\varphi_h \colon E_{0.h} \cup \partial_0 E_h \to R,\ \psi_h \colon E_{0.h} \cup \partial_0 E_h \to R^n,\ \psi_h = \big(\psi_{h.1}, \ldots, \psi_{h.n}\big),$$

are given functions and

$$\partial_w f(P^{(r,m)}[z,u])(T_h[u])_{\alpha^{(r,m)}} =$$
$$= (\partial_w f(P^{(r,m)}[z,u])(T_h[u_1])_{\alpha^{(r,m)}}, \ldots, \partial_w f(P^{(r,m)}[z,u])(T_h[u_n])_{\alpha^{(r,m)}}).$$

The difference operator δ_0 is defined by

$$\begin{aligned} \delta_0 z^{(r,m)} &= \frac{1}{h_0}\Big(z^{(r+1,m)} - z^{(r,m)}\Big), \\ \delta_0 u_j^{(r,m)} &= \frac{1}{h_0}\Big(u_j^{(r+1,m)} - u_j^{(r,m)}\Big), \quad 1 \le j \le n. \end{aligned} \tag{23}$$

The difference operator δ for the spatial variables is defined in the following way. Suppose that $(t^{(r)}, x^{(m)}) \in E'_h$ and that the functions (z,u) are known on the set $B_{r.h}$. If

$$\partial_{q_j} f(P^{(r,m)}[z,u]) \ge 0, \tag{24}$$

then

$$\delta_j z^{(r,m)} = \frac{1}{h_j}\Big(z^{(r,m+e_j)} - z^{(r,m)}\Big), \tag{25}$$

and

$$\delta_j u_\tau^{(r,m)} = \frac{1}{h_j}\Big(u_\tau^{(r,m+e_j)} - u_\tau^{(r,m)}\Big), \quad 1 \le \tau \le n. \tag{26}$$

If

$$\partial_{q_j} f(P^{(r,m)}[z,u]) < 0, \tag{27}$$

then

$$\delta_j z^{(r,m)} = \frac{1}{h_j}\Big(z^{(r,m)} - z^{(r,m-e_j)}\Big), \tag{28}$$

and

$$\delta_j u_\tau^{(r,m)} = \frac{1}{h_j}\Big(u_\tau^{(r,m)} - u_\tau^{(r,m-e_j)}\Big), \quad 1 \le \tau \le n. \tag{29}$$

The difference problem (20)–(22) has exactly one solution (z_h, u_h), where $z_h \colon B_{r.h} \to R,\ u_h \colon B_{r.h} \to R^n$. Indeed, if we assume that the solution of the above problem is defined on the set $B_{r.h}$, $0 \le r < K$, and $(t^{(r)}, x^{(m)}) \in E_h$, then we have

$$(t^{(r)}, x^{(m+e_j)}),\ (t^{(r)}, x^{(m-e_j)}) \in B_{r.h} \ \text{ for } \ 1 \le j \le n.$$

The above relations and the conditions (24)–(29) imply that the values $z_h^{(r+1,m)}$ and $u_h^{(r+1,m)}$ can be calculated from (20)–(22) and the solution is defined on $B_{r+1.h}$. Then, by induction, the solution (z_h, u_h) of (20)–(22) exists and it is unique on B_h.

The numerical method consisting of the system (20)–(21) with the initial condition (22), where the difference operators are defined by (24)–(29) is called a generalized Euler method for problem (1),(2).

The difference problem (20)–(22) is obtained in the following way. Suppose that Assumptions $\mathrm{H}_0[\mathrm{f}]$ and $\mathrm{H}[\alpha_0, \alpha']$ are satisfied and that the derivative $\partial_x \varphi$ exists on $E_0 \cup \partial_0 E$. Let (z, u) be unknown functions of the variables $(t, x) \in E$, where $u = (u_1, \dots, u_n)$. Write

$$U[z, u; t, x] = (t, x, z_{\alpha(t,x)}, u(t, x)).$$

Let us consider the quasilinear differential system

$$\partial_t z(t, x) = f(U[z, u; t, x]) + \\ + \partial_q f(U[z, u; t, x]) \circ (\partial_x z(t, x) - u(t, x)), \tag{30}$$

$$\partial_t u(t, x) = \partial_x f(U[z, u; t, x]) + \partial_w f(U[z, u; t, x])\, u_{\alpha(t,x)} \star \partial_x \alpha'(t, x) + \\ + \partial_q f(U[z, u; t, x]) \star [\partial_x u(t, x)]^T \tag{31}$$

with the initial condition

$$z(t, x) = \varphi(t, x),\ u(t, x) = \partial_x \varphi(t, x) \text{ on } E_0 \cup \partial_0 E. \tag{32}$$

Under natural assumptions on the given functions the above problem has the following property: if $(\tilde{z}, \tilde{u})$ is a solution of (30)–(32), then $\partial_x \tilde{z} = \tilde{u}$ and the conditions

(A) *the function $v \colon D \to R$ is a classical solution of (1),(2),*

and

(B) *the functions $(v, \partial_x v)$ are a classical solution of (30)–(32),*

are equivalent. The difference problem (20)–(22) is a discretization of (30)–(32). The quasilinear system (30),(31) has the following property: differential equations of the bicharacteristic for (30) and for (31) are the same and they have the form

$$\eta'(t) = -\partial_q f(t, \eta(t), z_{\alpha(t,\eta(t))}, u(t, \eta(t))),$$

where $\eta = (\eta_1, \dots, \eta_n)$. This property of the system (30),(31) is very important in the investigation of the stability of the problem (20)–(22). It is important in our considerations that we approximate classical solutions of nonlinear equation (1) by solutions of a quasilinear difference system and the method of discretization of the system (30),(31) at the point $(t^{(r)}, x^{(m)}) \in E'_h$ depends on the properties of the functions

$$\partial_q f = (\partial_{q_1} f, \dots, \partial_{q_n} f),$$

and on the previous values of the unknown functions (z, u) in (20),(21).

3. Convergence of the Generalized Euler Method

We formulate next assumptions on the given functions.

Assumption H[σ]. Suppose that the function $\sigma \colon [0, a] \times R_+ \to R_+$ is continuous and

1) σ is nondecreasing with respect to both variables and $\sigma(t,0)=0$ for $t\in[0,a]$,

2) for each $c\in R_+$ and $d\geq 1$ the maximal solution of the Cauchy problem

$$\eta'(t)=c\eta(t)+d\sigma(t,\eta(t))\,,\ \eta(0)=0,$$

is $\tilde\eta(t)=0\,,\ t\in[0,a]$.

Assumption H[f]. Suppose that Assumption H_0[f] is satisfied and

1) there is $A\in R_+$ such that

$$\|\partial_x f(P)\|,\ \|\partial_w f(P)\|_\star,\ \|\partial_q f(P)\|\leq A,$$

where $P=(t,x,w,q)\in\Omega$,

2) there is a function $\sigma\colon[0,a]\times R_+\to R_+$ such that Assumption H[σ] is satisfied and the terms

$$\|\partial_x f(t,x,w,q)-\partial_x f(t,x,\bar w,\bar q)\|,$$

$$\|\partial_w f(t,x,w,q)-\partial_w f(t,x,\bar w,\bar q)\|_\star,\ \|\partial_q f(t,x,w,q)-\partial_q f(t,x,\bar w,\bar q)\|$$

are bounded from above by $\sigma(t,\|w-\bar w\|_0+\|q-\bar q\|)$.

Theorem 1. *Suppose that Assumptions H[α_0,α'] and H[f] are satisfied and*

1) *$h\in H$ and for $P=(t,x,w,q)\in\Omega$ we have*

$$1-h_0\sum_{j=1}^n\frac{1}{h_j}\Big|\partial_{q_j}f(P)\Big|\geq 0, \tag{33}$$

2) *the operator $T_h\colon\mathbf{F}(B_h,R)\to C(B,R)$ satisfies the Assumption H[T_h],*

3) *the function $\varphi:E_0\cup\partial_0E\to R$ is of the class C^2, $v:B\to R$ is a solution of the problem* (1), (2) *and v is of the class C^2 on B,*

4) *the functions (z_h,u_h) where $z_h:B_h\to R$, $u_h:B_h\to R^n$, $u_h=(u_{h.1},\dots,u_{h.n})$, satisfy* (20)–(22) *with δ_0 and δ given by* (23)–(29) *and there is a function $\beta_0\colon H\to R_+$ such that*

$$|\varphi^{(r,m)}-\varphi_h^{(r,m)}|+\|\partial_x\varphi^{(r,m)}-\psi_h^{(r,m)}\|\leq\beta_0(h)\ \text{ on }\ E_{0.h}\cup\partial_0E_h \tag{34}$$

and $\lim_{h\to0}\beta_0(h)=0$.

Then there is a number $\varepsilon_0>0$ and a function $\beta\colon H\to R_+$ such that we have for $\|h\|\leq\varepsilon_0$

$$|v^{(r,m)}-z_h^{(r,m)}|+\|\partial_x v^{(r,m)}-u_h^{(r,m)}\|\leq\beta(h) \tag{35}$$

on E_h and $\lim_{h\to0}\beta(h)=0$.

Proof. Let us denote by $w\colon B\to R^n$ the function defined by

$$w=\partial_x v\,,\ w=(w_1,\dots,w_n).$$

Then the functions (v,w) satisfy the quasilinear system (30)–(31) and the initial condition (32). Write

$$\xi_h^{(r,m)}=v^{(r,m)}-z_h^{(r,m)}, \tag{36}$$

$$\lambda_h^{(r,m)} = w^{(r,m)} - u_h^{(r,m)}, \quad \lambda_h^{(r,m)} = (\lambda_{h.1}^{(r,m)}, \ldots, \lambda_{h.n}^{(r,m)}), \tag{37}$$

where $(t^{(r)}, x^{(m)}) \in B_h$. Let the functions $\omega_{h.0}, \omega_{h.1} : I_h \to R_+$ be defined by

$$\omega_{h.0}^{(r)} = \|\xi_h\|_{r.h}\,,\ \omega_{h.1}^{(r)} = \|\lambda_h\|_{r.h}, \tag{38}$$

and $\omega_h = \omega_{h.0} + \omega_{h.1}$. We will write a difference inequality for the function ω_h. Set

$$Q^{(r,m)}[v,w] = U[v,w;t^{(r)},x^{(m)}].$$

Let the functions $\Gamma_h,\ \Lambda_h : E_h' \to R$ be defined by

$$\Gamma_h^{(r,m)} = \delta_0 v^{(r,m)} - \partial_t v^{(r,m)} + \\ +\partial_q f(Q^{(r,m)}[v,w]) \circ \Big[\partial_x v^{(r,m)} - \delta v^{(r,m)}\Big], \tag{39}$$

and

$$\Lambda_h^{(r,m)} = f(Q^{(r,m)}[v,w]) - f(P^{(r,m)}[z_h,u_h]) - \\ -\partial_q f(Q^{(r,m)}[v,w]) \circ w^{(r,m)} + \partial_q f(P^{(r,m)}[z_h,u_h]) \circ u_h^{(r,m)} + \\ +\Big[\partial_q f(Q^{(r,m)}[v,w]) - \partial_q f(P^{(r,m)}[z_h,u_h])\Big] \circ \delta v^{(r,m)}. \tag{40}$$

It follows from (30) and from (20) that the function ξ_h satisfies the difference equation

$$\xi_h^{(r+1,m)} = \xi_h^{(r,m)} + h_0 \partial_q f(P^{(r,m)}[z_h,u_h]) \circ \delta \xi_h^{(r,m)} + \\ +h_0 \Big[\Lambda_h^{(r,m)} + \Gamma_h^{(r,m)}\Big]. \tag{41}$$

Put

$$J^+[r,m] = \Big\{\, j \in \{1,\ldots,n\} : \partial_{q_j} f(P^{(r,m)}[z_h,u_h]) \ge 0 \Big\}, \\ J^-[r,m] = \{1,\ldots,n\} \backslash J^+[r,m].$$

Consider the operator $W_h : \mathbf{F}(B_h,R) \to \mathbf{F}(E_h',R)$ defined by

$$W_h[\theta]^{(r,m)} = \theta^{(r,m)} \left[1 - h_0 \sum_{j=1}^{n} \frac{1}{h_j} \big|\partial_{q_j} f(P^{(r,m)}[z_h,u_h])\big|\right] + \\ + h_0 \sum_{j \in J^+[r,m]} \frac{1}{h_j} \partial_{q_j} f(P^{(r,m)}[z_h,u_h]) \theta^{(r,m+e_j)} - \\ - h_0 \sum_{j \in J^-[r,m]} \frac{1}{h_j} \partial_{q_j} f(P^{(r,m)}[z_h,u_h]) \theta^{(r,m-e_j)}, \tag{42}$$

where $\theta \in \mathbf{F}(B_h,R)$ and $(t^{(r)}, x^{(m)}) \in E_h'$. It follows from (24)–(29) that the relation (41) is equivalent to

$$\xi_h^{(r+1,m)} = W_h[\xi_h]^{(r,m)} + h_0 \big[\Lambda_h^{(r,m)} + \Gamma_h^{(r,m)}\big]. \tag{43}$$

According to the assumption (33), we have

$$\big|W_h[\xi_h]^{(r,m)}\big| \le \omega_{h.0}^{(r)} \text{ for } (t^{(r)}, x^{(m)}) \in E_h'. \tag{44}$$

Let $\tilde{c}, \tilde{s} \in R_+$ be such constants that

$$\begin{gathered} |\partial_t v(t,x)|\,,\ \|\partial_x v(t,x)\|\,,\ |\partial_{tt} v(t,x)| \le \tilde{c}, \\ \|\partial_{xx} v(t,x)\|,\ |\partial_{tx_i} v(t,x)| \le \tilde{c}, \end{gathered} \tag{45}$$

where $(t,x) \in B, i = 1,\dots,n$, and

$$|\partial_t \alpha_0(t,x)|,\ \|\partial_t \alpha'(t,x)\|,\ \|\partial_x \alpha'(t,x)\| \le \tilde{s},\ (t,x) \in \bar{E}. \tag{46}$$

It follows from Assumption H[f] and from the condition 2) of the theorem that there is $\gamma_0 \colon H \to R_+$ such that

$$\left|\Gamma_h^{(r,m)}\right| \le \gamma_0(h) \text{ on } E'_h \tag{47}$$

and $\lim_{h\to 0} \gamma_0(h) = 0$. According to Assumptions H[f] and H[T_h], there is $\tilde{\gamma}_0 \colon H \to R_+$ such that

$$\|v_{\alpha^{(r,m)}} - (T_h[z_h])_{\alpha^{(r,m)}}\|_0 \le \omega_{h.0}^{(r)} + \tilde{\gamma}_0(h) \ \text{ and } \ \lim_{h\to 0} \tilde{\gamma}_0(h) = 0 \tag{48}$$

and

$$|f(Q^{(r,m)}[v,w]) - f(P^{(r,m)}[z_h,u_h])| \le A\omega_h^{(r)} + A\tilde{\gamma}_0(h),$$

where $(t^{(r)}, x^{(m)}) \in E'_h$. In the same manner we can see that

$$\|\partial_q f(Q^{(r,m)}[v,w]) - \partial_q f(P^{(r,m)}[z_h,u_h])\| \le \sigma(t^{(r)}, \omega_h^{(r)} + \tilde{\gamma}_0(h)), \tag{49}$$

where $(t^{(r)}, x^{(m)}) \in E'_h$. An easy computation shows that

$$\left|\Lambda_h^{(r,m)}\right| \le 2\tilde{c}\sigma(t^{(r)}, \omega_h^{(r)} + \tilde{\gamma}_0(h)) + 2A\omega_h^{(r)} + A\tilde{\gamma}_0(h),$$

where $(t^{(r)}, x^{(m)}) \in E'_h$. According to the above estimates and (43), (44), we have

$$\begin{gathered} \omega_{h.0}^{(r+1)} \le \omega_{h.0}^{(r)} + 2h_0\tilde{c}\sigma(t^{(r)}, \omega_h^{(r)} + \tilde{\gamma}_0(h)) + \\ +2h_0 A\omega_h^{(r)} + h_0(\gamma_0(h) + A\tilde{\gamma}_0(h)),\ \ 0 \le r \le K-1. \end{gathered} \tag{50}$$

Now we write a difference inequality for $\omega_{h.1}$. Let the functions U_h, V_h : $E'_h \to R^n$ be defined by

$$\begin{gathered} U_h^{(r,m)} = \delta_0 w^{(r,m)} - \partial_t w^{(r,m)} \\ +\partial_q f(Q^{(r,m)}[v,w]) \star \left[\partial_x w^{(r,m)} - \delta w^{(r,m)}\right]^T \end{gathered} \tag{51}$$

and

$$\begin{gathered} V_h^{(r,m)} = \partial_x f(Q^{(r,m)}[v,w]) - \partial_x f(P^{(r,m)}[z_h,u_h]) + \\ +\partial_w f(Q^{(r,m)}[v,w]) A[w]^{(r,m)} \star \partial_x {\alpha'}^{(r,m)} - \\ -\partial_w f(P^{(r,m)}[z_h,u_h]) T_h A[u_h]^{(r,m)} \star \partial_x {\alpha'}^{(r,m)} + \\ +\left[\partial_q f(Q^{(r,m)}[v,w]) - \partial_q f(P^{(r,m)}[z_h,u_h])\right] \star \left[\delta w^{(r,m)}\right]^T. \end{gathered} \tag{52}$$

It follows from (31) and (21) that the function λ_h satisfy the difference equation

$$\lambda_h^{(r+1,m)} = \lambda_h^{(r,m)} + h_0 \partial_q f(P^{(r,m)}[z_h, u_h]) \star \Big[\delta \lambda_h^{(r,m)}\Big]^T + \\ + h_0 \Big[U_h^{(r,m)} + V_h^{(r,m)}\Big], \tag{53}$$

where $(t^{(r)}, x^{(m)}) \in E'_h$. For the function $\lambda_h = (\lambda_{h.1}, \ldots, \lambda_{h.n})$ we write

$$W_h[\lambda_h]^{(r,m)} = \Big(W_h[\lambda_{h.1}]^{(r,m)}, \ldots, W_h[\lambda_{h.n}]^{(r,m)}\Big)$$

where W_h is defined by (42). It follows from (24)–(29) that the relation (53) is equivalent to

$$\lambda_h^{(r+1,m)} = W_h[\lambda_h]^{(r,m)} + h_0 \Big[U_h^{(r,m)} + V_h^{(r,m)}\Big], \tag{54}$$

where $(t^{(r)}, x^{(m)}) \in E'_h$. According to the assumption (33), we have

$$\|W_h[\lambda_h]^{(r,m)}\| = \sum_{\tau=1}^{n} \Big| W_h[\lambda_{h.\tau}]^{(r,m)} \Big| \leq$$

$$\leq \Big[1 - h_0 \sum_{j=1}^{n} \frac{1}{h_j} \Big|\partial_{q_j} f(P^{(r,m)}[z_h, u_h])\Big|\Big] \, \|\lambda_h^{(r,m)}\| +$$

$$+ h_0 \sum_{j \in J^+[r,m]} \frac{1}{h_j} \partial_{q_j} f(P^{(r,m)}[z_h, u_h]) \|\lambda_h^{(r,m+e_j)}\| -$$

$$- h_0 \sum_{j \in J^-[r,m]} \frac{1}{h_j} \partial_{q_j} f(P^{(r,m)}[z_h, u_h]) \|\lambda_h^{(r,m-e_j)}\|$$

and consequently

$$\|W_h[\lambda_h]^{(r,m)}\| \leq \omega_{h.1}^{(r,m)} \text{ for } (t^{(r)}, x^{(m)}) \in E'_h. \tag{55}$$

It follows from Assumption H[f] and from the condition 2) of the theorem that there is $\gamma_1 \colon H \to R_+$ such that

$$\|U_h^{(r,m)}\| \leq \gamma_1(h) \text{ on } E'_h \,, \text{ and } \lim_{h \to 0} \gamma_1(h) = 0. \tag{56}$$

It is easy to see that the conclusion analogous to (49) can be drawn for the derivatives $\partial_x f$, $\partial_w f$. According to Assumption H$[T_h]$ there is $\tilde{\gamma}_1 \colon H \to R_+$ such that

$$\|w_{\alpha^{(r,m)}} - (T_h[u_h])_{\alpha^{(r,m)}}\|_0 \leq \tilde{\gamma}_1(h) + \omega_{h.1}^{(r)} \text{ and } \lim_{h \to 0} \tilde{\gamma}_1(h) = 0, \tag{57}$$

where $(t^{(r)}, x^{(m)}) \in E'_h$. Then we have the estimate

$$\|V_h^{(r,m)}\| \leq (1 + \tilde{c} + \tilde{c}\tilde{s}) \, \sigma(t^{(r)}, \omega_h^{(r)} + \tilde{\gamma}_0(h)) + A\tilde{s}\tilde{\gamma}_1(h) + A\tilde{s}\omega_h^{(r)},$$

and consequently

$$\omega_{h.1}^{(r+1)} \leq \omega_{h.1}^{(r)} + h_0(1 + \tilde{c} + \tilde{s}\tilde{c})\sigma(t^{(r)}, \omega_h^{(r)} + \tilde{\gamma}_0(h)) \tag{58}$$

$$+h_0 A\tilde{s}\omega_h^{(r)} + h_0(\gamma_1(h) + A\tilde{s}\tilde{\gamma}_1(h)),\ 0 \le r \le K-1.$$

Adding the inequalities (50) and (58), we get

$$\omega_h^{(r+1)} \le \omega_h^{(r)} + h_0\bar{d}\omega_h^{(r)} + h_0\tilde{d}\sigma(t^{(r)}, \omega_h^{(r)} + \tilde{\gamma}_0(h)) + h_0\tilde{\gamma}(h), \tag{59}$$

where $0 \le r \le K-1$ and

$$\begin{aligned} \bar{d} = A(2+\tilde{s}),\ \tilde{d} = 1 + 3\tilde{c} + \tilde{s}\tilde{c}, \\ \tilde{\gamma}(h) = \gamma_0(h) + \gamma_1(h) + A(\tilde{\gamma}_0(h) + \tilde{s}\tilde{\gamma}_1(h)). \end{aligned} \tag{60}$$

Consider the Cauchy problem

$$\eta'(t) = \bar{d}\eta(t) + \tilde{d}\sigma(t, \eta(t) + \tilde{\gamma}_0(h)) + \tilde{\gamma}(h),\ \eta(0) = \beta_0(h). \tag{61}$$

It follows from Assumption H[σ] that there is $\varepsilon_0 > 0$ such that for $\|h\| < \varepsilon_0$ there exists the maximum solution η_h of (61) and η_h is defined on $[0, a]$. Moreover, we have

$$\lim_{h\to 0} \eta_h(t) = 0 \text{ uniformly on } [0, a].$$

The function η_h satisfies the difference inequality

$$\eta_h^{(r+1)} \ge \eta_h^{(r)} + h_0\bar{d}\eta_h^{(r)} + \tilde{d}h_0\sigma(t^{(r)}, \eta_h^{(r)} + \tilde{\gamma}_0(h)) + h_0\tilde{\gamma}(h),\ 0 \le r \le K-1.$$

By the above inequality and (59) we have

$$\omega_h^{(r)} \le \eta_h^{(r)} \text{ for } 0 \le r \le K. \tag{62}$$

Then we get (35) for $\beta(h) = \eta_h(a)$. This completes the proof.

Remark. If we set $\sigma(t, s) = Ls$ for $(t, s) \in [0, a] \times R_+$, where $L \in R_+$, then the function σ satisfies Assumption H[σ]. In other words, we can assume that the functions $\partial_x f$, $\partial_w f$, $\partial_q f$ satisfy the Lipschitz condition with respect to (w, q) on Ω.

Remark. In the Assumption H[T_h] we can put $\tilde{\gamma}(h) = \tilde{c}\|h\|$, where $\tilde{c} \in R_+$ is such constant that

$$|\partial_t z(t, x)| \le \tilde{c},\ |\partial_{x_i} z(t, x)| \le \tilde{c},\ 1 \le i \le n,\ (t, x) \in B.$$

There exists an interpolating operator satisfying the so modified Assumption H[T$_h$]. It has been proposed in [8].

Lemma 1. *Suppose that all the assumptions of Theorem* 1 *are satisfied with* $\sigma(t, \tau) = L\tau$ *on* $[0, a] \times R_+$*, where* $L \in R_+$*. Then we have the following error estimate of the method* (20)–(29)*:*

$$|v^{(r,m)} - z_h^{(r,m)}| + \|\partial_x v^{(r,m)} - u_h^{(r,m)}\| \le \bar{\beta}(h) \text{ on } E_h,$$

where

$$\bar{\beta}(h) = \beta_0(h)e^{\tilde{L}a} + \bar{\gamma}(h)\frac{e^{\tilde{L}a} - 1}{\tilde{L}} \quad if \quad \tilde{L} > 0,$$
$$\bar{\beta}(h) = \beta_0(h) + a\bar{\gamma}(h) \quad if \quad \tilde{L} = 0$$

with

$$\tilde{L} = \bar{d} + L\tilde{d},\ \bar{\gamma}(h) = \tilde{\gamma}(h) + \tilde{c}\tilde{d}L\|h\|,$$

$$\gamma_0(h) = \gamma_1(h) = \frac{\tilde{c}}{2}(1 + A)\|h\|,\ \tilde{\gamma}_0(h) = \tilde{\gamma}_1(h) = \tilde{c}\|h\|,$$

and $\tilde{\gamma}, \bar{d}, \tilde{d}, \tilde{s}$ *are given by* (46), (60).

Proof. It follows that the estimates (47), (56), (48) and (57) are satisfied with the above given γ_0, γ_1, $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$, respectively, and we obtain the lemma from (35) and (62) and by solving of the problem (61).

Remark. In Lemma 1 on the error estimate we need estimates for the derivatives of the solution v of the problem (1),(2). One may obtain them by the method of differential inequalities, see [8], Chapter V.

4. Numerical Experiments

Let $n = 2$. Consider the mixed problem

$$\begin{aligned}\partial_t z(t,x,y) = x[\partial_x z(t,x,y) + \cos(\partial_x z(t,x,y) - 2txz(t,x,y))] + \\ + y[\partial_y z(t,x,y) - \sin(\partial_y z(t,x,y) + 2tyz(t,x,y))] + \\ + z\Big(t, \frac{x}{2}, \frac{y}{2}\Big) + f(t,x,y),\end{aligned} \tag{63}$$

$$z(0,x,y) = 1 \ \text{ for } \ (x,y) \in [-1,1] \times [-1,1], \tag{64}$$

$$z(t,1,y) = z(t,-1,y) = e^{t(1-y^2)} \ \text{ for } \ t \in [0,a],\ y \in [-1,1],$$

$$z(t,x,1) = z(t,x,-1) = e^{t(x^2-1)} \ \text{ for } \ t \in [0,a],\ x \in [-1,1],$$

where

$$f(t,x) = (1-2t)e^{t(x^1-y^2)}(x^2-y^2) - x - e^{\frac{t}{4}(x^2-y^2)}.$$

The solution of the problem is given by $v(t,x) = e^{t(x^2-y^2)}$. The classical difference method for (63),(64) has the form

$$\begin{aligned}\delta_0 z^{(r,m)} = x^{(m_1)}(\delta_1 z^{(r,m)(r,m)} + \cos(\delta_1 z^{(r,m)(r,m)} - 2tx^{(m_1)} z^{(r,m)})) + \\ + y^{(m_2)}(\delta_2 z^{(r,m)(r,m)} - \sin(\delta_2 z^{(r,m)(r,m)} + 2ty^{(m_2)} z^{(r,m)})) + \\ + A^{(r,m)} + f^{(r,m)},\end{aligned} \tag{65}$$

$$z^{(0,m_1,m_2)} = 1 \ \text{ for } \ (x^{(m_1)}, y^{(m_2)}) \in [-1,1] \times [-1,1], \tag{66}$$

$$z^{(r,N_1,m_2)} = z^{(r,-N_1,m_2)} = e^{t^{(r)}(1-(y^{(m_2)})^2)} \ \text{ for } \ t^{(r)} \in [0,a],\ y^{(m_2)} \in [-1,1],$$

$$z^{(r,m_1,N_2)} = z^{(r,m_1,-N_2)} = e^{t^{(r)}((x^{(m_1)})^2-1)} \ \text{ for } \ t^{(r)} \in [0,a],\ x^{(m_1)} \in [-1,1],$$

where $m = (m_1, m_2)$,

$$\delta_0 z^{(r,m)} = \frac{1}{h_0}\left[z^{(r+1,m)} - \Delta z^{(r,m)}\right],$$

$$\Delta z^{(r,m)} = \frac{1}{4}\left(z^{(r,m_1-1,m_2)} + z^{(r,m_1+1,m_2)} + z^{(r,m_1,m_2-1)} + z^{(r,m_1,m_2+1)}\right),$$

and

$$\delta_1 z^{(r,m)} = \frac{1}{2h_1}\Big[z^{(r,m_1+1,m_2)} - z^{(r,m_1-1,m_2)}\Big],$$

$$\delta_2 z^{(r,m)} = \frac{1}{2h_2}\Big[z^{(r,m_1,m_2+1)} - z^{(r,m_1,m_2-1)}\Big].$$

Moreover, we put

$$A^{(r,m)} = T_h[z](t^{(r)}, 0.5x^{(m_1)}, 0.5y^{(m_2)}), \tag{67}$$

and T_h is the interpolating operator for $n = 2$. The convergence of the method (65),(66) follows from [7]. Let us denote by $z_h : E_h \to R$ the solution of the problem (65),(66) and by $\zeta_h : E_h \to R$ the solution given by the generalized Euler method for the problem (63),(64).

Suppose that $t^{(r)}$ is fixed for some $0 \le r \le K$. Then we put

$$\varepsilon_h^{(r)} = \max\{|{z_h}^{(r,m)} - v^{(r,m)}|, -N \le (m_1, m_2) \le N\},$$

$$\nu_h^{(r)} = \frac{1}{(2N_1-1)(2N_2-1)} \sum_{-N \le (m_1,m_2) \le N} |{z_h}^{(r,m)} - v^{(r,m)}|.$$

The numbers $\varepsilon_h^{(r)}$ and $\nu_h^{(r)}$ can be called the maximal and the average error of the classical method for fixed $t^{(r)}$. In the similar way we define maximal and average errors $\bar{\varepsilon}_h^{(r)}$, $\bar{\nu}_h^{(r)}$ for the generalized Euler method.

We put $a = 1$, $b = 1$, $h_0 = 0.0001$, $h_1 = 0.04$, and we have the following experimental values for the above defined errors.

Table of maximal errors ε_h, $\bar{\varepsilon}_h$ and average errors ν_h, $\bar{\nu}_h$.

$t^{(r)}$	$\varepsilon_h^{(r)}$	$\bar{\varepsilon}_h^{(r)}$	$\nu_h^{(r)}$	$\bar{\nu}_h^{(r)}$
0.2	$9.01 \cdot 10^{-3}$	$8.72 \cdot 10^{-4}$	$6.61 \cdot 10^{-3}$	$4.39 \cdot 10^{-4}$
0.4	$4.53 \cdot 10^{-2}$	$4.10 \cdot 10^{-3}$	$3.26 \cdot 10^{-2}$	$1.96 \cdot 10^{-3}$
0.6	$1.14 \cdot 10^{-1}$	$1.06 \cdot 10^{-2}$	$7.99 \cdot 10^{-2}$	$4.87 \cdot 10^{-3}$
0.8	$2.21 \cdot 10^{-1}$	$2.18 \cdot 10^{-2}$	$1.50 \cdot 10^{-1}$	$9.55 \cdot 10^{-3}$
1.0	$3.70 \cdot 10^{-1}$	$3.94 \cdot 10^{-2}$	$2.46 \cdot 10^{-1}$	$1.65 \cdot 10^{-2}$

Note that $\bar{\varepsilon}_h^{(r)} < \varepsilon_h^{(r)}$ and $\bar{\nu}_h^{(r)} < \nu_h^{(r)}$ for all values of $t^{(r)}$.

Thus we see that the errors of the method (65),(66) are larger than the errors of the generalized Euler method. This is due to the fact that the approximation of the spatial derivatives of z in the generalized Euler method is better than the respective approximation of $\partial_x z$, $\partial_y z$ in (65),(66).

The amount of time of computing the approximate solution of the same problem with equal steps using the generalized method and using the classical method is comparable (in fact, the former is about 150 % of the latter), which is what we expected.

The method described in Theorem 1. have a potential for applications in the numerical solving of first order nonlinear differential equations with deviated variables.

References

1. P. Brandi, Z. Kamont, and A. Salvadori, Approximate solutions of mixed problems for first order partial differential-functional equations. *Atti Sem. Mat. Fis. Univ. Modena* **39** (1991), No. 1, 277–302.

2. M. Cinquini Cibrario, A class of systems of nonlinear partial differential equations in several independent variables. (Italian) *Ann. Mat. Pura Appl. (4)* **140** (1985), 223–253.
3. S. Cinquini, On hyperbolic systems of (nonlinear) partial differential equations in several independent variables. (Italian) *Ann. Mat. Pura Appl. (4)* **120** (1979), 201–214.
4. E. Godlewski and P.-A. Raviart, Numerical approximation of hyperbolic systems of conservation laws. *Applied Mathematical Sciences,* 118. *Springer-Verlag, New York,* 1996.
5. I. Győri, On the method of lines for the solutions of nonlinear partial differential equations. *Preprint,* 166. *Akad. Nauk SSSR, Otdel Vychisl. Mat., Moscow,* 1987. 48 pp.
6. D. Jaruszewska-Walczak and Z. Kamont, Numerical methods for hyperbolic functional differential problems on the Haar pyramid. *Computing* **65** (2000), No. 1, 45–72.
7. Z. Kamont, Finite-difference approximation of first-order partial differential-functional equations. *Ukraïn. Mat. Zh.* **46** (1994), No. 8, 985–996; English transl.: *Ukrainian Math. J.* **46** (1994), No. 8, 1079–1092 (1996).
8. Z. Kamont, Hyperbolic functional differential inequalities and applications. *Mathematics and its Applications,* 486. *Kluwer Academic Publishers, Dordrecht,* 1999.
9. Z. Kamont, Functional differential and difference inequalities with impulses. *Mem. Differential Equations Math. Phys.* **24** (2001), 5–82.
10. H. Leszczyński, On the method of lines for a non-linear heat equation with functional dependence. *Ann. Polon. Math.* **69** (1998), No. 1, 61–74.
11. M. Malec and M. Rosati, A convergent scheme for nonlinear systems of differential functional equations of parabolic type. *Rend. Mat. (7)* **3** (1983), No. 2, 211–227.
12. M. Malec and A. Schaffino, Méthode aux différences finies pour une équation nonlinéaire différentielle fonctionnelle du type parabolique avec une condition initiale de Cauchy. *Boll. Un. Mat. Ital. B (7)* **1** (1987), No. 1, 99–109.
13. C. V. Pao, Monotone methods for a finite difference system of reaction diffusion equation with time delay. Advances in difference equations, II. *Comput. Math. Appl.* **36** (1998), No. 10-12, 37–47.
14. C. V. Pao, Numerical methods for systems of nonlinear parabolic equations with time delays. *J. Math. Anal. Appl.* **240** (1999), No. 1, 249–279.
15. K. Prządka, Convergence of one-step difference methods for first order partial differential-functional equations. *Atti Sem. Mat. Fis. Univ. Modena* **35** (1987), No. 2, 263–288 (1988).
16. K. Prządka, Difference methods for nonlinear partial differential-functional equations of the first order. *Math. Nachr.* **138** (1988), 105–123.
17. A. Vande Wouwer, Ph. Saucez, and W. E. Schiesser, Adaptive method of lines. *Chapman & Hall/CRC, Boca Raton, FL,* 2001.
18. W. Voigt, Nonlinear parabolic differential-functional inequalities with boundary-functional conditions. *Beiträge Anal.* No. 18 (1981), 85–89.
19. W. Voigt, On finite-difference methods for parabolic functional-differential equations on unbounded domains. *Numerical methods and applications* (*Sofia,* 1989), 559–567, *Publ. House Bulgar. Acad. Sci., Sofia,* 1989.
20. B. Zubik-Kowal, The method of lines for first order partial differential-functional equations. *Studia Sci. Math. Hungar.* **34** (1998), No. 4, 413–428.

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