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EXPLICIT SOLUTION OF THE FIRST BVP OF THE ELASTIC MIXTURE FOR HALF-SPACE

Abstract. We consider the first BVP of elastic mixture theory for a transversally-isotropic half-space. The solution of the first BVP for the transversally-isotropic half-space is given in [1]. The present paper is an attempt to use this result for the BVP of elastic mixture theory for a transversally-isotropic elastic body. Using the potential method and the theory of integral equations, the uniqueness theorem is proved for a half-space and the first BVP previously is solved effectively (in quadratures), which has not been solved.

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[illegible]

The first BVP and the uniqueness theorem for a half-space. Let the plane ox_1x_2 be the boundary of a half-space $x_3 > 0$. Let the upper half-space be denoted by D and the boundary of D by S. Let the axis ox_3 be directed vertically upwards and the normal be $n(0,0,1)$.

A basic homogeneous equation of statics of transversally-isotropic elastic mixture theory can be written in the form [2]

$$C(\partial x)U = \begin{pmatrix} C^{(1)}(\partial x) & C^{(3)}(\partial x) \\ C^{(3)}(\partial x) & C^{(2)}(\partial x) \end{pmatrix} U = 0, \tag{1}$$

where the components of the matrix $C^{(j)}(\partial x) = \|C^{(j)}_{pq}(\partial x)\|_{3x3}$ are given in the form

$$\begin{aligned}
&C^{(j)}_{pq} = C^{(j)}_{qp}, \quad j = 1,2,3; \quad p,q = 1,2,3, \\
&C^{(j)}_{11}(\partial x) = c^{(j)}_{11}\frac{\partial^2}{\partial x_1^2} + c^{(j)}_{66}\frac{\partial^2}{\partial x_2^2} + c^{(j)}_{44}\frac{\partial^2}{\partial x_3^2}, \\
&C^{(j)}_{12}(\partial x) = (c^{(j)}_{11} - c^{(j)}_{66})\frac{\partial^2}{\partial x_1 \partial x_2}, \\
&C^{(j)}_{k3}(\partial x) = (c^{(j)}_{13} + c^{(j)}_{44})\frac{\partial^2}{\partial x_k \partial x_3}, \quad k = 1,2, \\
&C^{(j)}_{22}(\partial x) = c^{(j)}_{66}\frac{\partial^2}{\partial x_1^2} + c^{(j)}_{11}\frac{\partial^2}{\partial x_2^2} + c^{(j)}_{44}\frac{\partial^2}{\partial x_3^2}, \\
&C^{(j)}_{33}(\partial x) = c^{(j)}_{44}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) + c^{(j)}_{33}\frac{\partial^2}{\partial x_3^2},
\end{aligned}$$

$c^{(k)}_{pq}$ are the constants characterizing physical properties of the mixture and satisfying certain inequalities obtained due to positive definiteness of the potential energy. $U = U^T(x) = (u', u'')$ is a six-dimensional displacement vector-function, $u'(x) = (u'_1, u'_2, u'_3)$ and $u''(x) = (u''_1, u''_2, u''_3)$ are partial displacement vectors. Throughout this paper "T" denotes transposition.

Definition. A vector-function $U(x)$ defined in the domain D is called regular if it has integrable continuous second derivatives in D and $U(x)$ itself and its first derivatives are continuously extendable at every point of the boundary of D , i.e. $U(x) \in C^2(D) \cap C^1(D)$ and satisfies the following conditions at infinity

$$U(x) = O(|x|^{-1}), \quad \frac{\partial U}{\partial x_k} = O(|x|^{-2}), \quad |x|^2 = x_1^2 + x_2^2 + x_3^2, \quad k = 1,2,3.$$

For the equation (1) we pose the following BVP. Find a regular function $U(x)$ satisfying the equation (1) in D if on the boundary S the displacement vector U is given in the form

$$U^+ = f(z), \quad z \in S. \tag{2}$$

where $(.)^+$ denotes the limiting value from D and f is a given vector.

$$\begin{aligned} &|f_k| < AR, \quad R = \sqrt{z_1^2 + z_2^2} \leq 1, \quad |f_k| < AR^{-\alpha}, \\ &\alpha > 0, \quad R > 1, \quad k = 1, \dots, 6, \quad A = const > 0. \end{aligned} \tag{3}$$

The Uniqueness Theorem. Let us prove that the first homogeneous BVP has only a trivial solution. Note that if U is a regular solution of the equation (1) and satisfies the following conditions at infinity

$$U(x) = O(|x|^{-\alpha}), \quad P(\partial x, n)U = O(|x|^{-1-\alpha}), \quad \alpha > 0,$$

then we have the formula

$$U(x) = \\ = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[(P(\partial y, n)\Gamma)^* u^+ - \Gamma(y-z)(P(\partial y, n)u)^+ \right] dy_1\, dy_2, \quad x \in D, \tag{4}$$

where $P(\partial y, n)U$ is the generalized stress vector

$$\begin{aligned} &(P(\partial y, n)U)_k = c_{44}^{(1)} \frac{\partial u_k'}{\partial x_3} + c_{44}^{(3)} \frac{\partial u_k''}{\partial x_3} + \delta^{(1)} \frac{\partial u_3'}{\partial x_k} + \delta^{(3)} \frac{\partial u_3''}{\partial x_k}, \quad k = 1, 2, \\ &(P(\partial y, n)U)_3 = \beta^{(1)} \left(\frac{\partial u_1'}{\partial x_1} + \frac{\partial u_2'}{\partial x_2} \right) + \beta^{(3)} \left(\frac{\partial u_1''}{\partial x_1} + \frac{\partial u_2''}{\partial x_2} \right) + \\ &\qquad + c_{33}^{(1)} \frac{\partial u_3'}{\partial x_3} + c_{33}^{(3)} \frac{\partial u_3''}{\partial x_3}, \\ &(P(\partial y, n)U)_k = c_{44}^{(3)} \frac{\partial u_{k-3}'}{\partial x_3} + c_{44}^{(2)} \frac{\partial u_{k-3}''}{\partial x_3} + \\ &\qquad + \delta^{(4)} \frac{\partial u_3'}{\partial x_{k-3}} + \delta^{(2)} \frac{\partial u_3''}{\partial x_{k-3}}, \quad k = 4, 5, \\ &(P(\partial y, n)U)_6 = \beta^{(4)} \left(\frac{\partial u_1'}{\partial x_1} + \frac{\partial u_2'}{\partial x_2} \right) + \beta^{(2)} \left(\frac{\partial u_1''}{\partial x_1} + \frac{\partial u_2''}{\partial x_2} \right) + \\ &\qquad + c_{33}^{(3)} \frac{\partial u_3'}{\partial x_3} + c_{33}^{(2)} \frac{\partial u_3''}{\partial x_3}, \\ &\beta^{(j)} + \delta^{(j)} = \alpha_{13}^{(j)}, \quad j = 1, 2, 3, \quad \beta^{(4)} + \delta^{(4)} = \alpha_{13}^{(3)}, \\ &c_{13}^{(j)} + c_{44}^{(j)} = \alpha_{13}^{(j)}. \end{aligned} \tag{5}$$

$\Gamma(y-x)$ is the symmetric matrix of the fundamental solution of the equation (1)

$$\Gamma(x-y) = \begin{pmatrix} \Gamma^{(1)} & \Gamma^{(3)} \\ \Gamma^{(3)T} & \Gamma^{(2)} \end{pmatrix}, \tag{6}$$

where

$$\Gamma^{(j)}(x-y) = \sum_{k=1}^{6} \|\Gamma_{pq}^{j(k)}\|_{3x3}, \quad j = 1, 2, 3, \quad \Gamma_{pq}^{j(k)} = \Gamma_{qp}^{j(k)},$$

$$\Gamma^{1(k)}_{pq} = \delta_{pq}\frac{A^{(k)}_{11}}{r_k} + A^{(k)}_{12}\frac{\partial^2\Phi_k}{\partial x_p \partial x_q}, \ \ p=1,2; \ \ q=1,2;$$
$$\delta_{pq}=1, \ \ p=q, \quad \delta_{pq}=0, \ \ p\neq q,$$
$$\Gamma^{1(k)}_{p3} = A^{(k)}_{13}\frac{\partial^2\Phi_k}{\partial x_p \partial x_3}, \quad \Gamma^{1(k)}_{33} = \frac{A^{(k)}_{33}}{r_k}, \quad \Gamma^{3(k)}_{pq} = \delta_{pq}\frac{A^{(k)}_{14}}{r_k} + A^{(k)}_{42}\frac{\partial^2\Phi_k}{\partial x_p \partial x_q},$$
$$\Gamma^{3(k)}_{p3} = A^{(k)}_{16}\frac{\partial^2\Phi_k}{\partial x_p \partial x_3}, \ \ p=1,2, \quad \Gamma^{3(k)}_{33} = \frac{A^{(k)}_{36}}{r_k}, \quad \Gamma^{3(k)}_{3p} = A^{(k)}_{34}\frac{\partial^2\Phi_k}{\partial x_p \partial x_3},$$
$$\Gamma^{2(k)}_{pq} = \delta_{pq}\frac{A^{(k)}_{44}}{r_k} + A^{(k)}_{45}\frac{\partial^2\Phi_k}{\partial x_p \partial x_q},$$
$$\Gamma^{2(k)}_{p3} = A^{(k)}_{46}\frac{\partial^2\Phi_k}{\partial x_p \partial x_3}, \ \ p=1,2, \quad \Gamma^{2(k)}_{33} = \frac{A^{(k)}_{66}}{r_k}.$$

The coefficients $A^{(k)}_{pq}$ are defined as follows

$$\begin{aligned}
&A^{(k)}_{11} = (-1)^k(c^{(2)}_{44} - c^{(2)}_{66}a_k)r'_0, \quad A^{(k)}_{14} = -(-1)^k(c^{(3)}_{44} - c^{(3)}_{66}a_k)r'_0,\\
&A^{(k)}_{12} = \frac{A^{(k)}_{11}}{a_k}, \quad A^{(k)}_{24} = \frac{A^{(k)}_{14}}{a_k}, \quad A^{(k)}_{45} = \frac{A^{(k)}_{44}}{a_k},\\
&A^{(k)}_{44} = (-1)^k(c^{(1)}_{44} - c^{(1)}_{66}a_k)r'_0, \ \ k=1,2, \ \ r'_0 = [r_0(a_1-a_2)]^{-1},\\
&A^{(k)}_{12} = \frac{\delta_k}{a_k}[-q_3c^{(2)}_{44} + a_kt_{12} - a_k^2t_{11} + c^{(2)}_{11}q_4a_k^3],\\
&A^{(k)}_{42} = \frac{\delta_k}{a_k}[q_3c^{(3)}_{44} + a_kt_{13} - a_k^2t_{22} - c^{(3)}_{11}q_4a_k^3],\\
&A^{(k)}_{45} = \frac{\delta_k}{a_k}[-q_3c^{(1)}_{44} + a_kt_{23} - a_k^2t_{33} + c^{(1)}_{11}q_4a_k^3],\\
&A^{((k))}_{33} = \delta_k[q_4c^{(2)}_{33} - a_kt_{42} + a_k^2t_{44} - c^{(2)}_{44}q_1a_k^3],\\
&A^{(k)}_{36} = \delta_k[-q_4c^{(3)}_{33} - a_kt_{62} + a_k^2t_{66} + c^{(3)}_{44}q_1a_k^3],\\
&A^{(k)}_{66} = \delta_k[q_4c^{(1)}_{33} - a_kt_{52} + a_k^2t_{55} - c^{(1)}_{44}q_1a_k^3],\\
&A^{(k)}_{13} = \delta_k[v_{13} - v_{11}a_k + v_{12}a_k^2], \quad A^{(k)}_{16} = \delta_k[w_{13} - w_{12}a_k + w_{11}a_k^2],\\
&A^{(k)}_{34} = \delta_k[v_{23} - v_{21}a_k + v_{22}a_k^2], \quad A^{(k)}_{46} = \delta_k[w_{34} - w_{14}a_k + w_{24}a_k^2],\\
&\delta_k = d_k(a_1-a_k)(a_2-a_k)b_0^{-1}, \quad k=3,\dots,6,
\end{aligned} \tag{7}$$

where a_k are the positive roots of the characteristic equations

$$(r_0a^2 - c_0a + q_4)(b_0a^4 - b_1a^3 + b_2a^2 - b_3a + b_4) = 0,$$
$$r_0 = c^{(1)}_{66}c^{(2)}_{66} - c^{(3)2}_{66}, \quad c_0 = c^{(1)}_{66}c^{(2)}_{44} + c^{(1)}_{44}c^{(2)}_{66} - 2c^{(3)}_{66}c^{(3)}_{44}.$$

The coefficients d_k, b_k, v_{ij}, w_{ij}, t_{ij} are given in [3]. The singular matrix $[P(\partial y, n)\Gamma]^* = \sum\limits_{k=1}^{6}(M^{(k)}_{pq})_{6x6}$, which is obtained from $P(\partial x, n)\Gamma(x-y)$ by transposition of the columns and rows and the variables x and y, has the form

$$[P(\partial x)\Gamma(x-y)]^* = \sum_{k=1}^{6} \begin{pmatrix} M^{(1k)} & M^{(3k)} \\ M^{(4k)} & M^{(2k)} \end{pmatrix}, \tag{8}$$

where the elements of the matrix $M^{(jk)} = \|M_{pq}^{(jk)}\|_{3x3}, j = 1,2,3,4,$ are written as

$$\begin{aligned}
&M_{pj}^{(1k)} = \delta_{pj} R_{11}^{(k)} \frac{\partial}{\partial x_3}\frac{1}{r_k} + R_{12}^{(k)} \frac{\partial^3 \Phi_k}{\partial x_p \partial x_j \partial x_3}, \\
&\delta_{pj} = 1, \ \ p = j, \quad \delta_{pj} = 0, \ \ p \neq j, \quad p, j = 1, 2, \\
&M_{p3}^{(1k)} = R_{31}^{(k)} \frac{\partial}{\partial x_p}\frac{1}{r_k}, \quad M_{3p}^{(1k)} = R_{13}^{(k)} \frac{\partial}{\partial x_p}\frac{1}{r_k}, \\
&M_{33}^{(1k)} = R_{33}^{(k)} \frac{\partial}{\partial x_3}\frac{1}{r_k}, \quad M_{pj}^{(3k)} = \delta_{pj} R_{14}^{(k)} \frac{\partial}{\partial x_3}\frac{1}{r_k} + R_{24}^{(k)} \frac{\partial^3 \Phi_k}{\partial x_j \partial x_p \partial x_3}, \\
&M_{p3}^{(3k)} = R_{61}^{(k)} \frac{\partial}{\partial x_p}\frac{1}{r_k}, \quad M_{3p}^{(3k)} = R_{43}^{(k)} \frac{\partial}{\partial x_p}\frac{1}{r_k}, \quad M_{33}^{(3k)} = R_{63}^{(k)} \frac{\partial}{\partial x_3}\frac{1}{r_k}, \\
&M_{pj}^{(4k)} = \delta_{pj} R_{41}^{(k)} \frac{\partial}{\partial x_3}\frac{1}{r_k} + R_{42}^{(k)} \frac{\partial^3 \Phi_k}{\partial x_j \partial x_p \partial x_3}, \quad M_{p3}^{(4k)} = R_{34}^{(k)} \frac{\partial}{\partial x_p}\frac{1}{r_k}, \\
&M_{3p}^{(4k)} = R_{16}^{(k)} \frac{\partial}{\partial x_p}\frac{1}{r_k}, \quad M_{33}^{(4k)} = R_{36}^{(k)} \frac{\partial}{\partial x_3}\frac{1}{r_k}, \\
&M_{pj}^{(2k)} = \delta_{pj} \mu_{44}^{(k)} \frac{\partial}{\partial x_3}\frac{1}{r_k} + R_{44}^{(k)} \frac{\partial^3 \Phi_k}{\partial x_p \partial x_j \partial x_3}, \quad M_{p3}^{(2k)} = R_{64}^{(k)} \frac{\partial}{\partial x_p}\frac{1}{r_k}, \\
&M_{3p}^{(2k)} = R_{46}^{(k)} \frac{\partial}{\partial x_p}\frac{1}{r_k}, \quad M_{33}^{(2k)} = R_{66}^{(k)} \frac{\partial}{\partial x_3}\frac{1}{r_k}, \quad p = 1, 2.
\end{aligned}$$

The coefficients $R_{pq}^{(k)}$ satisfy the following conditions

$$\sum_{k=1}^{2} \frac{R_{11}^{(k)}}{a_k} = \sum_{k=3}^{6} \frac{R_{33}^{(k)}}{a_k} = \sum_{k=3}^{6} \frac{R_{66}^{(k)}}{a_k} = \sum_{k=1}^{2} \frac{\mu_{44}^{(k)}}{a_k} = 1,$$

$$\sum_{k=1}^{2} \frac{R_{14}^{(k)}}{a_k} = \sum_{k=1}^{6} R_{12}^{(k)} = 0,$$

$$\sum_{k=1}^{2} \frac{R_{41}^{(k)}}{a_k} = \sum_{k=3}^{6} \frac{R_{36}^{(k)}}{a_k} = \sum_{k=1}^{6} R_{24}^{(k)} = \sum_{k=3}^{6} \frac{R_{63}^{(k)}}{a_k} = \sum_{k=1}^{2} R_{44}^{(k)} = \sum_{k=1}^{6} R_{42}^{(k)} = 0$$

and, after elementary calculations the coefficients $R_{13}^{(k)},\dots,R_{64}^{(k)}$ take the form

$$\begin{aligned}
R_{13}^{(k)} &= \delta_0^{(1)}A_{33}^{(k)}+\delta_0^{(3)}A_{36}^{(k)}+c_{44}^{(1)}A_{13}^{(k)}+c_{44}^{(3)}A_{43}^{(k)},\\
R_{16}^{(k)} &= \delta_0^{(1)}A_{36}^{(k)}+\delta_0^{(3)}A_{66}^{(k)}+c_{44}^{(1)}A_{16}^{(k)}+c_{44}^{(3)}A_{46}^{(k)},\\
R_{31}^{(k)} &= -a_k\beta_0^{(1)}A_{12}^{(k)}-a_k\beta_0^{(3)}A_{42}^{(k)}+c_{33}^{(1)}A_{13}^{(k)}+c_{33}^{(3)}A_{16}^{(k)},\\
R_{34}^{(k)} &= -a_k\beta_0^{(1)}A_{42}^{(k)}-a_k\beta_0^{(3)}A_{45}^{(k)}+c_{33}^{(1)}A_{43}^{(k)}+c_{33}^{(3)}A_{46}^{(k)},\\
R_{43}^{(k)} &= \delta_0^{(4)}A_{33}^{(k)}+\delta_0^{(2)}A_{36}^{(k)}+c_{44}^{(3)}A_{13}^{(k)}+c_{44}^{(2)}A_{43}^{(k)},\\
R_{46}^{(k)} &= \delta_0^{(4)}A_{36}^{(k)}+\delta_0^{(2)}A_{66}^{(k)}+c_{44}^{(3)}A_{16}^{(k)}+c_{44}^{(2)}A_{46}^{(k)},\\
R_{61}^{(k)} &= -a_k\beta_0^{(4)}A_{12}^{(k)}-a_k\beta_0^{(2)}A_{42}^{(k)}+c_{33}^{(3)}A_{13}^{(k)}+c_{33}^{(2)}A_{16}^{(k)},\\
R_{64}^{(k)} &= -a_k\beta_0^{(4)}A_{42}^{(k)}-a_k\beta_0^{(2)}A_{45}^{(k)}+c_{33}^{(3)}A_{43}^{(k)}+c_{33}^{(2)}A_{46}^{(k)},\quad k=3,\dots,6.
\end{aligned}\tag{9}$$

We can easily prove that every column of the matrix $[P(\partial x,n)\Gamma]^*$ is a solution of the system (1) with respect to the point x if $x\neq y$ and all elements $M_{pq}^{(k)}$ have a singularity of type $|x|^{-2}$.

We choose $\delta_0^{(J)},\beta_0^{(j)}$, $j=1,\dots,4$, so that

$$\begin{aligned}
&\sum_{k=3}^{6}\frac{R_{13}^{(k)}}{\sqrt{a_k}}=0,\quad \sum_{k=3}^{6}\frac{R_{31}^{(k)}}{\sqrt{a_k}}=0,\quad \sum_{k=3}^{6}\frac{R_{16}^{(k)}}{\sqrt{a_k}}=0,\quad \sum_{k=3}^{6}\frac{R_{34}^{(k)}}{\sqrt{a_k}}=0,\\
&\sum_{k=3}^{6}\frac{R_{43}^{(k)}}{\sqrt{a_k}}=0,\quad \sum_{k=3}^{6}\frac{R_{46}^{(k)}}{\sqrt{a_k}}=0,\quad \sum_{k=3}^{6}\frac{R_{61}^{(k)}}{\sqrt{a_k}}=0,\quad \sum_{k=3}^{6}\frac{R_{64}^{(k)}}{\sqrt{a_k}}=0,
\end{aligned}\tag{10}$$

After some simplification, we find from (10) that

$$\begin{aligned}
\Delta &= \sum_{k=3}^{6}A_{12}^{(k)}\sqrt{a_k}\sum_{k=3}^{6}A_{45}^{(k)}\sqrt{a_k}-\left(\sum_{k=3}^{6}A_{42}^{(k)}\sqrt{a_k}\right)^2=\\
&\qquad=\sqrt{a_3a_4a_5a_6}\left[\sum_{k=3}^{6}\frac{A_{33}^{(k)}}{\sqrt{a_k}}\sum_{k=3}^{6}\frac{A_{66}^{(k)}}{\sqrt{a_k}}-\left(\sum_{k=3}^{6}\frac{A_{36}^{(k)}}{\sqrt{a_k}}\right)^2\right]=\\
&=\frac{B_0}{b_0^2}\big[[(\delta_{11}\delta_{22}+b_0m_1m_3)q_4+q_1b_4+\delta_{22}b_0m_2](\sqrt{a_3a_4a_5a_6})^{-1}+\\
&\qquad\qquad+q_1(\delta_{11}\delta_{22}+b_0m_1m_3-k_1)+b_0\delta_{11}m_2\big],
\end{aligned}$$

where

$$\begin{aligned}
&q_1=c_{11}^{(1)}c_{11}^{(2)}-c_{11}^{(3)2},\quad q_4=c_{44}^{(1)}c_{44}^{(2)}-c_{44}^{(3)2},\quad b_0=q_1q_4,\\
&m_1=\sum_{k=3}^{6}\sqrt{a_k},\quad m_2=\sum_{p\neq q}\sqrt{a_pa_q},\\
&m_3=\sum_{p\neq q\neq j}\sqrt{a_pa_qa_j},\quad p,q,j=3,\dots,6,
\end{aligned}$$

$$\delta_{11}=c_{11}^{(1)}c_{44}^{(2)}+c_{44}^{(1)}c_{11}^{(2)}-2c_{11}^{(3)}c_{44}^{(3)}>0,$$
$$\delta_{22}=c_{33}^{(1)}c_{44}^{(2)}+c_{44}^{(1)}c_{33}^{(2)}-2c_{33}^{(3)}c_{44}^{(3)}>0,$$
$$k_1+k_2=2(\alpha_{13}^{(1)}\alpha_{13}^{(2)}-\alpha_{13}^{(3)2})-\alpha_{13}^{(1)}v_{11}-\alpha_{13}^{(2)}w_{14}-\alpha_{13}^{(3)}(w_{12}+v_{21}),$$
$$k_1=\frac{1}{c_{44}^{(2)2}}\left[c_{44}^{(2)2}c_{13}^{(3)}-2c_{44}^{(2)}c_{44}^{(3)}c_{13}^{(3)}+c_{44}^{(3)2}c_{13}^{(2)}+c_{44}^{(2)}\right]^2+$$
$$+\frac{2q_4}{c_{44}^{(2)2}}\left[c_{44}^{(2)}c_{13}^{(3)}-c_{44}^{(3)}c_{13}^{(2)}\right]^2+\frac{q_4^2}{c_{44}^{(2)2}}\alpha_{13}^{(2)2},$$
$$B_0^{-1}=\prod_{p\neq q}(\sqrt{a_p}+\sqrt{a_q}),\quad p,q=3,\dots,6.$$

Taking into account the inequalities obtained from the positive definiteness the energy $E(u,u)$, we conclude that $\Delta\neq 0$. When $\delta_0^{(j)},\beta_0^{(j)}$ are solutions of the system (10), we denote the vector $P(\partial y,n)U$, by $N(\partial y,n)U$. Then from (4), when $U^+=0$, we have

$$U(x)=-\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Gamma(y-x)N(\partial y,n)U\,dy_1\,dy_2.$$

Hence for the vector NU as $x(x_1,x_2,x_3)\longrightarrow z(z_1,z_2,0)$ we find

$$[N(\partial z,n)U]^+ +\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}N\Gamma(y-z)(NU)^+dy_1\,dy_2=0.$$

Note that $N\Gamma(z-y)=0$, $z\in S$. ATherefore $(NU)^+=0$, and from (4) we have $U=0, x\in D$. Therefore the homogeneous equation has only the trivial solution. Thus we formulate the following

Theorem. *The first BVP has at most one regular solution.*

The first BVP. A solution of the first BVP will be sought in the domain D in terms of the double layer potential

$$U(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[N(\partial y,n)\Gamma(y-x]^*g(y)dy_1\,dy_2, \tag{11}$$

where g is an unknown real vector. Taking into account the properties of the double layer potential and the boundary condition for determining g, we obtain the following Fredholm integral equation of second kind:

$$g(z)+\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[N(\partial y,n)\Gamma(y-z)]^*g(y)dy_1\,dy_2=f(z),$$

Taking into account the fact that $[N\Gamma]^* = 0$, $x_3 = 0$, from the latter equation we have $g(z) = f(z)$ and (11) takes the form

$$U(x) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} [N(\partial y, n)\Gamma(y - x)]^* f(y) dy_1\, dy_2. \tag{12}$$

Thus we have obtained the Poisson formula for the solution of the first BVP for the half-space. Note that (12) is valid if and only if $f \in C^{1,\alpha}(S)$ and satisfies the condition $f = O(\frac{A}{|x|^{1+\beta}})$ at infinity, where A is a constant vector and $\beta > 0$.

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