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ON THE PROBLEM WITH A SLOPING DERIVATIVE FOR A MIXED TYPE EQUATION IN THE CASE OF A TWO-DIMENSIONAL DEGENERATION DOMAIN

Abstract. The paper considers a mixed type equation when the parabolic degeneration is two-dimensional. For this equation we study the problem with a sloping derivative and show that this problem is Noetherian.

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[illegible]

We consider the equation

$$K(y)u_{xx} + u_{yy} + p(x,y)u_y + q(x,y)u = 0, \tag{1}$$

where

$$K(y) = \begin{cases} 1 & \text{for } y > 0, \\ 0 & \text{for } -\delta < y < 0, \\ -1 & \text{for } y < -\delta. \end{cases}$$

For $\delta > 0$ equation (1) is a mixed type model equation with two independent variables, whose domain of parabolicity, like the domains of ellipticity and hyperbolicity, is two-dimensional.

Let Ω be a finite domain bounded by a simple arc $\sigma \in C^2$ with ends $C_1(0,0)$ and $C_2(1,0)$ that lies in a half-plane $y > 0$, by the segments $x = 0$, $x = 1$ and by the characteristics $C_{1\delta}C : y = -x-\delta$ and $C_{2\delta}C : y = x-1-\delta$, $\delta = \mathit{const} > 0$, of equation (1), where p and q are the given functions. These characteristics outgo from the points $C_{1\delta}(0,-\delta)$ and $C_{2\delta}(1,-\delta)$.

Let further $\Omega_1 = \Omega \cap \{(x,y) : y > 0\}$, $\Omega_2 = \Omega \cap \{(x,y) : -\delta < y < 0\}$, $\Omega_3 = \Omega \cap \{(x,y) : y < -\delta\}$, $I_\delta = \{(x,-\delta) : \delta > 0,\ 0 < x < 1\}$.

Below it is assumed that the coefficients $p(x,y)$ and $q(x,y)$ of equation (1) are constant in the domain Ω_2.

Let us consider the problem formulated as follows: find a function $u(x,y)$ with the following properties: 1) $u(x,y)$ is a regular solution of equation (1) in the domains Ω_1, Ω_2, Ω_3; 2) $u(x,y)$ is continuous in the closed domain Ω and has continuous first derivatives in the same domain everywhere except perhaps for the points $C_1(0,0)$ and $C_2(1,0)$ in whose neighborhood u_x and u_y may reduce to infinity of order less than unity; 3) $u(x,y)$ satisfies the boundary conditions

$$(p_1u_x + q_1u_y + \lambda_1 u)|_\sigma = \varphi, \tag{2}$$

$$(p_2u_x + q_2u_y + \lambda_2 u)|_{C_{1\delta}C} = \psi, \tag{3}$$

where p_i, q_i, λ_i $(i = 1,2)$, φ, ψ are the given real functions.

Below it is assumed that $\partial\Omega_1 \in C^{2,h}$, $\varphi, p_i, q_i, \lambda_i \in C^{1,h}$ $(i = 1,2)$, $\psi \in C^{2,h}$, $0 < h < 1$.

We introduce the following notation: $u(x,-\delta) = \tau_\delta(x)$, $u_y(x,-\delta) = -\nu_\delta(x)$, $u(x,0) = \tau(x)$, $u_y(x,0) = \nu(x)$, $0 \le x \le 1$, $2\alpha = -p$, $2\beta = \sqrt{|p^2 - 4q|}$. Using this notation, the solution $u(x,y)$ of problem (1), (2), (3) is representable in the domain Ω_2 for $p^2 - 4q = 0$, $p^2 - 4q > 0$ and $p^2 - 4q < 0$, respectively, in the form

$$u(x,y) = [(1-\alpha y)\tau(x) + y\nu(x)]\exp(\alpha y),$$

$$u(x,y) = [(\beta \operatorname{ch}\beta y - \alpha \operatorname{sh}\beta y)\tau(x) + \nu(x)\operatorname{sh}\beta y]\,\beta^{-1}\exp(\alpha y),$$

$$u(x,y) = [(\beta \cos\beta y - \alpha \sin\beta y)\tau(x) + \nu(x)\sin\beta y]\,\beta^{-1}\exp(\alpha y).$$

Hence we easily conclude that the functions $\tau_\delta(x)$ and $\nu_0(x)$ are related to the functions $\tau(x)$ and $\nu(x)$ as follows:

a) for $p^2-4q=0$ by

$$\tau_\delta(x)=[(1+\alpha\delta)\tau(x)-\delta\nu(x)]\exp(-\alpha\delta),$$
$$\nu_\delta(x)=\big[\alpha^2\delta\tau(x)+(1-\alpha\delta)\nu(x)\big]\exp(-\alpha\delta);$$

b) for $p^2-4q>0$ by

$$\begin{aligned}\tau_\delta(x)&=[(\beta\,\mathrm{ch}\,\beta\delta+\alpha\,\mathrm{sh}\,\beta\delta)\tau(x)-\nu(x)\,\mathrm{sh}\,\beta\delta]\,\beta^{-1}\exp(-\alpha\delta),\\ \nu_\delta(x)&=\big[(\alpha^2-\beta^2)\tau(x)\,\mathrm{sh}\,\beta\delta+(\beta\,\mathrm{ch}\,\beta\delta-\alpha\,\mathrm{sh}\,\beta\delta)\nu(x)\big]\\ &\quad\times\beta^{-1}\exp(-\alpha\delta);\end{aligned}\tag{4}$$

c) for $p^2-4q<0$ by

$$\begin{aligned}\tau_\delta(x)&=[(\beta\cos\beta\delta+\alpha\sin\beta\delta)\tau(x)-\nu(x)\sin\beta\delta]\,\beta^{-1}\exp(-\alpha\delta),\\ \nu_\delta(x)&=\big[(\alpha^2+\beta^2)\tau(x)\sin\beta\delta+(\beta\cos\beta\delta-\alpha\sin\beta\delta)\nu(x)\big]\\ &\quad\times\beta^{-1}\exp(-\alpha\delta).\end{aligned}$$

If we assume that conditions (3) are fulfilled on the characteristic $C_{1\delta}C$ as in [1], then we obtain the following relation between $\tau_\delta(x)$ and $\nu_\delta(x)$ on I_δ:

$$\frac{1}{2}\,(p_2-q_2)\tau_\delta(x)-\frac{1}{2}\,(p_2-q_2)\nu_\delta(x)+[T(\tau_\delta,\nu_\delta)](x)=\widetilde{\psi}(x),\tag{5}$$

where $T(\tau_\delta,\nu_\delta)$ is a completely defined linear integral operator, $\widetilde{\psi}=\frac{\psi}{R(x,-x,x,0)}$, and $R(x,y,\xi,\eta)$ is a Riemann function [2].

As will be seen below, assuming that $p_1(C_1)=0$, $q_1(C_1)\neq 0$ at the point C_1 we can conclude that the derivatives u_x and u_y of the solution of problem (1), (2), (3) are continuous. In addition to this, we assume that $p_2(t)-q_2(t)\neq 0$, $t\in C_{1\delta}C$.

If $\tau_\delta(x)$ and $\nu_\delta(x)$ are sewn continuously on I_δ from (4), (5), then we obtain the equalities

$$\begin{aligned}&\text{a)}\ \ \delta\nu'(x)+(1-\alpha\delta)\nu(x)=\frac{2(\widetilde{T}(\tau,\nu)-\widetilde{\psi})}{p_2-q_2}\exp(\alpha\delta)\\ &\qquad+(1+\alpha\delta)\tau'(x)-\alpha^2\delta\tau(x)\ \text{ for }\ p^2-4q=0;\\ &\text{b)}\ \ \mathrm{sh}\,\beta\delta\nu'(x)+(\beta\,\mathrm{ch}\,\beta\delta-\alpha\,\mathrm{sh}\,\beta\delta)\nu(x)=\frac{2\beta(\widetilde{T}(\tau,\nu)-\widetilde{\psi})}{p_2-q_2}\exp(\alpha\delta)\\ &\qquad+(\beta\,\mathrm{ch}\,\beta\delta+\alpha\,\mathrm{sh}\,\beta\delta)\tau'(x)-(\alpha^2-\beta^2)\,\mathrm{sh}\,\beta\delta\tau(x)\\ &\qquad\qquad\text{for }\ p^2-4q>0;\\ &\text{c)}\ \ \sin\beta\delta\nu'(x)+(\beta\cos\beta\delta-\alpha\sin\beta\delta)\nu(x)=\frac{2\beta(\widetilde{T}(\tau,\nu)-\widetilde{\psi})}{p_2-q_2}\exp(\alpha\delta)\\ &\qquad+(\beta\cos\beta\delta+\alpha\sin\beta\delta)\tau'(x)-(\alpha^2+\beta^2)\sin\beta\delta\tau(x)\\ &\qquad\qquad\text{for }\ p^2-4q>0.\end{aligned}\tag{6}$$

Solving (6) as a linear differential equation with respect to $\nu(x)$, the relations between $\tau(x)$ and $\nu(x)$ after some transformations can be written in the form

$$\begin{aligned}
\text{a)}\ \ &\nu(t)\exp\left[\frac{1-\alpha\delta}{\delta}\,t\right]=\nu(0)\\
&+\int_0^t\left[\frac{2(\widetilde{T}(\tau,\nu)-\widetilde{\psi})}{\delta(p_2-q_2)}\,\exp(\alpha\delta)-\alpha^2\tau(x)\right]\,\exp\left[\frac{1-\alpha\delta}{\delta}\,x\right]dx\\
&+\frac{1+\alpha\delta}{\delta}\left[\tau(t)\exp\left(\frac{1-\alpha\delta}{\delta}\,t\right)-\tau(0)\right.\\
&\left.-\frac{1-\alpha\delta}{\delta}\int_0^t\tau(x)\exp\left[\frac{(1-\alpha\delta)x}{\delta}\right]dx\right]\quad\text{for}\quad p^2-4q=0;
\end{aligned}$$

$$\begin{aligned}
\text{b)}\ \ &\nu(t)\exp\left[\frac{(\beta\operatorname{ch}\beta\delta-\alpha\operatorname{sh}\beta\delta)}{\operatorname{sh}\beta\delta}\,t\right]=\nu(0)+\int_0^t\left[\frac{2(\widetilde{T}(\tau,\nu)-\widetilde{\psi})\beta}{\operatorname{sh}\beta\delta(p_2-q_2)}\,\exp(\alpha\delta)\right.\\
&\left.-(\alpha^2-\beta^2)\tau(x)\right]\,\exp\left[\frac{\beta\operatorname{ch}\beta\delta-\alpha\operatorname{sh}\beta\delta}{\operatorname{sh}\beta\delta}\,x\right]dx\\
&+\frac{\beta\operatorname{ch}\beta\delta+\alpha\operatorname{sh}\beta\delta}{\operatorname{sh}\beta\delta}\left[\tau(t)\exp\left(\frac{\beta\operatorname{ch}\beta\delta-\alpha\operatorname{sh}\beta\delta}{\operatorname{sh}\beta\delta}\,t\right)-\tau(0)\right.\qquad\qquad(7)\\
&\left.-\frac{\beta\operatorname{ch}\beta\delta-\alpha\operatorname{sh}\beta\delta}{\operatorname{sh}\beta\delta}\int_0^t\tau(x)\exp\left[\frac{\beta\operatorname{ch}\beta\delta-\alpha\operatorname{sh}\beta\delta}{\operatorname{sh}\beta\delta}\,x\right]dx\right]\\
&\qquad\qquad\text{for}\ \ p^2-4q>0;
\end{aligned}$$

$$\begin{aligned}
\text{c)}\ \ &\nu(t)\exp\left[\frac{(\beta\cos\beta\delta-\alpha\sin\beta\delta)}{\sin\beta\delta}\,t\right]=\nu(0)+\int_0^t\left[\frac{2(\widetilde{T}(\tau,\nu)-\widetilde{\psi})\beta}{\sin\beta\delta(p_2-q_2)}\,\exp(\alpha\delta)\right.\\
&\left.-(\alpha^2+\beta^2)\tau(x)\right]\,\exp\left[\frac{\beta\cos\beta\delta-\alpha\sin\beta\delta}{\sin\beta\delta}\,x\right]dx\\
&+\frac{\beta\cos\beta\delta+\alpha\sin\beta\delta}{\sin\beta\delta}\left[\tau(t)\exp\left(\frac{\beta\cos\beta\delta-\alpha\sin\beta\delta}{\sin\beta\delta}\,t\right)-\tau(0)\right.\\
&\left.-\frac{\beta\cos\beta\delta-\alpha\sin\beta\delta}{\sin\beta\delta}\int_0^t\tau(x)\exp\left[\frac{\beta\cos\beta\delta-\alpha\sin\beta\delta}{\sin\beta\delta}\,x\right]dx\right]\\
&\qquad\qquad\text{for}\ \ p^2-4q<0\ \ \text{and}\ \ \sin\beta\delta\neq 0.
\end{aligned}$$

When $p^2-4q<0$ and $\sin\beta\delta=0$, (6) immediately implies the relation

$$\nu(t)=\frac{2(\widetilde{T}(\tau,\nu)-\widetilde{\psi})}{(p_2-q_2)\cos\beta\delta}\,\exp(\alpha\delta)+\tau'(t).$$

Let us use a general representation of regular solutions of equation (1) in Ω_1 by analytic functions $\omega(z)$ [3]

$$u(x,y)=\mathrm{Re}\left\{\alpha(z,\overline{z})\omega(z)+\int_{p_0}^{z}\beta(z,\overline{z},t)\omega(t)\,dt\right\}, \tag{8}$$

where $\omega(z)$ is an arbitrary analytic function in Ω_1 that satisfies the condition $\mathrm{Im}\,\omega(p_0)=0$, $p_0\in\Omega_1$; $\alpha(z,\overline{z})$ and $\beta(z,\overline{z},t)$ are entire functions of their arguments. I. Vekua proved that if $\omega(z)\in C^{1,h}(\Omega_1)$ is an analytic function in a 1-connected domain Ω_1 satisfying the condition $\mathrm{Im}\,\omega(p_0)=0$, then there exists a unique real function $\mu(t)\in C^{0,h}$ such that the formula

$$\omega(z)=\int_{\partial\Omega_1}\mu(t)\log e\left(1-\frac{z}{t}\right)\,dS_t \tag{9}$$

holds, where dS_t are elements of an arc of the boundary $\partial\Omega_1$, while under $\log e\left(1-\frac{z}{t}\right)$, $z\in\Omega_1$, $t\in\partial\Omega_1$, we understand a branch if this function that is equal to zero for $z=0$. Assuming that (8) is a boundary condition [4], we can rewrite (2) equivalently as

$$\alpha_1(t)\mu(t)+\beta_1(t)\int_{\partial\Omega_1}\frac{\mu(t_1)\,dt_1}{t_1-t} + \int_{\partial\Omega_1}K(t,t_1)\mu(t_1)\,dt_1=\varphi(t),\quad t\in\partial\Omega_1\setminus C_1C_2, \tag{10}$$

where

$$\alpha_1(t)=\mathrm{Re}[-\pi t'\alpha(t,\overline{t})(p_1(t)+iq_1(t)),$$
$$\beta_1(t)=\mathrm{Im}[-i\overline{t}\,'(p_1(t)+iq_1(t))\alpha(t,\overline{t})],$$
$$\alpha(z,\overline{z})=\exp\left[\int_0^{\overline{z}}p(z,\overline{t})\,dt\right].$$

From (8) we find that [4]

$$\tau'(t)=\widetilde{\alpha}_1(t)\mu(t)+\widetilde{\beta}_1(t)\int_{\partial\Omega_1}\frac{\mu(t_1)\,dt_1}{t_1-t}+K_1(\mu), \tag{11}$$

$$\nu(t)=\widetilde{\alpha}_2(t)\mu(t)+\widetilde{\beta}_2(t)\int_{\partial\Omega_1}\frac{\mu(t_1)\,dt_1}{t_1-t}+K_2(\mu), \tag{12}$$

$$\widetilde{\alpha}_1(t)=\mathrm{Re}(-\pi i\alpha(t,\overline{t})t'),\quad \widetilde{\beta}_1(t)=\mathrm{Im}(-i\alpha(t,\overline{t})\overline{t}'),$$
$$\widetilde{\alpha}_2(t)=\mathrm{Re}(\pi\alpha(t,\overline{t})t'),\qquad \widetilde{\beta}_2(t)=\mathrm{Im}(\alpha(t,\overline{t})t').$$

Here $K_1(\mu)$, $K_2(\mu)$ are completely defined integral operators.

Using formulas (8), (11), (12), from the boundary condition (3) we obtain a singular integral equation with Cauchy kernel which, together with equation (10) can be rewritten in the form of a singular equation on the whole boundary $\partial\Omega_1$

$$\alpha_i^*(t)\mu(t)+\beta_i^*(t)\int_{\partial\Omega_1}\frac{\mu(t_1)\,dt_1}{t_1-t}+K_i^*=f_i^*(t)\quad(i=1,2,3,4), \tag{13}$$

where K_i^* $(i = 1,2,3,4)$ are completely defined compact linear integral operators, f_i^* $(i = 1,2,3,4)$ are the known functions, for $p^2 - 4q = 0$

$$\alpha_1^*(t) = \begin{cases} \alpha_1(t), & t \in \partial\Omega_1 \setminus AB, \\ \widetilde{\alpha}_2(t) \exp\left[\frac{1-\alpha\sigma}{\sigma} f(t)\right], & t \in AB, \end{cases}$$

$$\beta_1^*(t) = \begin{cases} \beta_1(t), & t \in \partial\Omega_1 \setminus AB, \\ \widetilde{\beta}_2(t) \exp\left[\frac{1-\alpha\sigma}{\sigma} f(t)\right], & t \in AB; \end{cases}$$

for $p^2 - 4q > 0$

$$\alpha_2^*(t) = \begin{cases} \alpha_1(t), & t \in \partial\Omega_1 \setminus AB, \\ \widetilde{\alpha}_2(t) \exp\left[(\alpha - \operatorname{cth}\beta\sigma) f(t)\right], & t \in AB, \end{cases}$$

$$\beta_2^*(t) = \begin{cases} \beta_1(t), & t \in \partial\Omega_1 \setminus AB, \\ \widetilde{\beta}_2(t) \exp\left[\frac{1-\alpha\sigma}{\sigma} f(t)\right], & t \in AB; \end{cases}$$

for $p^2 - 4q < 0$, $\sin\beta\sigma \neq 0$

$$\alpha_3^*(t) = \begin{cases} \alpha_1(t), & t \in \partial\Omega_1 \setminus AB, \\ \widetilde{\alpha}_2(t) \left[(\alpha - \operatorname{ctg}\beta\sigma) f(t)\right], & t \in AB, \end{cases}$$

$$\beta_3^*(t) = \begin{cases} \beta_1(t), & t \in \partial\Omega_1 \setminus AB, \\ \widetilde{\beta}_2(t) \left[(\alpha - \operatorname{ctg}\beta\sigma) f(t)\right], & t \in AB. \end{cases}$$

for $p^2 - 4q < 0$, $\sin\beta\sigma = 0$

$$\alpha_4^*(t) = \begin{cases} \beta_1(t), & t \in \partial\Omega_1 \setminus AB, \\ (p_2 + q_2)\cos\sigma\widetilde{\beta}_2(t) - 2(p_2 - q_2)\cos\beta\widetilde{\alpha}_1(t), & t \in AB, \end{cases}$$

$$f(x) = \int_0^x \frac{p_2 + q_2}{p_2 - q_2}\, dt.$$

So, in terms of solvability, problem (1), (2), (3) is equivalently reduced to the integral equation (13).

The solution $\mu(t)$ of the obtained singular integral equations is sought in the space $H^*(\partial\Omega_1)$ [5], assuming that the node of the curve $\partial\Omega_1$ is the point $C_2(1,0)$, while the index of (13) is calculated in the same manner as in [1].

Thus the following theorem is valid.

Theorem. *Let the conditions*

1) $H(t) = p_1(t) + iq_1(t) \neq 0$, $t \in \sigma$,
2) $p_1(C_1) = 0$,
3) $p_2^2(t) - q_2^2(t) \neq 0$, $t \in C_1C_2$, $\varphi(0) = \psi(0)$

be fulfilled. Then problem (1), (2), (3) *is Noetherian.*

In this direction a special mention should be made of work [6].

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