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A CONTRACTION PROCEDURE FOR A SEMI-LINEAR WAVE EQUATION ASSOCIATED WITH A FULL NONLINEAR DAMPING-SOURCE TERM AND A LINEAR INTEGRAL EQUATION AT THE BOUNDARY

Abstract. In this paper the unique solvability of a semi-linear wave equation associated with a full nonlinear damping-source term and a linear integral equation at the boundary is proved by a contraction procedure.

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1. Introduction

We study the solution $u(x,t)$ of the following semi-linear equation:

$$u_{tt} - \mu(t)u_{xx} + F(x,t,u,u_t) = 0, \quad 0 < x < 1, \quad 0 < t < T, \tag{1.1}$$

associated with initial-boundary values given by

$$u(0,t) = 0, \tag{1.2}$$

$$-\mu(t)u_x(1,t) = Q(t), \tag{1.3}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \tag{1.4}$$

where F, u_0, u_1, and μ are given real functions satisfying conditions specified later, and $Q(t)$ satisfies the following integral equation:

$$Q(t) = K_1(t)u(1,t) + \lambda_1(t)u_t(1,t) - g(t) - \int_0^t k(t-s)u(1,s)\,ds, \tag{1.5}$$

where g, k, K_1, and λ_1 are given functions. This problem is a mathematical model describing the shock of a rigid body and a viscoelastic bar (see [3], [7]–[13]) considered by several authors.

In [3], with $F(x,t,u,u_t) = Ku + \lambda u_t$ and $\mu(t) \equiv a^2$, A. Nguyen and T. Nguyen studied the equation (1.1) in the domain $[0,l] \times [0,T]$ when the initial data are homogeneous, namely $u(x,0) = u_t(x,0) = 0$, and the boundary conditions are given by

$$\begin{cases} Eu_x(0,t) = -f(t), \\ u(l,t) = 0, \end{cases} \tag{1.6}$$

where E is a constant.

In [7], Nguyen and Alain considered the problem (1.1)–(1.4) with $\lambda_1(t) \equiv 0$, $K_1(t) = h \geq 0$ and $\mu(t) = 1$, wherein the unknown function $u(x,t)$ and the unknown boundary value $Q(t)$ satisfy the integral equation

$$Q(t) = hu(1,t) - g(t) - \int_0^t k(t-s)u(1,s)\,ds. \tag{1.7}$$

We note that the equation (1.7) is deduced from a Cauchy problem for an ordinary differential equation at the boundary $x = 1$.

In [12] Santos studied the asymptotic behavior of the solution of the problem $\{(1.1), (1.2), (1.4)\}$ in the case where $F(x,t,u,u_t) = 0$, associated with the boundary condition of memory type at $x = 1$ as follows

$$u(1,t) + \int_0^t g(t-s)\mu(s)u_x(1,s)\,ds = 0, \quad t > 0. \tag{1.8}$$

Note that the boundary conditions (1.7) and (1.8) are similar since the formal differences between them can be crossed out after solving the Volterra equation with respect to the variable $u(1,t)$ given by (1.8).

In [8], [9], [10], Nguyen, Lê and Truc Nguyen proved the unique existence, stability, regularity in time variable and an asymptotic expansion for the solution of the problem (1.1)–(1.5) when $F(x,t,u,u_t) = Ku + \lambda u_t - f(x,t)$.

For a specific nonlinear case of $F(x,t,u,u_t)$, namely

$$K|u|^{p-2}u + \lambda|u_t|^{q-2}u_t - f(x,t)$$

with $p \geq 2$ and $q \geq 2$, Lê [11] proved the unique solvability of the problem under consideration. Furthermore the author also studied the stability of the weak solution with respect to some given parameters.

In [13], Sengul investigated the solvability of the equation (1.1) in the case where $F(x,t,u,u_t) = g(u) + \alpha u_t - f(x,t)$, associated with homogeneous boundary conditions, where the initial conditions are similar to (1.4).

Although there have been many publications related to the problem under consideration, the contraction procedure has not been much applied for proving the solvability, to our knowledge. Specifically, in [3], [7]–[12] et cetera, the authors only applied Galerkin approximation associated with a priori estimates, weak-convergence and compactness arguments (see [4], [6], [15]). Although this method is effective, it is occasionally quite difficult to understand.

In this paper, we apply a contraction procedure (see [5], [14]) to obtain the unique solvability of the problem (1.1)–(1.5), and the essential proofs are shorter and easier. The obtained result may be considered as a generalization of those in Nguyen and Alain [7], in Lê [11], in Santos [12], and in Sengul [13].

2. Preliminary Results and Notation

First we introduce some preliminary results and notation used in this paper. Let $\Omega := (0,1)$ and $Q_T := \Omega \times (0,T)$ for $T > 0$. We omit the definitions of usual function spaces: $C^m(\overline{\Omega})$, $L^p = L^p(\Omega)$, $W^{m,p}(\Omega)$. We denote $W^{m,p} = W^{m,p}(\Omega)$, $L^p = W^{0,p}(\Omega)$, and $H^m = W^{m,2}(\Omega)$ for $1 \leq p \leq \infty$ and $m = 0, 1, \ldots$.

The norm in L^2 is denoted by $\|\cdot\|$. We also denote by $\langle \cdot,\cdot \rangle$ the scalar product in L^2 or the dual scalar product of a continuous linear functional with an element of a function space. We denote by $\|\cdot\|_X$ the norm of a Banach space X and by X' the dual space of X. We denote by $L^p(0,T;X)$, $1 \leq p \leq \infty$, the Banach space of the real measurable functions $u : (0,T) \to X$ such that

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(\cdot,t)\|_X^p dt \right)^{1/p} < \infty \text{ for } 1 \leq p < \infty$$

and

$$\|u\|_{L^\infty(0,T;X)} = \operatorname*{ess\,sup}_{0 \leq t \leq T} \|u(\cdot,t)\|_X \text{ for } p = \infty.$$

In addition, we denote by $C\left([0,T];X\right)$ the space of all continuous functions

$$u:[0,T]\to X$$

with

$$\|u\|_{C([0,T];X)} := \max_{0\le t\le T}\|u(\cdot,t)\|_X < \infty,$$

and by $C^1\left([0,T];X\right)$ the space of all differentiable functions $u:[0,T]\to X$ with

$$\|u\|_{C^1([0,T];X)} := \max_{0\le t\le T}\left(\|u(\cdot,t)\|_X + \|u'(\cdot,t)\|_X\right) < \infty.$$

Let $u(t)$, $u'(t)=u_t(t)$, $u''(t)=u_{tt}(t)$, $u_x(t)$, and $u_{xx}(t)$ denote $u(x,t)$, $\frac{\partial u}{\partial t}(x,t)$, $\frac{\partial^2 u}{\partial t^2}(x,t)$, $\frac{\partial u}{\partial x}(x,t)$, and $\frac{\partial^2 u}{\partial x^2}(x,t)$, respectively.

We put

$$V=\{v\in H^1 : v(0)=0\}, \tag{2.1}$$

$$a(u,v)=\left\langle \frac{\partial u}{\partial x},\frac{\partial v}{\partial x}\right\rangle = \int_0^1 \frac{\partial u}{\partial x}\,\frac{\partial v}{\partial x}\,dx. \tag{2.2}$$

Here V is a closed subspace of H^1 and a is a scalar product on V. Additionally, $\|v\|_{H^1}$ and $\|v\|_V=\sqrt{a(v,v)}$ are two equivalent norms. Then we have the following lemma.

Lemma 1. *The imbedding $V\hookrightarrow C^0([0,1])$ is compact and*

$$\|v\|_{C^0([0,1])}\le \|v\|_V \tag{2.3}$$

for all $v\in V$.

We omit the detailed proof because of its obviousness. Moreover, there are the following results whose proofs are also omitted.

Lemma 2. *Suppose $u\in L^2\left(0,T;V\right)$ with $u'\in L^2\left(0,T;H^{-1}(\Omega)\right)$. Then*

$$u\in C\left([0,T];L^2(\Omega)\right)$$

(*after possibly being redefined on a set of measure zero*).

Lemma 3. *Let m be a nonnegative integer. Suppose*

$$u\in L^2\left(0,T;H^{m+2}(\Omega)\right)$$

with $u'\in L^2\left(0,T;H^m(\Omega)\right)$. Then

$$u\in C\left([0,T];H^{m+1}(\Omega)\right)$$

(*after possibly being redefined on a set of measure zero*).

3. Unique Solvability

First and foremost we formulate the following assumptions:

(A_μ) $\mu \in H^2(0,T), \mu(t) \geq \mu_0 > 0$;

$(A_F^{(1)})$ $F(\cdot,\cdot,v,w)$, $F_t(\cdot,\cdot,v,w) \in L^2(Q_T)$ for arbitrarily $(v,w) \in \mathbb{R}^2$;

$(A_F^{(2)})$ There exists $K > 0$ such that

$$\left|F(x,t,v,w) - F(x,t,\widetilde{v},\widetilde{w})\right| \leq K\big(|v-\widetilde{v}| + |w-\widetilde{w}|\big)$$

for arbitrary $(x,t) \in Q_T$ and $(v,\widetilde{v},w,\widetilde{w}) \in \mathbb{R}^4$;

(A_{K_1}) $K_1 \in H^1(0,T)$, $K_1(t) \geq 0$;

(A_{λ_1}) $\lambda_1 \in H^1(0,T)$, $\lambda_1(t) \geq \lambda_0 > 0$;

(A_g) $g \in H^1(0,T)$;

(A_k) $k \in H^1(0,T)$;

$(A_{0,1})$ $u_0 \in V \cap H^2$, $u_1 \in H^1$, and u_0, u_1, K_1, λ_1, g and k satisfy the compatibility condition

$$-\mu(0)u_0'(1) = K_1(0)u_0(1) + \lambda_1(0)u_1(1) - g(0).$$

In this paper we say that a function

$$u \in C^1\left([0,T];L^2\right) \cap C\left([0,T];V\right)$$

is a weak solution of the problem (1.1)–(1.5) if

$$\begin{cases} \dfrac{d}{dt}\left\langle u'(t),v\right\rangle + \mu(t)\left\langle u_x(t),v'\right\rangle + Q(t)v(1) + \big\langle F(x,t,u(t),u'(t)),v\big\rangle = 0, \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x), \\ Q(t) = K_1(t)u(1,t) + \lambda_1(t)u'(1,t) - g(t) - \displaystyle\int_0^t k(t-s)u(1,s)\,ds \end{cases}$$

for each $v \in V$ and almost all $0 \leq t \leq T$, and $\frac{d}{dt}\left\langle u'(t),v\right\rangle$ is the derivative in the sense of distributions on $(-\infty,T)$ of the function

$$\begin{cases} \left\langle u'(t),v\right\rangle, & t > 0, \\ 0, \ t < 0 \end{cases}$$

(see also [1], [2]). In addition, we can say that the problem (1.1)–(1.5) is solvable in $C^1\left([0,T];L^2\right) \cap C\left([0,T];V\right)$ in the above weak sense.

We have the following theorem.

Theorem 1. *Let (A_μ), $(A_F^{(1)})$, $(A_F^{(2)})$, (A_{K_1}), (A_{λ_1}), (A_g), (A_k), and $(A_{0,1})$ hold. Then, for $T > 0$, the problem* (1.1)–(1.5) *has a unique weak solution $u(x,t)$ satisfying*

$$u \in C^1\left([0,T];L^2\right) \cap C\left([0,T];V\right). \tag{3.1}$$

Proof. In this proof, to deal with the nonlinear damping-source F easily, we use a contraction procedure (see [5, p. 500, Theorem 2]) which consists of several steps as follows.

Step 1. *The solvability in* $C^1\left([0,T];L^2\right)$.

We will define the operator Ⓢ as follows. Given a function $u \in C^1\left([0,T];L^2\right)$, let $f(x,t) := F\left(x,t,u(t),u_t(t)\right)$ for $(x,t) \in Q_T$. From $(A_F^{(1)})$, we deduce that

$$f, f_t \in L^2(Q_T). \tag{3.2}$$

Then the following lemma is valid whose proof is similar to that in [11].

Lemma 4. *Under the condition* (3.2) *and the assumptions* (A_μ), (A_{K_1}), (A_{λ_1}), (A_g), (A_k) *and* $(A_{0,1})$, *the linear initial-boundary value problem*

$$\begin{cases} w_{tt} - \mu(t)w_{xx} = -f \ \text{ in } Q_T, \\ w(0,t) = 0, \\ -\mu(t)w_x(1,t) = P(t), \\ w(x,0) = u_0(x), w_t(x,0) = u_1(x), \\ P(t) = K_1(t)w(1,t) + \lambda_1(t)w_t(1,t) - g(t) - \int\limits_0^t k(t-s)w(1,s)\,ds \end{cases} \tag{3.3}$$

has a unique weak solution $w(x,t)$ *such that*

$$w \in L^\infty\left(0,T;V\cap H^2\right), \quad w' \in L^\infty\left(0,T;V\right), \quad w'' \in L^\infty\left(Q_T\right). \tag{3.4}$$

In addition, $w(x,t)$ *satisfies the following estimate:*

$$E_w(t) \le M_0 \tag{3.5}$$

for $t \in [0,T]$, *where*

$$E_w(t) = \|w'(t)\|^2 + \mu_0\|w_x(t)\|^2 + K_1(t)w^2(1,t) + 2\int\limits_0^t \lambda_1(s)|w'(1,s)|^2\,ds \tag{3.6}$$

and M_0 *is a non-negative constant independent of* t.

Remark 1. The unique solvability of the problem (3.3) is independent of $(A_F^{(2)})$.

Using the embedding $H^2(0,T) \hookrightarrow C^1\left([0,T]\right)$ and applying Lemma 2 and Lemma 3, we deduce from (3.4) that

$$w \in C^1\left([0,T];L^2\right) \cap C\left([0,T];V\right). \tag{3.7}$$

In addition, w satisfies

$$\begin{cases} \langle w''(t), v\rangle + \mu(t)\left\langle w_x(t), v'\right\rangle + P(t)v(1) + \langle f(t), v\rangle = 0, \\ w(x,0) = u_0(x), \quad w'(x,0) = u_1(x), \\ P(t) = K_1(t)w(1,t) + \lambda_1(t)w'(1,t) - g(t) - \int\limits_0^t k(t-s)w(1,s)\,ds \end{cases} \tag{3.8}$$

for each $v \in V$ and almost all $0 \le t \le T$.

Define Ⓢ $: C^1\left([0,T];L^2\right) \to C^1\left([0,T];L^2\right)$ by setting

$$\text{Ⓢ}\, u = w. \tag{3.9}$$

It is claimed that if $T > 0$ is small enough, then Ⓢ is a strict contraction. To prove this, take $u, \widetilde{u} \in C^1\left([0,T];L^2\right)$ arbitrarily, and define $w :=$ Ⓢ$\,u$ and $\widetilde{w} :=$ Ⓢ$\,\widetilde{u}$ as above. As a result, w verifies (3.8) for $f(x,t) = F\left(x,t,u(x,t),u_t(x,t)\right)$, and $\widetilde{w}$ satisfies a system similar to (3.8) for

$$\begin{cases} \widetilde{f}(x,t) := F\left(x,t,\widetilde{u}(x,t),\widetilde{u}_t(x,t)\right), \\ \widetilde{P}(t) = K_1(t)\widetilde{w}(1,t) + \lambda_1(t)\widetilde{w}'(1,t) - g(t) - \int\limits_0^t k(t-s)\widetilde{w}(1,s)\,ds. \end{cases} \tag{3.10}$$

In addition, we have

$$\langle w''(t) - \widetilde{w}''(t), v\rangle + \mu(t)\langle w_x(t) - \widetilde{w}_x(t), v_x\rangle + \\ + \left(P(t) - \widetilde{P}(t)\right)v(1) + \left\langle f(t) - \widetilde{f}(t), v\right\rangle = 0 \tag{3.11}$$

for each $v \in V$ and almost all $0 \le t \le T$.

Now, in (3.11), replacing v by $w' - \widetilde{w}'$ and then integrating with respect to t, we get

$$E(t) = \int\limits_0^t \mu'(s)\left\|w_x(s) - \widetilde{w}_x(s)\right\|^2 ds + \int\limits_0^t K_1'(s)\left[w(1,s) - \widetilde{w}(1,s)\right]^2 ds + \\ + 2\int\limits_0^t \left[w'(1,s) - \widetilde{w}'(1,s)\right]\left(\int\limits_0^s k(s-\tau)\left[w(1,\tau) - \widetilde{w}(1,\tau)\right] d\tau\right) ds - \\ - 2\int\limits_0^t \left\langle f(s) - \widetilde{f}(s), w'(s) - \widetilde{w}'(s)\right\rangle ds, \tag{3.12}$$

where

$$E(t) = \left\|w'(t) - \widetilde{w}'(t)\right\|^2 + \mu(t)\left\|w_x(t) - \widetilde{w}_x(t)\right\|^2 + K_1(t)\left[w(1,t) - \widetilde{w}(1,t)\right]^2 + \\ + 2\int\limits_0^t \lambda_1(s)\left[w'(1,s) - \widetilde{w}'(1,s)\right]^2 ds. \tag{3.13}$$

From (2.3), (3.2), (3.12), (3.13) and the assumptions (A_μ), (A_{K_1}), (A_λ), (A_k), we deduce the following estimates

$$\int\limits_0^t \mu'(s)\left\|w_x(s) - \widetilde{w}_x(s)\right\|^2 ds \le \frac{1}{\mu_0}\int\limits_0^t \left|\mu'(s)\right| E(s)\,ds, \tag{3.14}$$

$$\int_0^t K_1'(s)\left[w(1,s)-\widetilde{w}(1,s)\right]^2\,ds \le \int_0^t \frac{|K_1'(s)|}{\mu_0}E(s)\,ds, \tag{3.15}$$

$$2\int_0^t \left[w'(1,s)-\widetilde{w}'(1,s)\right]\left(\int_0^s k(s-\tau)\big[w(1,\tau)-\widetilde{w}(1,\tau)\big]\,d\tau\right)ds \le$$
$$\le \varepsilon\frac{E(t)}{2\lambda_0}+\frac{T}{\varepsilon\mu_0}\|k\|^2_{L^2(0,T)}\int_0^t E(s)\,ds, \tag{3.16}$$

$$-2\int_0^t \big\langle f(s)-\widetilde{f}(s), w'(s)-\widetilde{w}'(s)\big\rangle\,ds \le$$
$$\le \int_0^t \|f(s)-\widetilde{f}(s)\|^2\,ds+\int_0^t E(s)\,ds \tag{3.17}$$

for some $\varepsilon > 0$.

With the relevant choice of ε, namely $\varepsilon = \lambda_0$, and using Gronwall's inequality, we conclude from (3.12)–(3.17) that

$$E(t) \le \left(2\int_0^t \|f(s)-\widetilde{f}(s)\|^2\,ds\right)\exp\left[D(t)\right], \tag{3.18}$$

where

$$D(t) = 2\int_0^t \left[1+\frac{1}{\mu_0}\left(\mu'(s)+K_1'(s)+\frac{T}{\lambda_0}\,\|k\|^2_{L^2(0,T)}\right)\right]ds. \tag{3.19}$$

Using the assumptions $(A_\mu), (A_{K_1}), (A_k)$, we deduce from (3.19) that there exists a constant $M > 0$ (independent of t but dependent on T) such that

$$\exp\left[D(t)\right] \le M \tag{3.20}$$

for $t \in [0,T]$. From the assumption $(A_F^{(2)})$, it is clear that (3.18) is equivalent to

$$E(t) \le 2TMK^2\|u-\widetilde{u}\|^2_{C^1([0,T];L^2)} \tag{3.21}$$

for arbitrary u, $\widetilde{u} \in C^1\left([0,T];L^2\right)$, and $t \in [0,T]$.

Combining (3.13) and (3.21), we obtain

$$\|w'(t)-\widetilde{w}'(t)\|^2 \le 2TMK^2\|u-\widetilde{u}\|^2_{C^1([0,T];L^2)} \tag{3.22}$$

for arbitrary u, $\widetilde{u} \in C^1\left([0,T];L^2\right)$ and $t \in [0,T]$. Moreover, it is not difficult to see that

$$\|w(t)-\widetilde{w}(t)\|^2 \le 2T^3MK^2\|u-\widetilde{u}\|^2_{C^1([0,T];L^2)} \tag{3.23}$$

for arbitrary $u,\, \widetilde{u} \in C^1\left([0,T];L^2\right)$ and $t \in [0,T]$. Hence, after maximizing the left hand sides of (3.22) and (3.23) with respect to t, we find

$$\|w-\widetilde{w}\|^2_{C^1([0,T];L^2)} \le 2MT(T^2+1)K^2\|u-\widetilde{u}\|^2_{C^1([0,T];L^2)} \tag{3.24}$$

for arbitrary $u,\, \widetilde{u} \in C^1\left([0,T];L^2\right)$. Thus

$$\|Ⓢ\, u - Ⓢ\, \widetilde{u}\|_{C^1([0,T];L^2)} \le K\sqrt{2MT(T^2+1)}\, \|u-\widetilde{u}\|_{C^1([0,T];L^2)} \tag{3.25}$$

for arbitrary $u,\, \widetilde{u} \in C^1\left([0,T];L^2\right)$. Then Ⓢ is a strict contraction, provided $T>0$ is so small that

$$K\sqrt{2MT(T^2+1)} = \alpha < 1.$$

As a result, with the application of Banach's fixed point theorem, we conclude that the problem (1.1)–(1.5) is solvable in $C^1\left([0,T];L^2\right)$ in the weak sense.

Step 2. *The solvability in* $C\left([0,T];V\right)$.

From (3.5) and (3.6), we obtain that

$$\|w_x(t)\|^2 \le M_0 \tag{3.26}$$

for $t \in [0,T]$. However, Step 1 shows that the operator Ⓢ defined by Ⓢ $u := w$ for all $u \in C^1(0,T;L^2)$ has at least one fixed point. Hence there exists $u \in C^1(0,T;L^2)$ such that $w = Ⓢ\, u \equiv u$. Then we deduce from (3.26) that

$$\|u_x(t)\|^2 \le M_0 \tag{3.27}$$

for $t \in [0,T]$. From (2.1) and (2.2), after maximizing the left side of (3.27) with respect to t, we get

$$u \in C(0,T;V). \tag{3.28}$$

As a result, the problem (1.1)–(1.5) is also solvable in $C(0,T;V)$.

Remark 2. In the case $T>0$ is given, we select $T_1>0$ so small that

$$K\sqrt{2MT_1(T_1^2+1)} < 1.$$

Then we are able to apply Banach's fixed point theorem to find a weak solution u of the problem (1.1)–(1.5) existing on the time interval $[0,T_1]$; namely, we obtain

$$u \in C^1(0,T_1;L^2) \cap C(0,T_1;V).$$

In addition, from the assumptions (A_μ), $(A_F^{(1)})$, (A_{K_1}), (A_{λ_1}), (A_g), (A_k), and $(A_{0,1})$, we deduce that

$$\begin{cases} \|u'(t)\|^2 + \mu(t)\|u_x(t)\|^2 < +\infty, \\ \|u''(t)\|^2 + \mu(t)\|u'_x(t)\|^2 < +\infty, \\ u_{xx} = \dfrac{1}{\mu(t)}\left(u_{tt} + F(\cdot,t,u,u_t)\right) \in L^2 \end{cases}$$

for all $(x,t) \in Q_{T_1}$ (see also [11, Theorem 1]). Hence we obtain the regularity of the weak solution u on Q_{T_1} as follows

$$u \in L^\infty\left(0, T_1; V \cap H^2\right), u' \in L^\infty\left(0, T_1; V\right), u'' \in L^\infty\left(Q_{T_1}\right).$$

Then we can continue, by redefining T_1 if necessary, and assuming $u'_x(T_1)$, $u_{xx}(T_1) \in L^2$ or $(u(T_1), u'(T_1)) \in \left(V \cap H^2\right) \times H^1$. Now, by repeating the argument above, we can extend our solution to the time interval $[T_1, 2T_1]$. Continuing, after finitely many steps we construct a weak solution existing on the full interval $[0,T]$.

Step 3. *The uniqueness of the weak solution.*

To prove the uniqueness, suppose both u and $\widetilde{u}$ are two weak solutions of the problem (1.1)–(1.5). Then we have $w = u, \widetilde{w} = \widetilde{u}$ in (3.12)–(3.17). Hence we compute

$$E(t) \le 2\widetilde{M}K^2\left(\int_0^t \|u(s) - \widetilde{u}(s)\|^2\,ds\right), \tag{3.29}$$

where $\widetilde{M}$ is a positive constant independent of t but dependent on T such that

$$\widetilde{M} \ge 2\exp\left(\int_0^t \left[3 + \frac{1}{\mu_0}\left(\mu'(s) + K_1'(s) + \frac{T}{\lambda_0}\,\|k\|^2_{L^2(0,T)}\right)\right] ds\right) \tag{3.30}$$

for $t \in [0,T]$. From (3.13), (3.29) and (3.30), we find

$$\|u(t) - \widetilde{u}(t)\|_V^2 \le \frac{2\widetilde{M}K^2}{\mu_0}\int_0^t \|u(s) - \widetilde{u}(s)\|_V^2\,ds. \tag{3.31}$$

Because of Gronwall's inequality, we deduce from (3.31) that $u \equiv \widetilde{u}$.

The above three steps show that the theorem is proved completely. □

Remark 3. Note that one can study the unique solvability of the problem under consideration by applying Galerkin approximation associated with a priori estimates, weak-convergence and compactness arguments. Obviously, stronger assumptions on the nonlinear damping-source term will be needed, and many technical arguments must be modified.

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