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ON A METHOD OF CONSTRUCTION OF THE SOLUTION OF THE MULTIPOINT BOUNDARY VALUE PROBLEM FOR A SYSTEM OF GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

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Let $t_1,\dots,t_n \in [a,b]$, $A=(a_{ik})_{i,k=1}^n \in BV_{n\times n}(a,b)$, $a_{ik}(t)\equiv a_{1ik}(t)-a_{2ik}(t)$, where a_{jik} $(j=1,2)$ are functions nondecreasing on the intervals $[a,t_i[$ and $]t_i,b]$, $A_j(t)\equiv (a_{jik}(t))_{i,k=1}^n$ $(j=1,2)$, $f=(f_k)_{k=1}^n \in \bigcap\limits_{j=1}^{2} K_n(a,b;A_j)$, and let $\varphi_i : BV_n(a,b)\to R$ $(i=1,\dots,n)$ be continuous functionals, nonlinear, in general.

We consider a method of construction of the solution of the problem

$$dx(t)=dA(t)\cdot f(t,x(t)), \tag{0}$$
$$x_i(t_i)=\varphi_i(x) \quad (i=1,\dots,n). \tag{1}$$

Take an arbitrary vector-function $(x_{i0})_{i=1}^n \in BV_n(a,b)$ as the initial approximation to the solution of the problem (1), (2). If the $(m-1)$-th approximation has been constructed, then for the m-th approximation we will take a vector-function $(x_{im})_{i=1}^n \in BV_n(a,b)$ whose i-th component is the solution of the Cauchy problem

$$dx_{im}(t)=\sum_{l=1}^n f_l(t,x_{1\,m-1}(t),\dots,x_{i-1\,m-1}(t),x_{im}(t),x_{i+1\,m-1}(t),\dots$$
$$\dots,x_{n\,m-1}(t))da_{il}(t), \tag{2}$$
$$x_{im}(t_i)=\varphi_i(x_{1\,m-1},\dots,x_{n\,m-1}) \quad (i=1,\dots,n). \tag{3}$$

The use will be made of the following notation and definitions: $R=]-\infty,\infty[$, $R_+=[0,\infty[$; $R^{n\times m}$ is the space of all real $n\times m$-matrices $X=(x_{ij})_{i,j=1}^{n,m}$ with the norm $\|X\|=\max\{\sum\limits_{i=1}^n |x_{ij}| : j=1,\dots,m\}$; $R^n=R^{n\times 1}$. $BV_{n\times m}(a,b)$ is the space of all matrix-functions of bounded variation $X:[a,b]\to R^{n\times m}$ with the norm $\|X\|_s=\sup\{\|X(t)\| : t\in[a,b]\}$; $d_1X(t)=X(t)-X(t-0)$, $d_2X(t)=X(t+0)-X(t)$.

If $g:[a,b]\to R$ is nondecreasing, $x:[a,b]\to R$ and $a\le s<t\le b$, then $\int\limits_s^t x(\tau)dg(\tau)=\int\limits_{]s,t[} x(\tau)dg(\tau)+x(t)d_1g(t)+x(s)d_2g(s)$, where $\int\limits_{]s,t[} x(\tau)dg(\tau)$ is the Lebesque–Stieltjes integral over the open interval $]s,t[$ with respect to the measure μ_g corresponding to g. $L_{n\times m}(a,b;A_\sigma)$ is the set of all matrix-functions $(x_{jk}(t))_{j,k=1}^{n,m}$ such that x_{jk} is integrable with respect to $a_{\sigma ij}$ $(i=1,\dots,n;\ \sigma=1,2)$. $K_n(a,b;A_\sigma)$ is the Caratheodory class, i.e. the set of all vector-functions $f=(f_k)_{k=1}^n : [a,b]\times R^n\to R^n$ such that: (a) $f_k(\cdot,x)$ is $\mu_{a_{\sigma ik}}$-measurable for $x\in R^n$ $(i=1,\dots,n)$; (b) $f_k(t,\cdot):[a,b]\to R^n$ is continuous for $t\in[a,b]$, and $\sup\{|f_k(\cdot,x)| : x\in D\}\in L_n(a,b;A_\sigma)$ for every compact $D\subset R^n$ $(\sigma=1,2)$.

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If B_1 and B_2 are normed spaces, then an operator $\varphi : B_1 \to B_2$ is called positively homogeneous if $\varphi(\lambda x) = \lambda\varphi(x)$ for every $\lambda \in R_+$ and $x \in B_1$. An operator $\varphi : BV_n(a,b) \to R^n$ is called nondecreasing if for every $x, y \in BV_n(a,b)$ such that $x(t) \leq y(t)$ for $t \in [a,b]$, the inequality $\varphi(x)(t) \leq \varphi(y)(t)$ is fulfilled for $t \in [a,b]$.

A vector-function $x = (x_i)_{i=1}^n \in BV_n(a,b)$ is said to be a solution of the system (1) (of the system $dx(t) \leq dA(t) \cdot f(t, x(t))$) if $x_i(t) - x_i(s) - \sum\limits_{k=1}^n \int\limits_s^t f_k(\tau, x(\tau)) da_{ik}(\tau) = 0$ (≤ 0) for $a \leq s \leq t \leq b$ ($i = 1, \dots, n$).

We say that the pair $((c_{il})_{i,l=1}^n; (\varphi_{0i})_{i=1}^n)$ consisting of a matrix-function $(c_{il})_{i,l=1}^n \in BV_{n\times n}(a,b)$ and of a positively homogeneous nondecreasing operator $(\varphi_{0i})_{i=1}^n : BV_n(a,b) \to R_+^n$ belongs to the set $U(t_1, \dots, t_n)$ if the functions c_{il} ($i \neq l$; $i, l = 1, \dots, n$) are nondecreasing on $[a,b]$ and continuous at the point t_i, $d_j c_{ii}(t) \geq 0$ for $t \in [a,b]$ ($j = 1, 2$; $i = 1, \dots, n$), and the problem $[dx_i(t) - \operatorname{sign}(t-t_i) \sum\limits_{l=1}^n x_l(t) dc_{il}(t)] \operatorname{sign}(t-t_i) \leq 0$ ($i = 1, \dots, n$), $(-1)^j d_j x_i(t_i) \leq x_i(t_i) d_j c_{ii}(t_i)$ ($j = 1, 2$; $i = 1, \dots, n$), $x_i(t_i) \leq \varphi_{0i}(|x_1|, \dots, |x_n|)$ ($i = 1, \dots, n$) has no nontrivial nonnegative solution.

Theorem. *Let the conditions*

$$(-1)^{\sigma+1}[f_k(t, x_1, \dots, x_n) - f_k(t, y_1, \dots, y_n)] \operatorname{sign}[(t - t_i)(x_i - y_i)] \leq$$
$$\leq \sum_{l=1}^n p_{\sigma ikl}(t)|x_l - y_l| \quad \textit{for } \mu_{a_{\sigma ik}} - \textit{almost every } t \in [a,b] \setminus \{t_i\} \quad (i, k = 1, \dots, n)$$

and

$$\Big\{(-1)^{\sigma+j+1}[f_k(t_i, x_1, \dots, x_n) - f_k(t_i, y_1, \dots, y_n)] \operatorname{sign}(x_i - y_i) -$$
$$- \sum_{l=1}^n \alpha_{\sigma ikjl}|x_l - y_l|\Big\} d_j a_{\sigma ik}(t_i) \leq 0 \quad (j = 1, 2; \;\; i, k = 1, \dots, n)$$

be fulfilled on R^n for $\sigma \in \{1, 2\}$, and let the inequalities

$$|\varphi_i(x_1, \dots, x_n) - \varphi_i(y_1, \dots, y_n)| \leq \varphi_{0i}(|x_1 - y_1|, \dots, |x_n - y_n|) \quad (i = 1, \dots, n)$$

be fulfilled on $BV_n(a,b)$, where $\alpha_{\sigma ikjl} \in R$, $(p_{\sigma ikl})_{k,l=1}^n \in L_{n\times n}(a, b; A_\sigma)$. Let, moreover, there exist a matrix-function $(c_{il})_{i,l=1}^n \in BV_{n\times n}(a,b)$ such that $d_j c_{ii}(t) < 1$ for $(-1)^j(t - t_i) < 0$ ($j = 1, 2$; $i = 1, \dots, n$), $((c_{il})_{i,l=1}^n; (\varphi_{0i})_{i=1}^n) \in U(t_1, \dots, t_n)$,

$$\sum_{\sigma=1}^2 \sum_{k=1}^n \int_s^t p_{\sigma ikl}(\tau) da_{\sigma ik}(\tau) \leq c_{il}(t) - c_{il}(s)$$

for $a \leq s < t < t_i$ and $t_i < s < t \leq b$ ($i, l = 1, \dots, n$), and

$$\sum_{\sigma=1}^2 \sum_{k=1}^n \alpha_{\sigma ikjl} d_j a_{\sigma ik}(t_i) \leq d_j c_{il}(t_i) \quad (j = 1, 2; \; i, l = 1, \dots, n).$$

Then the problem (1), (2) has a unique solution $(x_i)_{i=1}^n$ and for every $(x_{i0})_{i=1}^n \in BV_n(a,b)$, there exists a unique sequence $(x_{im})_{i=1}^n \in BV_n(a,b)$ ($m = 1, 2, \dots$) such that the function x_{im} is a solution of the problem (3), (4) for every natural m and

$$\sum_{i=1}^n |x_i(t) - x_{im}(t)| \leq r_0 \delta^m \quad \textit{for} \quad t \in [a,b] \;\; (m = 1, 2, \dots),$$

where $r_0 > 0$ *and* $\delta \in]0,1[$ *are numbers independent of* m.

Similar results for ordinary differential equations can be found in [1].

References

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