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NONLOCAL BOUNDARY VALUE PROBLEMS FOR FRACTIONAL DIFFERENTIAL EQUATIONS

Dedicated to the memory of Professor Temuri Chanturia

Abstract. We present the existence principle which can be used for a large class of nonlocal fractional boundary value problems of the form $({}^c\!D^\alpha x)(t) = f(t, x(t), x'(t), ({}^c\!D^\mu x)(t))$, $\Lambda(x) = 0$, $\Phi(x) = 0$, where ${}^c\!D$ is the Caputo fractional derivative. Here, $\alpha \in (1,2)$, $\mu \in (0,1)$, f is a L^q-Carathéodory function, $q > \frac{1}{\alpha-1}$, and $\Lambda, \Phi : C^1[0,T] \to \mathbb{R}$ are continuous and bounded ones. The proofs are based on the Leray–Schauder degree theory. Applications of our existence principle are given.

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რეზიუმე. შემოთავაზებულია ამოხსნადობის პრინციპი წილადურ სასაზღვრო ამოცანათა ფართო კლასისათვის, რომელთაც აქვთ შემდეგი სახე $({}^c\!D^\alpha x)(t) = f(t, x(t), x'(t), ({}^c\!D^\mu x)(t))$, $\Lambda(x) = 0$, $\Phi(x) = 0$, სადაც ${}^c\!D$ არის კაპუტოს წილადური წარმოებული, $\alpha \in (1,2)$, $\mu \in (0,1)$, f არის L^q-კარათეოდორის ფუნქცია, $q > \frac{1}{\alpha-1}$, ხოლო $\Lambda, \Phi : C^1[0,T] \to \mathbb{R}$ უწყვეტი, შემოსაზღვრული ფუნქციონალებია. მოყვანილია, აგრეთვე, აღნიშნული პრინციპის გამოყენებები. დამტკიცებები ეფუძნება ლერე–შაუდერის თეორიას ასახვათა ხარისხების შესახებ.

1. Introduction

Let $T > 0$ and $\mathbb{R}_+ = [0, \infty)$. As usual, L^q $(q \geq 1)$ is the set of functions whose qth powers of modulus are integrable on $[0, T]$ equipped with the norm $\|x\|_q = \left(\int_0^T |x(t)|^q \,\mathrm{d}t\right)^{\frac{1}{q}}$. $C[0, T]$ is equipped with the norm $\|x\| = \max\{|x(t)| : t \in [0, T]\}$.
Let $\mathcal{A}$ be a set of functionals $\Lambda : C^1[0, T] \to \mathbb{R}$, which are

(a) continuous,

(b) bounded, that is, $\Lambda(\Omega)$ is bounded for any bounded $\Omega \subset C^1[0, T]$.

We say that $\Lambda, \Phi \in \mathcal{A}$ satisfy *the compatibility condition* if for each $\nu \in [0, 1]$ there exists a solution of the problem

$$x'' = 0, \quad \Lambda(x) - \nu\Lambda(-x) = 0, \quad \Phi(x) - \nu\Phi(-x) = 0.$$

This is true if and only if the system

$$\begin{aligned} \Phi(a + bt) - \nu\Phi(-a - bt) = 0, \\ \Psi(a + bt) - \nu\Psi(-a - bt) = 0 \end{aligned} \tag{1.1}$$

has a solution $(a, b) \in \mathbb{R}^2$ for each $\nu \in [0, 1]$.

We say that the functionals $\Phi, \Psi \in \mathcal{A}$ satisfy *the admissible compatibility condition* if $\Phi and \Psi$ satisfy the compatibility condition and there exists a positive constant $L = L(\Phi, \Psi)$ such that $|a| \leq L$ and $|b| \leq L$ for each $\nu \in [0, 1]$ and each solution $(a, b) \in \mathbb{R}^2$ of system (1.1).

Remark 1.1. If the functionals $\Phi, \Psi : C^1[0, T] \to \mathbb{R}$ are linear and continuous, then $\Phi, \Psi \in \mathcal{A}$ and satisfy the compatibility condition. Indeed, system (1.1) is of the form

$$\begin{aligned} a\Phi(1) + b\Phi(t) = 0, \\ a\Psi(1) + b\Psi(t) = 0 \end{aligned}$$

for each $\nu \in [0, 1]$, and we see that it is always solvable in $\mathbb{R}^2$. The set of all its solutions (a, b) is bounded (that is, Φ, Ψ satisfy the admissible compatibility condition) if and only if $\Phi(1)\Psi(t) - \Phi(t)\Psi(1) \neq 0$.

We investigate the fractional boundary value problem

$$({}^c\!D^\alpha x)(t) = f(t, x(t), x'(t), ({}^c\!D^\mu x)(t)), \tag{1.2}$$

$$\Phi(x) = 0, \quad \Psi(x) = 0, \tag{1.3}$$

where $\alpha \in (1, 2)$, $\mu \in (0, 1)$, f is an L^q-Carathéodory function on $[0, T] \times \mathbb{R}^3$, $q > \frac{1}{\alpha - 1}$, and where $\Phi, \Psi \in \mathcal{A}$ satisfy the admissible compatibility condition.

We say that a function $x \in C^1[0, T]$ is *a solution of problem* (1.2), (1.3) if ${}^c\!D^\alpha x \in L^q[0, T]$, x satisfies the boundary conditions (1.3), and (1.2) holds for a.e. $t \in [0, T]$.

Note that if x is a solution of problem (1.2), (1.3), then ${}^c\!D^\mu x \in C[0, T]$ (see Lemma 2.5).

The Caputo fractional derivative ${}^{c}D^{\gamma}v$ *of order* $\gamma > 0$, $\gamma \notin \mathbb{N}$, of a function $v : [0,T] \to \mathbb{R}$ is defined by the formula [10, 15, 18]

$$({}^{c}D^{\gamma}v)(t) = \frac{1}{\Gamma(n-\gamma)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t (t-s)^{n-\gamma-1} \left(v(s) - \sum_{k=0}^{n-1} \frac{v^{(k)}(0)}{k!} s^k \right) \mathrm{d}s,$$

where $n = [\gamma] + 1$ and $[\gamma]$ denotes the integral part of γ, and Γ is the Euler gamma function.

We recall that a function $f : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ is an *L^q-Carathéodory function on* $[0,T] \times \mathbb{R}^3$ if

(i) for each $(x,y,z) \in \mathbb{R}^3$, the function $f(\cdot,x,y,z) : [0,T] \to \mathbb{R}$ is measurable,

(ii) for a.e. $t \in [0,T]$, the function $f(t,\cdot,\cdot,\cdot) : \mathbb{R}^3 \to \mathbb{R}$ is continuous,

(iii) for each compact set $\mathcal{U} \subset \mathbb{R}^3$, there exists $w_{\mathcal{U}} \in L^q[0,T]$ such that $|f(t,x,y,z)| \le w_{\mathcal{U}}(t)$ for a.e. $t \in [0,T]$ and all $(x,y,z) \in \mathcal{U}$.

Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. We can find numerous applications in porous media, electromagnetic, fluid mechanics, viskoelasticity, edge detection, and so on. (For examples and details, see [7, 8, 10, 13, 14, 15, 18, 23, 27] and references therein). There has been a significant development in the study of fractional differential equations in recent years. The authors discuss regular (see, e.g., [4, 6, 11, 12, 17]) and singular (see, e.g., [2, 5, 19, 26, 28]) fractional boundary value problems. These problems are usually investigated with the two-point boundary conditions, multipoint boundary conditions and also with nonlocal boundary conditions (see, e.g., [3, 6]). Paper [3] deals with the integral boundary conditions

$$ax(0) + bx'(0) = \int_0^1 q_1(x(s))\,\mathrm{d}s, \quad ax(1) + bx'(1) = \int_0^1 q_2(x(s))\,\mathrm{d}s,$$

while that of [6] with the conditions

$$x(0) = \Theta(x), \quad x(T) = x_T,$$

where $\Theta : C[0,T] \to \mathbb{R}$ is a continuous functional and $x_T \in \mathbb{R}$. The existence results are proved by: the Banach, Schauder, Krasnosel'skii and Leggett-Williams fixed point theorems, fixed point theorems on cones, a mixed monotone method, the Leray–Schauder nonlinear alternative, the lower and upper solution method and by fixed point index theory.

The aim of the present paper is to give the existence principle for solving the problem (1.2), (1.3) and to show its applications. We note that unlike the paper dealing with fractional differential equations for $1 < \alpha < 2$ (with the exception of [2, 16]), the nonlinearity f in (1.2) depends on the derivative of x. Due to this fact, we have to assume that f is an L^q-Carathéodory function with $q > \frac{1}{\alpha-1}$. The existence principle is proved by the Leray–Schauder degree theory (see, e.g., [9]). Note that our existence principle is closely related to that given in [24] for n-order differential equations, in [1, 20, 21, 22] for second-order differential equations and in [25] for second-order differential systems.

From now on, we assume that

$$(P) \qquad \mu \in (0,1), \quad \alpha \in (1,2), \quad q > \frac{1}{\alpha - 1} \quad \text{and} \quad p = \frac{q}{q-1}.$$

Then $\frac{1}{p} + \frac{1}{q} = 1$ and $(\alpha - 2)p + 1 > 0$.

The paper is organized as follows. Section 2 contains technical lemmas that are used in the subsequent sections. Section 3 presents the existence principle for solving the problem (1.2), (1.3). It is shown that the solvability of this problem is reduced to the existence of a fixed point of an integral operator. The existence of its fixed point is proved by the Leray–Schauder degree theory. In Section 4, we apply the existence principle for two sets of admissible boundary conditions. Examples demonstrate our results.

2. Preliminaries

In this section we state technical lemmas and results which are used in the subsequent sections. Lemmas 2.1, 2.2 and 2.4–2.6 are proved in [2]. Note that condition (P) holds in this and in the next sections.

Lemma 2.1. *Suppose $\gamma \in L^q[0,T]$. Then*

$$(a) \quad \int_0^t (t-s)^{\alpha-2}\gamma(s)\,\mathrm{d}s \quad \textit{is continuous on } [0,T],$$

$$(b) \quad \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (t-s)^{\alpha-1}\gamma(s)\,\mathrm{d}s = (\alpha-1)\int_0^t (t-s)^{\alpha-2}\gamma(s)\,\mathrm{d}s \quad \textit{for } t \in [0,T].$$

Lemma 2.2. *Let $\{\rho_n\} \subset L^q[0,T]$ be L^q-convergent and let $\lim_{n\to\infty} \rho_n = \rho$. Then*

$$\lim_{n\to\infty} \int_0^t (t-s)^{\alpha-2}\rho_n(s)\,\mathrm{d}s = \int_0^t (t-s)^{\alpha-2}\rho(s)\,\mathrm{d}s \quad \textit{uniformly on } [0,T].$$

Corollary 2.3. *Suppose the assumptions of Lemma 2.2 are satisfied. Let $\{\lambda_n\} \subset [0,1]$ be convergent and $\lim_{n\to\infty} \lambda_n = \lambda$. Then*

$$\lim_{n\to\infty} \lambda_n \int_0^t (t-s)^{\alpha-2}\rho_n(s)\,\mathrm{d}s = \lambda \int_0^t (t-s)^{\alpha-2}\rho(s)\,\mathrm{d}s \quad \textit{uniformly on } [0,T].$$

Proof. The result follows from Lemma 2.2, where ρ_n is replaced by $\lambda_n\rho_n$ (note that $\lim_{n\to\infty} \lambda_n\rho_n = \lambda\rho$ in $L^q[0,T]$). $\square$

Lemma 2.4. *Let $\gamma \in L^q[0,T]$. Then solutions of the fractional differential equation*

$$^{c}D^{\alpha}x(t) = \gamma(t) \tag{2.1}$$

belong to the class $C^1[0,T]$, and

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\gamma(s) + x(0) + x'(0)t$$

are all solutions of (2.1).

Lemma 2.5. *Let $x \in C^1[0,T]$. Then*

$$^{c}D^{\mu}x(t) = \frac{1}{\Gamma(1-\mu)} \int_0^t (t-s)^{-\mu}x'(s)\,\mathrm{d}s \quad \text{for } t \in [0,T]$$

and $^{c}D^{\mu}x \in C[0,T]$.

Lemma 2.6. *Suppose that $\eta \in L^q[0,T]$ and $0 \le t_1 < t_2 \le T$. Then*

$$\left| \int_0^{t_2} (t_2-s)^{\alpha-2}\eta(s)\,\mathrm{d}s - \int_0^{t_1} (t_1-s)^{\alpha-2}\eta(s)\,\mathrm{d}s \right| \le$$
$$\le \left(\frac{t_1^d + (t_2-t_1)^d - t_2^d}{d} \right)^{\frac{1}{p}} \|\eta\|_q + \left(\frac{(t_2-t_1)^d}{d} \right)^{\frac{1}{p}} \|\eta\|_q,$$

where $d = (\alpha-2)p + 1$.

3. An Existence Principle

Suppose

$$f \text{ is a } L^q\text{-Carathéodory function on } [0,T]\times\mathbb{R}^3. \tag{3.1}$$

If $x \in C^1[0,T]$, then $^{c}D^{\mu}x \in C[0,T]$ by Lemma 2.5. Therefore the function $f(t, x(t), x'(t), (^{c}D^{\mu}x)(t))$ belongs to the set $L^q[0,T]$. Hence by Lemma 2.4, $x \in C^1[0,T]$ is a solution of (1.2) if and only if

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), x'(s), (^{c}D^{\mu}x)(s))\,\mathrm{d}s + a + bt, \quad t \in [0,T], \tag{3.2}$$

where $a, b \in \mathbb{R}$. Let $\Phi, \Psi \in \mathcal{A}$. Define an operator $\mathcal{S} : C^1[0,T] \to C^1[0,T]$ by the formula

$$(\mathcal{S}x)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), x'(s), (^{c}D^{\mu}x)(s))\,\mathrm{d}s +$$
$$+ x(0) - \Phi(x) + (x'(0) - \Psi(x))t.$$

It is easy to check that if x is a fixed point of the operator $\mathcal{S}$, then equality (3.2) is fulfilled with $a = x(0) - \Phi(x)$, $b = x'(0) - \Psi(x)$, and $\Phi(x) = 0$, $\Psi(x) = 0$. Consequently, any fixed point x of $\mathcal{S}$ is a solution of problem (1.2), (1.3).

The following result is the existence principle for solving the problem (1.2), (1.3).

Theorem 3.1. *Let $\Phi, \Psi \in \mathcal{A}$ satisfy the admissible compatibility condition. Suppose that* (3.1) *holds and there exists a positive constant S such that*

$$\|x\| < S, \quad \|x'\| < S$$

for each $\lambda \in [0,1]$ and each solution x of the problem

$$\left.\begin{aligned} ({}^c\!D^\alpha x)(t) &= \lambda f(t, x(t), x'(t), ({}^c\!D^\mu x)(t)), \\ \Phi(x) = 0&, \quad \Psi(x) = 0. \end{aligned}\right\} \tag{3.3}$$

Then problem (1.2), (1.3) *has a solution.*

Proof. We first note that since $\Phi, \Psi \in \mathcal{A}$ satisfy the admissible compatibility condition, system (1.1) has a solution for each $\nu \in [0,1]$ and there is a positive constant K such that $|a| \le K$ and $|b| \le K$ for each $\nu \in [0,1]$ and each solution $(a,b) \in \mathbb{R}^2$ of (1.1). Set

$$\Omega = \{x \in C^1[0,T] : \|x\| < S + (1+T)K, \ \|x'\| < S + K\}.$$

Then Ω is an open, bounded and symmetric with respect to $0 \in C^1[0,T]$ subset of the Banach space $C^1[0,T]$. We know that any fixed point of $\mathcal{S}$ is a solution of problem (1.2), (1.3). If

$$\mathrm{D}(\mathcal{I} - \mathcal{S}, \Omega, 0) \ne 0, \tag{3.4}$$

where "D" stands for the Leray–Schauder degree and $\mathcal{I}$ is the identical operator on $C^1[0,T]$, then $\mathcal{S}$ has a fixed point by the Leray–Schauder degree method. Hence to prove our theorem we need to show that (3.4) holds. To this end, define an operator $\mathcal{K} : [0,1] \times \overline{\Omega} \to C^1[0,T]$ by the formula

$$\begin{aligned} \mathcal{K}(\lambda, x)(t) = \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), x'(s), ({}^c\!D^\mu x)(s)) \, \mathrm{d}s + \\ + x(0) - \Phi(x) + (x'(0) - \Psi(x))t. \end{aligned}$$

Then $\mathcal{K}(1, \cdot) = \mathcal{S}$. We prove that $\mathcal{K}$ is a compact operator. We start with the proof that $\mathcal{K}$ is continuous. Let $\{\lambda_n\} \subset [0,1]$ and $\{x_n\} \subset \overline{\Omega}$ be convergent and let $\lim_{n\to\infty} \lambda_n = \lambda$, $\lim_{n\to\infty} x_n = x$. Let us put

$$\gamma_n(t) = f(t, x_n(t), x_n'(t), ({}^c\!D^\mu x_n)(t)), \quad \gamma(t) = f(t, x(t), x'(t), ({}^c\!D^\mu x)(t)),$$

$$z_n(t) = \frac{\lambda_n}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \gamma_n(s) \, \mathrm{d}s, \quad z(t) = \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \gamma(s) \, \mathrm{d}s.$$

We conclude from Lemma 2.5 that $\lim_{n\to\infty} {}^c\!D^\mu x_n = {}^c\!D^\mu x$ in $C[0,T]$ and

$$\|{}^c\!D^\mu x_n\| \le \frac{\|x_n'\|}{\Gamma(1-\mu)} \max\left\{ \int_0^t (t-s)^{-\mu}\,\mathrm{d}s : t \in [0,T] \right\} \le$$
$$\le \frac{(S+K)T^{1-\mu}}{\Gamma(2-\mu)} \tag{3.5}$$

for $n \in \mathbb{N}$. Hence

$$\lim_{n\to\infty} \gamma_n(t) = \gamma(t) \quad \text{for a.e. } t \in [0,T], \tag{3.6}$$

and since f fulfils (3.1), $\{x_n\}$ is bounded in $C^1[0,T]$ and $\{{}^c\!D^\mu x_n\}$ is bounded in $C[0,T]$, there exists $w \in L^q[0,T]$ such that

$$|\gamma_n(t)| \le w(t) \quad \text{for a.e. } t \in [0,T] \text{ and all } n \in \mathbb{N}. \tag{3.7}$$

Therefore, $\lim_{n\to\infty} \|\gamma_n - \gamma\|_q = 0$ by the dominated convergence theorem in $L^q[0,T]$. Consequently, by Corollary 2.3 and Lemma 2.1(b),

$$\lim_{n\to\infty} z_n'(t) = \lim_{n\to\infty} \frac{\lambda_n}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2}\gamma_n(s)\,\mathrm{d}s =$$
$$= \frac{\lambda}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2}\gamma(s)\,\mathrm{d}s =$$
$$= z'(t) \qquad \text{uniformly on } [0,T].$$

As a result, $\lim_{n\to\infty} z_n = z$ in $C^1[0,T]$ since $z_n(0) = z(0) = 0$. The continuity of $\mathcal{K}$ follows now from the equalities $\mathcal{K}(\lambda_n, x_n)(t) = z_n(t) + x_n(0) - \Phi(x_n) + (x_n'(0) - \Psi(x_n))t$, $\mathcal{K}(\lambda, x)(t) = z(t) + x(0) - \Phi(x) + (x'(0) - \Psi(x))t$ and from

$$\lim_{n\to\infty} (x_n(0) - \Phi(x_n)) = x(0) - \Phi(x), \quad \lim_{n\to\infty} (x_n(0) - \Psi(x_n)) = x(0) - \Psi(x).$$

We now prove that the set $\mathcal{K}([0,1] \times \overline{\Omega})$ is relatively compact in $C^1[0,T]$. Since the set $\{x(0) - \Phi(x) + (x'(0) - \Psi(x))t : x \in \overline{\Omega}\}$ is relatively compact in $\mathbb{R}$, which immediately follows from the properties of Φ and Ψ, it suffices to show that the set

$$\mathcal{B} = \left\{ \lambda \int_0^t (t-s)^{\alpha-1} f(s, x(s), x'(s), ({}^c\!D^\mu x)(s))\,\mathrm{d}s : \lambda \in [0,1],\, x \in \overline{\Omega} \right\}$$

is relatively compact in $C^1[0,T]$. Since ${}^c\!D^\mu x \in C[0,T]$ for $x \in C^1[0,T]$ and (cf. (3.5)) $\|{}^c\!D^\mu x\| \le \frac{T^{1-\mu}\|x'\|}{\Gamma(2-\mu)}$ for $x \in \overline{\Omega}$, there exists $\rho \in L^q[0,T]$ such that

$$|f(t, x(t), x'(t), ({}^c\!D^\mu x)(t))| \le \rho(t) \quad \text{for a.e. } t \in [0,T] \text{ and all } x \in \overline{\Omega}. \tag{3.8}$$

The boundedness of $\mathcal{B}$ in $C^1[0,T]$ follows from the relations (for $t \in [0,T]$ and $x \in \overline{\Omega}$)

$$\left| \int_0^t (t-s)^{\alpha-1} f(s, x(s), x'(s), ({}^c\!D^\mu x)(s))\, \mathrm{d}s \right| \le T^{\alpha-1} \|\rho\|_1$$

and

$$\begin{aligned} \left| \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t (t-s)^{\alpha-1} f(s, x(s), x'(s), ({}^c\!D^\mu x)(s))\, \mathrm{d}s \right| &= \\ = (\alpha-1) \left| \int_0^t (t-s)^{\alpha-2} f(s, x(s), x'(s), ({}^c\!D^\mu x)(s))\, \mathrm{d}s \right| &\le \\ \le (\alpha-1) \int_0^t (t-s)^{\alpha-2} \rho(s)\, \mathrm{d}s &\le \\ \le (\alpha-1) \left(\int_0^t (t-s)^{(\alpha-2)p}\, \mathrm{d}s \right)^{\frac{1}{p}} \left(\int_0^t \rho^q(s)\, \mathrm{d}s \right)^{\frac{1}{q}} &\le \\ \le (\alpha-1) \left(\frac{T^{(\alpha-2)p+1}}{(\alpha-2)p+1} \right)^{\frac{1}{p}} \|\rho\|_q, & \end{aligned}$$

where the Hölder inequality is used. Furthermore, for $0 \le t_1 < t_2 \le T$ and $x \in \overline{\Omega}$, Lemma 2.6 (for $\eta(t) = f(t, x(t), x'(t), ({}^c\!D^\mu x)(t))$) gives

$$\begin{aligned} \Bigg| \int_0^{t_2} (t_2-s)^{\alpha-2} f(s, x(s), x'(s), ({}^c\!D^\mu x)(s))\, \mathrm{d}s \, - & \\ - \int_0^{t_1} (t_1-s)^{\alpha-2} f(s, x(s), x'(s), ({}^c\!D^\mu x)(s))\, \mathrm{d}s \Bigg| &\le \\ \le \left(\frac{t_1^d + (t_2-t_1)^d - t_2^d}{d} \right)^{\frac{1}{p}} \|\rho\|_q + \left(\frac{(t_2-t_1)^d}{d} \right)^{\frac{1}{p}} \|\rho\|_q & \end{aligned}$$

since $\|\eta\|_q \le \|\rho\|_q$. Here $d = (\alpha-2)p+1$. Hence the set $\{y' : y \in \mathcal{B}\}$ is equicontinuous on $[0,T]$, and thus $\mathcal{B}$ is relatively compact in $C^1[0,T]$ by the Arzelà-Ascoli theorem. To summarize, $\mathcal{K}$ is a compact operator.

Suppose now that $\mathcal{K}(\lambda_*, x_*) = x_*$ for some $\lambda_* \in [0,T]$ and some $x_* \in \overline{\Omega}$. Let $\gamma_*(t) = f(t, x_*(t), x_*'(t), ({}^c\!D^\mu x_*)(t))$ for a.e. $t \in [0,T]$. Then $\gamma_* \in L^q[0,T]$ and the equality

$$x_*(t) = \frac{\lambda_*}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \gamma_*(s)\, \mathrm{d}s + x_*(0) - \Phi(x_*) + (x_*'(0) - \Psi(x_*))t \quad (3.9)$$

holds for $t \in [0,T]$. Hence $\Phi(x_*) = 0$, $\Psi(x_*) = 0$ and, by Lemma 2.4,

$$({}^c\!D^\alpha x_*)(t) = \lambda_* \gamma_*(t) \quad \text{for a.e. } t \in [0,T].$$

Hence x_* is a solution of problem (3.3) with $\lambda = \lambda_*$, and $\|x_*\| < S$, $\|x_*'\| < S$ by the assumptions. As a result, $\mathcal{K}(\lambda, x) \neq x$ for each $\lambda \in [0,1]$ and each $x \in \partial\Omega$. Therefore, by the homotopy property,

$$\mathrm{D}(\mathcal{I} - \mathcal{K}(0,\cdot), \Omega, 0) = \mathrm{D}(\mathcal{I} - \mathcal{K}(1,\cdot), \Omega, 0). \tag{3.10}$$

We will now proceed to showing that

$$\mathrm{D}(\mathcal{I} - \mathcal{K}(0,\cdot), \Omega, 0) \neq 0. \tag{3.11}$$

Let us define a compact operator $\mathcal{L} : [0,1] \times \overline{\Omega} \to C^1[0,T]$ as

$$\mathcal{L}(\nu, x) = x(0) + \Phi(x) - \nu\Phi(-x) + (x'(0) + \Psi(x) - \nu\Psi(-x))t.$$

Then $\mathcal{L}(1,\cdot)$ is odd (i.e., $\mathcal{L}(1,-x) = -\mathcal{L}(1,x)$ for $x \in \overline{\Omega}$) and

$$\mathcal{L}(0,\cdot) = \mathcal{K}(0,\cdot). \tag{3.12}$$

If $\mathcal{L}(\nu_1, x_1) = x_1$ for some $(\nu_1, x_1) \in [0,1] \times \overline{\Omega}$, then

$$x_1(t) = x_1(0) + \Phi(x_1) - \nu_1\Phi(-x_1) + (x_1'(0) + \Psi(x_1) - \nu_1\Psi(-x_1))t, \tag{3.13}$$

and therefore $x_1(t) = a + bt$ for $t \in [0,T]$, where $a = x_1(0) + \Phi(x_1) - \nu_1\Phi(-x_1)$ and $b = (x_1'(0) + \Psi(x_1) - \nu_1\Psi(-x_1))t$. Let $t = 0$ in $x_1(t)$ and $x_1'(t)$, where x_1 is given in (3.13), and have

$$\Phi(x_1) - \nu_1\Phi(-x_1) = 0, \quad \Psi(x_1) - \nu_1\Psi(-x_1) = 0,$$

which is system (1.1) with $\nu = \nu_1$. Hence due to the first part of the proof, the inequalities $|a| \le K$ and $|b| \le K$ are fulfilled. Consequently, $\|x_1\| \le (1+T)K$ and $\|x_1'\| \le K$, and thus $x_1 \notin \partial\Omega$. Next, by the homotopy property and the Borsuk antipodal theorem,

$$\mathrm{D}(\mathcal{I} - \mathcal{L}(0,\cdot), \Omega, 0) = \mathrm{D}(\mathcal{I} - \mathcal{L}(1,\cdot), \Omega, 0) \text{ and } \mathrm{D}(\mathcal{I} - \mathcal{L}(1,\cdot), \Omega, 0) \neq 0.$$

The last relations together with (3.12) give that (3.11) holds. Finally, we conclude that from (3.10) and (3.11) follows (3.4). □

4. Applications of the Existence Principle

4.1. **Functionals satisfying the admissible complementary condition.** We give two sets of nonlinear functionals $\Lambda, \Phi \in \mathcal{A}$ satisfying the admissible complementary condition. Such functionals in the nonlocal boundary conditions (1.3) will be used in the next subsection for solving problem (1.2), (1.3) by means of our existence principle.

For $j = 0, 1$, let $\mathcal{B}_j$ be the set of functionals $\Lambda \in \mathcal{A}$ for which there exists a positive constant $K = K(\Lambda)$ such that

$$x \in C^1[0,T],\ |x^{(j)}| \ge K \text{ on } [0,T] \Rightarrow \Lambda(x)\,\mathrm{sign}(x^{(j)}) > 0$$

Remark 4.1. The functionals from the set $\mathcal{B}_j$ have the following important property: If $\Lambda \in \mathcal{B}_j$ and $\Lambda(x) = 0$ for some $x \in C^1[0,T]$ and $j \in \{0,1\}$, then there exists $\xi \in [0,T]$ such that $|x^{(j)}(\xi)| < K(\Lambda)$.

Example 4.2. Let $0 \le a < b \le T$, $j \in \{0,1\}$, $\Theta : C^1[0,T] \to \mathbb{R}$ be continuous and $\sup\{|\Theta(x)| : x \in C^1[0,T]\} < \infty$. Then the functionals

$$\Lambda_1(x) = \min\{x^{(j)}(t) : a \le t \le b\} + \Theta(x),$$
$$\Lambda_2(x) = \max\{x^{(j)}(t) : a \le t \le b\} + \Theta(x),$$
$$\Lambda_3(x) = \int_a^b \max\{x^{(j)}(s) : a \le s \le t\}\,\mathrm{d}t + \Theta(x)$$

belong to the set $\mathcal{B}_j$. If $0 \le t_1 < t_2 < \cdots < t_n \le T$, $b_i > 0$, $g, f_i \in C(\mathbb{R})$ and $\lim_{u\to\pm\infty} g(u) = \lim_{u\to\pm\infty} f_i(u) = \pm\infty$, $i = 1, 2, \ldots, n$, then the functionals

$$\Lambda_4(x) = \sum_{i=1}^{n} b_i g(x^{(j)}(t_i)) + \Theta(x),$$
$$\Lambda_5(x) = \int_a^b \left(\sum_{i=1}^{n} b_i f_i(x^{(j)}(s))\right) \mathrm{d}s + \Theta(x),$$
$$\Lambda_6(x) = \int_a^b \left(\int_a^s \left(\sum_{i=1}^{n} b_i f_i(x^{(j)}(\xi))\right) \mathrm{d}\xi\right) \mathrm{d}s + \Theta(x)$$

also belong to $\mathcal{B}_j$.

Lemma 4.3. *Let $\Phi \in \mathcal{B}_0$ and $\Psi \in \mathcal{B}_1$. Then Φ, Ψ satisfy the admissible compatibility condition.*

Proof. Since $\Phi \in \mathcal{B}_0$ and $\Psi \in \mathcal{B}_1$, there exists a positive constant K such that for each $\nu \in [0,1]$ we have $[\Phi(a+bt) - \nu\Phi(-a-bt)]\mathrm{sign}(a+bt) > 0$ if $|a+bt| \ge K$ for $t \in [0,T]$ and $[\Psi(a+bt) - \nu\Psi(-a-bt)]\mathrm{sign}(b) > 0$ if $|b| \ge K$. Hence if $(a_0, b_0) \in \mathbb{R}^2$ is a solution of system (1.1) for some $\nu \in [0,1]$, then (see Remark 4.1) $|b_0| < K$ and $|a_0 + b_0\xi| < K$ for some $\xi \in [0,T]$. From the inequality $|a_0| \le |a_0 + b_0\xi| + |b_0\xi| < (1+K)T$ we see that for each $\nu \in [0,1]$, any solution $(a,b) \in \mathbb{R}^2$ of (1.1) satisfies the estimate

$$|a| < (1+K)T, \quad |b| < K. \tag{4.1}$$

Let $M = \{(a,b) \in \mathbb{R}^2 : |a| < (1+K)T,\ |b| < K\}$ and $\mathcal{F} : [0,1] \times \overline{M} \to \mathbb{R}^2$ be defined as

$$\mathcal{F}(\nu, a, b) = (\Phi(a+bt) - \nu\Phi(-a-bt),\ \Psi(a+bt) - \nu\Psi(-a-bt)).$$

Then $\mathcal{F}$ is a continuous operator and M is an open, bounded and symmetric with respect to $(0,0) \in \mathbb{R}^2$ subset of $\mathbb{R}^2$. We have also $\mathcal{F}(\nu, a, b) \ne (0,0)$ for $\nu \in [0,1]$ and $(a,b) \in \partial M$, and $\mathcal{F}(1,\cdot,\cdot)$ is an odd operator (that is, $\mathcal{F}(1,-a,-b) = -\mathcal{F}(1,a,b)$ for $(a,b) \in \overline{M}$). Hence by the Borsuk antipodal theorem and the homotopy property,

$$\deg(\mathcal{F}(1,\cdot,\cdot), M, 0) \ne 0,$$
$$\deg(\mathcal{F}(1,\cdot,\cdot), M, 0) = \deg(\mathcal{F}(\nu,\cdot,\cdot), M, 0) \quad \text{for } \nu \in [0,1],$$

where "deg" stands for the Brower degree. Consequently, the operator equation $\mathcal{F}(\nu, a, b) = (0, 0)$ has a solution for each $\nu \in [0, 1]$. Hence for each $\nu \in [0, 1]$ system (1.1) has a solution and any its solution (a, b) satisfies (4.1), and therefore Φ, Ψ satisfy the admissible complementary condition. $\square$

Remark 4.4. The special cases of the boundary conditions (1.3) are:

(a) the Dirichlet conditions $x(0) = A$, $x(T) = B$ (for $\Phi(x) = x(0) - A$, $\Psi(x) = \int_0^T x'(s)\,\mathrm{d}s + A - B$),

(b) the mixed conditions $x(0) = A$, $x'(T) = B$ (for $\Phi(x) = x(0) - A$, $\Psi(x) = x'(T) - B$) and $x'(0) = A$, $x(T) = B$ (for $\Phi(x) = x(T) - B$, $\Psi(x) = x'(0) - A$),

(c) the antiperiodic conditions $x(0) + x(T) = 0$, $x'(0) + x'(T) = 0$ (for $\Phi(x) = x(0) + x(T)$, $\Psi(x) = x'(0) + x'(T)$),

(d) the initial conditions $x(\xi) = A$, $x'(\xi) = B$, where $\xi \in [0, T]$ (for $\Phi(x) = x(\xi) - A$, $\Psi(x) = x'(\xi) - B$),

(e) the multipoint conditions $\sum\limits_{j=0}^{n} a_j x^{2l_j - 1}(t_j) = A$, $\sum\limits_{i=0}^{m} b_i\,(x'(s_i))^{2k_i - 1} = B$, where $a_j, b_i \in (0, \infty)$, $l_j, k_i \in \mathbb{N}$ ($j = 0, \dots, n$, $i = 0, \dots, m$), $0 \le t_0 < t_1 < \dots < t_n \le T$, $0 \le s_0 < s_1 < \dots < s_m \le T$ (for $\Phi(x) = \sum\limits_{j=0}^{n} a_j x^{2l_j - 1}(t_j) - A$, $\Psi(x) = \sum\limits_{i=0}^{m} b_i (x'(s_i))^{2k_i - 1} - B$).

Let $\mathcal{C}$ be the set of functionals $\Lambda \in \mathcal{A}$ such that $\sup\{|\Lambda(x)| : x \in C^1[0, T]\} < \infty$.

Lemma 4.5. *Let $0 \le \xi < \eta \le T$, $\Lambda_1, \Lambda_2 \in \mathcal{C}$ and*

$$\Phi(x) = x(\xi) + \Lambda_1(x),\ \Psi(x) = x(\eta) + \Lambda_2(x) \quad \textit{for } x \in C^1[0, T].$$

Then Φ, Ψ satisfy the admissible compatibility condition.

Proof. Since $\Lambda_1, \Lambda_2 \in \mathcal{C}$, there is a positive constant S such that $|\Lambda_1(x)| < S$ and $|\Lambda_1(x)| < S$ for $x \in C^1[0, T]$. System (1.1) has the form

$$\begin{aligned} &(1 + \nu)(a + b\xi) + \Lambda_1(a + bt) - \nu\Lambda_1(-a - bt) = 0, \\ &(1 + \nu)(a + b\eta) + \Lambda_2(a + bt) - \nu\Lambda_2(-a - bt) = 0. \end{aligned} \tag{4.2}$$

Suppose that $(a_0, b_0) \in \mathbb{R}^2$ is a solution of (4.2) for some $\nu \in [0, 1]$. Then

$$(1+\nu)(\eta-\xi)b_0 = \Lambda_1(a_0+b_0t)-\nu\Lambda_1(-a_0-b_0t)-\Lambda_2(a_0+b_0t)+\nu\Lambda_2(-a_0-b_0t),$$

and consequently, $|b_0| < \frac{2S}{\eta - \xi}$. Since

$$a_0 = -b_0\xi + \frac{1}{1 + \nu}[\nu\Lambda_1(-a_0 - b_0t) - \Lambda_1(a_0 + b_0t)],$$

we have

$$|a_0| < \frac{2ST}{\eta - \xi} + S = S\left(1 + \frac{2T}{\eta - \xi}\right).$$

As a result, for each $\nu \in [0,1]$, any solution $(a,b) \in \mathbb{R}^2$ of (4.2) satisfies the estimate

$$|a| < S\left(1 + \frac{2T}{\eta - \xi}\right), \ |b| < \frac{2S}{\eta - \xi}. \tag{4.3}$$

Put $M = \left\{(a,b) \in \mathbb{R}^2 : |a| < S\left(1 + \frac{2T}{\eta-\xi}\right), \ |b| < \frac{2S}{\eta-\xi}\right\}$. In order to prove that Φ, Ψ satisfy the admissible compatibility condition we define a continuous operator $\mathcal{F} : [0,1] \times \overline{M} \to \mathbb{R}^2$ by

$$\begin{aligned}\mathcal{F}(\nu,a,b) = \Big((1+\nu)(a+b\xi) + \Lambda_1(a+bt) - \nu\Lambda_1(-a-bt),\\ (1+\nu)(a+b\eta) + \Lambda_2(a+bt) - \nu\Lambda_2(-a-bt)\Big).\end{aligned}$$

Then $\mathcal{F}(1,\cdot,\cdot)$ is an odd operator and $\mathcal{F}(\nu,a,b) \neq (0,0)$ for all $\nu \in [0,1]$ and $(a,b) \in \partial\mathcal{M}$. By the Borsuk antipodal theorem and by the homotopy property, we can prove just as in the proof of Lemma 4.3 that for each $\nu \in [0,1]$, the equation $\mathcal{F}(\nu,a,b) = (0,0)$ has a solution. Consequently, system (4.2) has a solution for each $\nu \in [0,1]$ and all its solutions (a,b) satisfy (4.3). Hence Φ, Ψ satisfy the admissible compatibility condition. □

4.2. **Existence results for nonlocal fractional BVPs.** Bearing in mind Section 4.1, we work with the boundary conditions

$$\Phi(x) = 0, \ \Psi(x) = 0, \quad \Phi \in \mathcal{B}_0, \ \Psi \in \mathcal{B}_1, \tag{4.4}$$

and

$$x(\xi) + \Lambda_1(x) = 0, \ x(\eta) + \Lambda_2(x) = 0, \quad 0 \le \xi < \eta \le T, \ \Lambda_1, \Lambda_2 \in \mathcal{C}. \tag{4.5}$$

Lemmas 4.3 and 4.5 show that the functionals Φ, Ψ in (4.4) and the functionals $x(\xi) + \Lambda_1(x)$, $x(\eta) + \Lambda_2(x)$ in (4.5) satisfy the admissible compatibility condition. We discuss the solvability of problems (1.2), (4.4) and (1.2), (4.5) by the existence principle (Theorem 3.1).

Theorem 4.6. *Let* (3.1) *hold. Suppose that the estimate*

$$\begin{aligned}&|f(t,x,y,z)| \le \rho(t)p(|x|,|y|,|z|)\\ &\qquad\qquad \textit{for a.e. } t \in [0,T] \textit{ and all } (x,y,z) \in \mathbb{R}^3\end{aligned} \tag{4.6}$$

is fulfilled, where $\rho \in L^q[0,T]$ and $p \in C(\mathbb{R}^3_+)$ are nonnegative, p is nondecreasing in all its arguments and

$$\lim_{u\to\infty} \frac{p(u,u,u)}{u} = 0. \tag{4.7}$$

Then problems (1.2), (4.4) *and* (1.2), (4.5) *are solvable.*

Proof. By Theorem 3.1, we have to prove that there exists a positive constant S such that

$$\|x\| < S, \quad \|x'\| < S \tag{4.8}$$

for each $\lambda \in [0,1]$ and each solution x of the problems

$$({}^c\!D^\alpha x)(t) = \lambda f(t,x(t),x'(t),({}^c\!D^\mu x)(t)), \quad (4.4), \tag{4.9}$$

and
$$({}^c\!D^\alpha x)(t) = \lambda f(t, x(t), x'(t), ({}^c\!D^\mu x)(t)), \quad (4.5). \tag{4.10}$$
We start with problem (4.9). Let $x \in C^1[0,T]$ be a solution of (4.9). Since $\Phi \in \mathcal{B}_0$ and $\Psi \in \mathcal{B}_1$, there exists a positive constant K such that (cf. Remark 4.1) $|x(\xi_0)| < K$, $|x'(\xi_1)| < K$ for some $\xi_0, \xi_1 \in [0,T]$. Furthermore, by Lemma 2.5, ${}^c\!D^\mu x \in C[0,T]$ and
$$\|{}^c\!D^\mu x\| \le \frac{\|x'\|}{\Gamma(1-\mu)} \left\| \int_0^t (t-s)^{-\mu}\, \mathrm{d}s \right\| \le V\|x'\|,$$
where $V = \frac{T^{1-\mu}}{\Gamma(1-\mu)}$. From the equality $x(t) = x(\xi_0) + \int_{\xi_0}^t x'(s)\,\mathrm{d}s$ we get
$$\|x\| < K + T\|x'\|. \tag{4.11}$$
From estimate (4.6) it follows now that
$$|f(t, x(t), x'(t), ({}^c\!D^\mu x)(t))| \le \rho(t) p(K + T\|x'\|, \|x'\|, V\|x'\|) \tag{4.12}$$
for a.e. $t \in [0,T]$. Since x is a solution of the equation in (4.9), we have (cf. (3.2) and Lemma 2.1)
$$x'(t) = \frac{\lambda}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} f(s, x(s), x'(s), ({}^c\!D^\mu x)(s))\,\mathrm{d}s + b \quad \text{for } t \in [0,T], \tag{4.13}$$
where $b \in \mathbb{R}$. From $|x'(\xi_1)| < K$, we obtain
$$|b| < K + \left| \frac{\lambda}{\Gamma(\alpha-1)} \int_0^{\xi_1} (\xi_1 - s)^{\alpha-2} f(s, x(s), x'(s), ({}^c\!D^\mu x)(s))\,\mathrm{d}s \right|. \tag{4.14}$$
By Lemma 2.6 (with $\eta = \rho, t_2 = t$ and $t_1 = 0$),
$$\int_0^t (t-s)^{\alpha-2} \rho(s)\,\mathrm{d}s \le \left(\frac{T^{(\alpha-2)p+1}}{(\alpha-2)p+1} \right)^{\frac{1}{p}} \|\rho\|_q =: W.$$
We conclude from the last inequality and from (4.12), (4.13) and (4.14) that
$$\|x'\| \le \frac{2W}{\Gamma(\alpha-1)} p(K + T\|x'\|, \|x'\|, V\|x'\|) + K. \tag{4.15}$$
In view of (4.7), there is a positive constant S_1 such that the inequality
$$\frac{2W}{\Gamma(\alpha-1)} p(K + Tv, v, Vv) + K < v$$
is fulfilled for all $v \ge S_1$. Hence (4.15) yields $\|x'\| < S_1$, and hence $\|x\| < K + TS_1$ by (4.11). Put $S = \max\{S_1, K + TS_1\}$. Then (4.8) holds for each $\lambda \in [0,1]$ and each solution x of problem (4.9).

We proceed now to discussing problem (4.10). Let x be a solution of(4.10). Due to $\Lambda_1, \Lambda_2 \in \mathcal{C}$ there is $L > 0$ such that $|\Lambda_1(x)| \le L$ and

$|\Lambda_1(x)| \le L$ for $x \in C^1[0,T]$. Therefore $|x(\xi)| \le L$ and $|x(\eta)| \le L$. It follows from $x(\xi) - x(\eta) = x'(\tau)(\eta - \xi)$, where $\tau \in (\xi, \eta)$ that $|x'(\tau)| \le \frac{2L}{\eta-\xi}$. We have proved that for each $\lambda \in [0,1]$ and any solution x of problem (4.10) there exists $\tau = \tau(\lambda, x) \in (\xi, \eta)$ such that $|x(\xi)| \le L$ and $|x'(\tau)| \le \frac{2L}{\eta-\xi}$. Essentially the same reasoning as in the first part of the proof $\Big($with $K > \max\Big\{L, \frac{2L}{\eta-\xi}\Big\}\Big)$ yields that there is a positive constant S such that (4.8) holds for each $\lambda \in [0,1]$ and each solution x of problem (4.10). $\square$

Example 4.7. Let $\gamma_i \in L^q[0,T]$ $(i = 0,1,2,3)$, $h \in C([0,T] \times \mathbb{R}^3)$ be bounded and $g_j \in C(\mathbb{R})$, $\lim_{u\to\pm\infty} \frac{g_j(u)}{u} = 0$ $(j = 1,2,3)$. Then the function

$$f(t,x,y,z) = \gamma_0(t)h(t,x,y,z) + \gamma_1(t)g_1(x) + \gamma_2(t)g_2(y) + \gamma_2(t)g_3(z)$$

satisfies the conditions of Theorem 4.6 with $\rho(t) = \sum_{i=0}^{3} |\gamma_i(t)|$ and

$$p(u_1,u_2,u_3) = \sum_{j=1}^{3} \max\Big\{|g_j(s)| : |s| \le u_j\Big\} + K \quad \text{for } (u_1,u_2,u_3) \in \mathbb{R}_+^3,$$

where $K = \sup\{|h(t,x,y,z)| : (t,x,y,z) \in [0,T] \times \mathbb{R}^3\}$. Hence Theorem 4.6 can be applied to problems $(1.2), (4.4)$ and $(1.2), (4.5)$.

In particular, equation (1.2) has solutions u_1 and u_2, where u_1 satisfies the boundary conditions

$$\min\{u(t) : t \in [0,T]\} = A, \quad \max\{u'(t) : t \in [0,T]\} = B, \quad A, B \in \mathbb{R},$$

and u_2 satisfies the boundary conditions

$$u(\xi) = \arctan(\|u\| - \|u'\|) + A, \quad u(\eta) = \int_0^T \sin(u'(t))\,\mathrm{d}t + B, \quad A, B \in \mathbb{R}.$$

where $0 \le \xi < \eta \le 1$.

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