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ON THE WEIGHTED INITIAL PROBLEM FOR SINGULAR FUNCTIONAL DIFFERENTIAL SYSTEMS

Abstract. For singular functional differential systems, sufficient conditions for solvability and well-posedness of the weighted initial problem are established.

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In a finite interval $]a,b[$ we consider the functional differential system

$$\frac{dx(t)}{dt} = f(x)(t) \tag{1}$$

with the weighted initial condition

$$\limsup_{t\to a} \big\|\phi^{-1}(t)x(t)\big\| < +\infty. \tag{2}$$

Here, $f : C([a,b];\mathbb{R}^n) \to L_{loc}(]a,b];\mathbb{R}^n)$ is a singular operator satisfying the local Carathéorory conditions, $\phi(t) = \operatorname{diag}\big(\varphi_1(t),\dots,\varphi_n(t)\big)$, and $\varphi_i : [a,b] \to \mathbb{R}_+$ $(i=1,\dots,n)$ are continuous non-decreasing functions such that $\varphi_i(a) = 0$, $\varphi_i(t) > 0$ for $a < t \le b$ $(i = 1,\dots,n)$.

The initial problem for the singular system (1) has been thoroughly investigated in the cases, in which f is either the Nemytski's operator [1]–[6], or the evolutionary operator [7]–[9]. The weighted initial problem for higher order singular functional differential equations is studied in [11]–[14]. As for the weighted singular problem (1), (2), it is not studied well enough. In the present paper unimprovable in a certain sense conditions are given which, respectively, guarantee solvability and well-posedness of this problem.

Throughout the paper, the use will be made of the following notation.

$\mathbb{R} =]-\infty,+\infty[$, $\mathbb{R}_+ = [0,+\infty[$.

$\mathbb{R}^n$ is the space of n-dimensional real column-vectors $x = (x_i)_{i=1}^n$ with the norm

$$\|x\| = \sum_{i=1}^n |x_i|.$$

If $x = (x_i)_{i=1}^n \in \mathbb{R}^n$, then

$$[x]_+ = \Big(\frac{x_i + |x_i|}{2}\Big)_{i=1}^n.$$

$r(X)$ is the spectral radius of the $n \times n$ matrix X, and X^{-1} is the inverse to X matrix.

$\operatorname{diag}(x_1, \dots, x_n)$ is the diagonal $n \times n$-matrix with diagonal elements $x_1, \dots, x_n$.

If $X = \operatorname{diag}(x_1, \dots, x_n)$, then $\operatorname{Sgn}(X) = \big(\operatorname{sgn}(x_1), \dots, \operatorname{sgn}(x_n)\big)$.

$\mathbb{R}_+^n$ and $\mathbb{R}_+^{n\times n}$ are the sets of n-dimensional vectors and $n \times n$-matrices with nonnegative components.

$C([a,b];\mathbb{R}^n)$ is the space of continuous vector functions $x : [a,b] \to \mathbb{R}^n$ with the norm

$$\|x\|_C = \max\Big\{\|x(t)\| : \ a \le t \le b\Big\}.$$

$C_\phi([a,b];\mathbb{R}^n)$ is the space of continuous vector functions $x : [a,b] \to \mathbb{R}^n$, satisfying the condition (2), with the norm

$$\|x\|_{C_\phi} = \sup\Big\{\big\|\phi^{-1}(t)x(t)\big\| : \ a < t \le b\Big\}.$$

If $x = (x_i)_{i=1}^n \in C_\phi([a,b];\mathbb{R}^n)$, then

$$|x|_{C_\phi} = \big(\|x_i\|_{C_{\varphi_i}}\big)_{i=1}^n.$$

$L([a,b];\mathbb{R}^n)$ is the space of vector functions with Lebesgue integrable on $[a,b]$ components.

$L_{loc}(]a,b];\mathbb{R}^n)$ is the space of vector functions whose components are Lebesgue integrable on $[a+\varepsilon, b]$ for an arbitrarily small $\varepsilon > 0$.

$K_{loc}(]a,b] \times \mathbb{R}^k;\mathbb{R}^m)$ and $K_{loc}(C([a,b];\mathbb{R}^k); L_{loc}(]a,b];\mathbb{R}^m))$ are the sets of vector functions $g :]a,b] \times \mathbb{R}^k \to \mathbb{R}^m$ and operators $f : C([a,b];\mathbb{R}^k) \to L_{loc}(]a,b];\mathbb{R}^m)$, satisfying the local Carathéodory conditions (see [15]).

An important particular case of the functional differential system (1) is the differential system with a deviating argument

$$\frac{dx(t)}{dt} = g\big(t, x(t), x(\tau(t))\big). \tag{3}$$

Along with the problem (1), (2), we consider the problem (3), (2). Everywhere below, when the question concerns these problems, it will be assumed that

$$f \in K_{loc}\big(C([a,b];\mathbb{R}^n); L_{loc}(]a,b];\mathbb{R}^n)\big), \quad g \in K_{loc}(]a,b] \times \mathbb{R}^{2n};\mathbb{R}^n),$$

and $\tau : [a,b] \to [a,b]$ is a measurable function.

We are mainly interested in the case, where the systems (1) and (3) are singular, i.e., in the case in which

$$\int_a^b f_\rho^*(t)\,dt = +\infty \ \text{ and } \ \int_a^b g_\rho^*(t)\,dt = +\infty \ \text{ for } \ \rho > 0,$$

where

$$f^*_\rho(t) = \sup\Big\{\big\|f(x)(t)\big\| : \ \|x\|_C \le \rho\Big\},$$

$$g^*_\rho(t) = \max\Big\{\big\|g(t,x,y)\big\| : \ \|x\| + \|y\| \le \rho\Big\}.$$

For an arbitrary positive number δ, we put

$$\chi(t,\delta,\lambda) = \begin{cases} 0 & \text{for } \ a \le t < a+\delta \\ \lambda & \text{for } \ t > a+\delta \end{cases},$$

and consider the auxiliary initial problem

$$\frac{dx(t)}{dt} = \chi(t,\delta,\lambda) f(x)(t), \tag{4}$$

$$x(a) = 0, \tag{5}$$

depending on the parameters $\lambda \in]0,1]$ and $\delta > 0$.

On the basis of Corollary 2 in [16], the following theorem can be proved.

Theorem 1. *Let there exist a positive number ρ_0 such that for arbitrary $\lambda \in]0,1]$ and $\delta > 0$ every solution x of the problem* (4), (5) *admits the estimate*

$$\|x\|_{C_\phi} \le \rho_0.$$

Then the problem (1), (2) *has at least one solution.*

This theorem allows one to get efficient sufficient conditions for the solvability of the problems (1), (2) and (3), (2). In particular, the following propositions are valid.

Theorem 2. *Let there exist a matrix $\mathcal{P} \in \mathbb{R}^{n\times n}_+$ and a vector function $q : \mathbb{R}_+ \to \mathbb{R}^n_+$ such that*

$$r(\mathcal{P}) < 1, \quad \lim_{\rho\to+\infty} \frac{\|q(\rho)\|}{\rho} = 0, \tag{6}$$

and for an arbitrary vector function $x \in C_\phi([a,b];\mathbb{R}^n)$ on the interval $[a,b]$ the inequality

$$\int\limits_a^t \Big[\operatorname{sgn}(x(s)) f(x)(s)\Big]_+ ds \le \phi(t)\Big(\mathcal{P}|x|_{C_\phi} + q\big(\|x\|_{C_\phi}\big)\Big)$$

is fulfilled. Then the problem (1), (2) *has at least one solution.*

Corollary 1. *Let the functions φ_i $(i = 1,\dots,n)$ be absolutely continuous and let there exist a set of zero measure $I_0 \subset [a,b]$, matrices $\mathcal{P}_k \in \mathbb{R}^{n\times n}_+$ $(k = 1,2)$ and a vector function $q : \mathbb{R}_+ \to \mathbb{R}^n_+$ with non-decreasing components such that on the set $([a,b]\setminus I_0)\times\mathbb{R}^{2n}$ the inequality*

$$\operatorname{Sgn}(x) g(t,x,y) \le \phi'(t)\Big(\mathcal{P}_1\phi^{-1}(t)|x| + \mathcal{P}_2\phi^{-1}(\tau(t))|y|\Big) + \\ + \phi'(t) q\Big(\big\|\phi^{-1}(t)|x| + \phi^{-1}(\tau(t))|y|\,\big\|\Big)$$

is fulfilled. If, moreover, the conditions (6) *are fulfilled, where* $\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2$, *then the problem* (3), (2) *has at least one solution.*

Remark 1. In Theorem 2 and Corollary 1, the condition $r(\mathcal{P}) < 1$ is unimprovable and it cannot be replaced by the condition $r(\mathcal{P}) \leq 1$. The validity of that fact follows directly from the theorem below.

Theorem 3. *Let the functions* φ_i $(i = 1, \dots, n)$ *be absolutely continuous and let there exist a set of zero measure* $I_0 \subset [a, b]$, *matrices* $\mathcal{P}_k \in \mathbb{R}_+^{n\times n}$ $(k = 1, 2)$ *and a vector* $q_0 = (q_{0i})_{i=1}^n$ *with positive components* q_{0i} $(i = 1, \dots, n)$ *such that on the set* $([a, b] \setminus I_0) \times \mathbb{R}^{2n}$ *the inequality*

$$g(t, x, y) \geq \phi'(t)\Big(\mathcal{P}_1\phi^{-1}(t)|x| + \mathcal{P}_2\phi^{-1}(\tau(t))|y| + q_0\Big)$$

is fulfilled. If, moreover, $r(\mathcal{P}_1 + \mathcal{P}_2) \geq 1$, *then the problem* (3), (2) *has no solution.*

Along with the problem (1), (2), we consider the perturbed problem

$$\frac{dy(t)}{dt} = f(y)(t) + h(t), \tag{7}$$

$$\limsup_{t\to a} \big\|\phi^{-1}(t)y(t)\big\| < +\infty, \tag{8}$$

and introduce the following

Definition. The problem (1), (2) is called well-posed if there exists a positive number ρ such that for an arbitrary function $h \in L([a, b]; \mathbb{R}^n)$, satisfying the condition

$$\nu_\phi(h) = \sup\left\{\left\|\phi^{-1}(t)\int_a^t |h(s)|\, ds\right\| :\ a < t \leq b\right\} < +\infty,$$

the problem (7), (8) is uniquely solvable and its solution admits the estimate

$$\|y - x\|_{C_\phi} \leq \rho\nu_\phi(h),$$

where x is a solution of the problem (1), (2).

Theorem 4. *Let there exist a matrix* $\mathcal{P} \in \mathbb{R}_+^{n\times n}$ *such that* $r(\mathcal{P}) < 1$, *and for arbitrary vector functions* x *and* $y \in C_\phi([a, b]; \mathbb{R}^n)$ *in the interval* $[a, b]$ *the inequality*

$$\int_a^t \Big[\operatorname{sgn}(y(s))\big(f(x + y)(s) - f(x)(s)\big)\Big]_+ ds \leq \phi(t)\mathcal{P}|y|_{C_\phi}$$

is fulfilled. If, moreover,

$$\sup\left\{\left\|\phi^{-1}(t)\int_a^t \big|f(0)(s)\big|\, ds\right\| :\ a < t \leq b\right\} < +\infty,$$

then the problem (1), (2) *is well-posed.*

Corollary 2. *Let the functions φ_i $(i = 1, \dots, n)$ be absolutely continuous and let there exist a set of zero measure $I_0 \subset [a, b]$ and matrices $\mathcal{P}_k \in \mathbb{R}_+^{n\times n}$ $(k = 1, 2)$ such that $r(\mathcal{P}_1 + \mathcal{P}_2) < 1$, and for any $t \in [a, b] \setminus I_0$, x, $\overline{x}$, y and $\overline{y} \in \mathbb{R}^n$ the inequality*

$$\operatorname{sgn}(\overline{x})\Big(g\big(t, x+\overline{x}, y+\overline{y}\big) - g(t, x, y)\Big) \le \phi'(t)\Big(\mathcal{P}_1\phi^{-1}(t)|\overline{x}| + \mathcal{P}_2\phi^{-1}(\tau(t))|\overline{y}|\Big)$$

is fulfilled. If, moreover,

$$\sup\left\{\left\|\phi^{-1}(t)\int\limits_a^t \big|g(s, 0, 0)\big|\,ds\right\| : \ a < t \le b\right\} < +\infty,$$

then the problem (3), (2) *is well-posed.*

From Theorem 3 and Corollary 2 it follows

Corollary 3. *Let the functions φ_i $(i = 1, \dots, n)$ be absolutely continuous and*

$$g(t, x, y) = \phi'(t)\Big(\mathcal{P}_1\phi^{-1}(t)|x| + \mathcal{P}_2\phi^{-1}(\tau(t))|y| + q_0\Big),$$

where $\mathcal{P}_k \in \mathbb{R}_+^{n\times n}$ $(k = 1, 2)$, and $q_0 \in \mathbb{R}_+^n$ is the vector with positive components. Then the problem (3), (2) *is well-posed if and only if*

$$r(\mathcal{P}_1 + \mathcal{P}_2) < 1.$$

Remark 2. According to Corollary 3, the inequality $r(\mathcal{P}) < 1$ ($r(\mathcal{P}_1 + \mathcal{P}_2) < 1$) in Theorem 4 (in Corollary 2) is unimprovable and it cannot be replaced by the inequality $r(\mathcal{P}) \le 1$ ($r(\mathcal{P}_1 + \mathcal{P}_2) \le 1$).

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