Mem. Differential Equations Math. Phys. 6(1995), 113–115

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FUNCTIONAL DIFFERENTIAL EQUATIONS WITH PULSES

(Reported on September 22, 1995)

The following equation is under consideration

$$\dot{x}(t) + \sum_{j=1}^{k} A_j(t)x(h_j(t)) = f(t), \quad t \in [0,b], \tag{1}$$
$$x(\xi) = 0, \quad \text{if} \quad \xi < 0$$
$$x(t_i) = \mathcal{B}_i x(t_i - 0), \quad i = 1, 2, \dots, m, \tag{2}$$

where

$$0 = t_0 < t_1 < \dots < t_m < t_{m+1} = b, \quad h_i(t) \le t, \quad t \in [0,b],$$
$$\det B_i \ne 0, \quad i = 1, 2, \dots, m.$$

Under a solution of (1)-(2) we understand an absolutely continuous on every interval $[t_{i-1}, t_i)$, $i = 1, \dots, m+1$, function $x : [0,b] \to \mathbf{R^n}$ satisfying at the points t_i the condition (2), and satisfying for almost all $t \in [0,b]$ the equation (1).

Let us point out that equations of type (1)-(2) are intensively studied. A large number of works are devoted to such equations. Among them there are several monographs (see, for example [1], [2], [3]) which have appeared recently.

Define by $\mathbf{D(0, t_1, \dots, t_m, b)}$ the Banach space of functions $x : [0,b] \to \mathbf{R^n}$ absolutely continuous on every interval $[t_i, t_{i+1})$, $i = 0, 1, \dots, m$, and satisfying at the points t_i, $i = 1, 2, \dots, m$, the condition (2). Denote by $\mathbf{D(0, b)}$ the Banach space of absolutely continuous functions $y : [0,b] \to \mathbf{R^n}$.

Assume that the $n \times n$ matricies A_j and the functions h_j, $j = 1, \dots, k$, are chosen such that the operator $\mathcal{L} : \mathbf{D(0, t_1, \dots, t_m, b)} \to \mathbf{L_p(0, b)}$, $1 \le p \le \infty$, defined by

$$(\mathcal{L}x)(t) = \dot{x}(t) + \sum_{j=1}^{k} A_j(t)x(h_j(t)) = f(t), \quad t \in [0,b],$$
$$x(\xi) = 0, \quad \text{if} \quad \xi < 0$$

is continuous. The specialty of the equation

$$\mathcal{L}x = f \tag{3}$$

in comparison with the types of the functional differential equations studied before is that the domain of the operator $\mathcal{L}$ consists not of absolutely continuous functions but of piecewise absolutely continuous ones.

1991 *Mathematics Subject Classification.* 47H05.

Key words and phrases. Functional differential equation with pulses, Cauchy Matrix.

Basing on this fact, we suggest the following scheme for investigating of the equation (3). Namely, between the two spaces $\mathbf{D(0, t_1, \dots, t_m, b)}$ and $\mathbf{D(0, b)}$ a linear isomorphism can be established. Indeed, let $J : \mathbf{D(0, b)} \to \mathbf{D(0, t_1, \dots, t_m, b)}$ be a linear isomorphism. Then the substitution

$$x = Jy \tag{4}$$

transforms (3) to

$$\widetilde{\mathcal{L}} y = f, \tag{5}$$

where $\widetilde{\mathcal{L}} = \mathcal{L}J$ is a linear continuous operator acting from the space of absolutely continuous functions $\mathbf{D(0, b)}$ into the Lebesgue space $\mathbf{L_p(0, b)}$, $1 \le p \le \infty$.

Lemma 1. *The equality*

$$x(t) = y(t) + \sum_{i=1}^{m} Q_i(t) y(t_i), \tag{6}$$

where

$$Q_i(t) = \Big[\chi_{[t_i, t_{i+1})}(t) + \sum_{j=1}^{m-i} \chi_{[t_{i+j}, t_{i+j+1})}(t) \prod_{k=1}^{j} B_{i+j-1-k} \Big] (B_i - E),$$
$$i = 1, 2, \dots, m,$$

establishes linear isomorphism between the spaces $\mathbf{D(0, t_1, \dots, t_m, b)}$ *and* $\mathbf{D(0, b)}$.

Here $\chi_{(\alpha,\beta)}$ is the characteristic function of the interval (α, β), E is the identity matrix, $B_0 = E$. Substituting (6) into (1), we obtain

$$\dot{y}(t) + \sum_{j=1}^{k+m} A_j(t) y(h_j(t)) = f(t), \quad t \in [0, b], \tag{7}$$
$$y(\xi) = 0, \quad \text{if} \quad \xi < 0,$$

where

$$A_{k+i}(t) = \sum_{j=1}^{k} A_j(t) Q_i(h_j(t)), \quad h_{k+i}(t) = t_i, \quad i = 1, 2, \dots, m.$$

Equation (7) is of the type of functional differential equations with delayed argument. Basics of general theory for such equations where introduced in [4]. The essential role in that investigations is assigned to the Cauchy matrix. Due to this, establishing connection between the Cauchy matrix $C(t, s)$ of (1)-(2) and the Cauchy matrix $\widetilde{C}(t, s)$ of (7) proves to be very useful.

Lemma 2. *The equality*

$$C(t, s) = \widetilde{C}(t, s) + \sum_{i=1}^{m} \chi_{[t_i, b)} Q_i(t) \widetilde{C}(t_i, s), \quad 0 \le s \le t \le b,$$

determines a relation between the Cauchy matrices of the equations (1) − (2) *and* (7).

The equations which $\widetilde{C}(t,s)$ satisfies as a function in both the first and the second arguments are found in the works of V. P. Maksimov (see, for example, [5]). With the help of the last lemma, it is possible to to take advantage of those statements for constructing the corresponding equations for $C(t,s)$.

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