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L. P. Castro, R. C. Guerra, and N. M. Tuan

ON INTEGRAL OPERATORS GENERATED BY THE FOURIER TRANSFORM AND A REFLECTION

Dedicated to the memory of Professor Boris Khvedelidze (1915–1993) on the 100th anniversary of his birthday

Abstract. We present a detailed study of structural properties for certain algebraic operators generated by the Fourier transform and a reflection. First, we focus on the determination of the characteristic polynomials of such algebraic operators, which, e.g., exhibit structural differences when compared with those of the Fourier transform. Then, this leads us to the conditions that allow one to identify the spectrum, eigenfunctions, and the invertibility of this class of operators. A Parseval type identity is also obtained, as well as the solvability of integral equations generated by those operators. Moreover, new convolutions are generated and introduced for the operators under consideration.

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რეზიუმე. ჩვენ წარმოგიდგენთ ფურიეს გარდაქმნითა და არეკვლის ოპერატორებით წარმოქნილი ზოგიერთი ალგებრული ოპერატორის სტრუქტურული თვისებების დეტალურ კვლევას. პირველ რიგში, ყურადღება გამახვილებულია ასეთი ალგებრული ოპერატორებისთვის მახასიათებელი პოლინომების განსაზღვრაზე, რომელიც დაგვანახებს სტრუქტურულ განსხვავებებს ფურიეს გარდაქმნასთან შედარებით. ამას მივყავართ პირობებთან, რომლებიც საშუალებას მოგვცემს მოვახდინოთ სპექტრის, საკუთრივი ფუნციებისა და ამ კლასში შებრუნებადი ოპერატორების იდენტიფიცირება. მიღებულია პარსევალის ტიპის იგივეობა და შესწავლილია ამ ოპერატორებით წარმოქმნილი ინტეგრალური განტოლებების ამოხსნადობა. განხილული ოპერატორებისთვის შემოტანილია ახალი ნახვევის ოპერატორის ცნება.

1. Introduction

In several types of mathematical applications it is useful to apply more than once the Fourier transformation (or its inverse) to the same object, as well as to use algebraic combinations of the Fourier transform. This is the case e.g. in wave diffraction problems which – although being initially modeled as boundary value problems – can be translated into single equations by applying operator theoretical methods and convenient operators upon the use of algebraic combinations of the Fourier transform (cf. [8–10]). Additionally, in such processes it is also useful to construct relations between convolution type operators [7], generated by the Fourier transform, and some simpler operators like the reflection operator; cf. [5, 6, 11, 21]. Some of the most known and studied classes of this type of operators are the Wiener–Hopf plus Hankel and Toeplitz plus Hankel operators.

It is also well-known that several of the most important integral transforms are involutions when considered in appropriate spaces. For instance, the Hankel transform J, the Cauchy singular integral operator S on a closed curve, and the Hartley transforms (typically denoted by H_1 and H_2, see [2–4, 17]) are involutions of order 2. Moreover, the Fourier transform F and the Hilbert transform $\mathcal{H}$ are involutions of order 4 (i.e. $\mathcal{H}^4 = I$, in this case simply because $\mathcal{H}$ is an anti-involution in the sense that $\mathcal{H}^2 = -I$).

Those involution operators possess several significant properties that are useful for solving problems which are somehow characterized by those operators, as well as several kinds of integral equations, and ordinary and partial differential equations with transformed argument (see [1, 15, 16, 18, 20, 22–26]).

Let $W : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ be the reflection operator defined by

$$(W\varphi)(x) := \varphi(-x),$$

and let now $\langle \cdot\, , \,\cdot \rangle_{L^2(\mathbb{R}^n)}$ denote the usual inner product in $L^2(\mathbb{R}^n)$. Moreover, let F denote the Fourier integral operator given by

$$(Ff)(x) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int\limits_{\mathbb{R}^n} e^{-i\langle x,y\rangle} f(y)\, dy.$$

In view of the above-mentioned interest, in the present work we propose a detailed study of some of the fundamental properties of the following operator, generated by the operators I (identity operator), F and W:

$$T := aI + bF + cW : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \tag{1.1}$$

where $a, b, c \in \mathbb{C}$. In very general terms, we can consider the operator T as a Fourier integral operator with reflection which allows to consider similar operators to the Cauchy integral operator with reflection (see [12–14, 19] and the references therein). Anyway, it is also well-known that $F^2 = W$. In this paper, the operator T, together with its properties, can be seen as a starting point to further studies of the Fourier integral operators with more general shifts that will be addressed in the forthcoming papers.

The paper is organized as follows. In the next section, we will justify that T is an algebraic operator and we will deduce their characteristic polynomials for distinct cases of the parameters a, b and c. Then, the conditions that allow to identify the spectrum, eigenfunctions, and the invertibility of the operator are obtained. Moreover, Parseval type identities are derived, and the solvability of integral equations generated by those operators is described. In addition, new operations for the operators under consideration are introduced such that they satisfy the corresponding property of the classical convolution.

2. Characteristic Polynomials

In order to have some global view on corresponding linear operators, we start by recalling the concept of algebraic operators.

An operator L defined on the linear space X is said to be algebraic if there exists a non-zero polynomial $P(t)$, with variable t and coefficients in the complex field $\mathbb{C}$, such that $P(L) = 0$. Moreover, the algebraic operator L is said be of order N if $P(L) = 0$ for a polynomial $P(t)$ of degree N, and $Q(L) \neq 0$ for any polynomial Q of degree less than N. In such a case, P is said to be the characteristic polynomial of L (and its roots are called the characteristic roots of L). As an example, for the operators J, S, H_1, H_2 and $\mathcal{H}$, mentioned in the previous section, we may directly identify their characteristic polynomials in the following corresponding way:

$$\begin{gathered} P_J(t) = t^2 - 1; \quad P_S(t) = t^2 - 1; \\ P_{H_1}(t) = t^2 - 1; \quad P_{H_2}(t) = t^2 - 1; \quad P_{\mathcal{H}}(t) = t^2 + 1. \end{gathered}$$

As above mentioned, it is well-known that the operator F is an involution of order 4 (thus $F^4 = I$, where I is the identity operator in $L^2(\mathbb{R}^n)$). In other words, F is an algebraic operator which has a characteristic polynomial given by $P_F(t) = t^4 - 1$. Such polynomial has obviously the following four characteristic roots: 1, $-i$, -1, i.

We will consider the following four projectors correspondingly generated with the help of F:

$$\begin{aligned} P_0 &= \frac{1}{4}\,(I + F + F^2 + F^3), \\ P_1 &= \frac{1}{4}\,(I + iF - F^2 - iF^3), \\ P_2 &= \frac{1}{4}\,(I - F + F^2 - F^3), \\ P_3 &= \frac{1}{4}\,(I - iF - F^2 + iF^3), \end{aligned}$$

and that satisfy the identities

$$\begin{cases} P_j P_k = \delta_{jk} P_k & \text{for } \ j,k = 0,1,2,3, \\ P_0 + P_1 + P_2 + P_3 = I, \\ F = P_0 - iP_1 - P_2 + iP_3, \end{cases} \tag{2.1}$$

where

$$\delta_{jk} = \begin{cases} 0, & \text{if } \ j \neq k, \\ 1, & \text{if } \ j = k. \end{cases}$$

Moreover, we have

$$F^2 = P_0 - P_1 + P_2 - P_3, \tag{2.2}$$

$$F^4 = P_0 + P_1 + P_2 + P_3 = I. \tag{2.3}$$

It is also clear that

$$\alpha P_0 + \beta P_1 + \gamma P_2 + \delta P_3 = 0$$

if and only if

$$\alpha = \beta = \gamma = \delta = 0.$$

Having in mind this property, in the sequel, for denoting the operator

$$A = \alpha P_0 + \beta P_1 + \gamma P_2 + \delta P_3,$$

we will use the notation $(\alpha;\beta;\gamma;\delta) = A$.

Obviously, $A^n = (\alpha^n;\beta^n;\gamma^n;\delta^n)$, for every $n \in \mathbb{N}$, where we admit that $A^0 = I$.

Theorem 2.1. *Let us consider the operator*

$$T = aI + bF + cW, \quad a,b,c \in \mathbb{C}. \tag{2.4}$$

The characteristic polynomial of this T is:

(i)

$$P_T(t) = t^2 - 2at + (a^2 - c^2) \tag{2.5}$$

if and only if

$$b = 0 \ \text{ and } \ c \neq 0; \tag{2.6}$$

(ii)

$$\begin{aligned} P_T(t) = t^3 &- \big[(3a+c) + ib\big]t^2 + \big[3(a^2-c^2) + 2a(c+ib)\big]t \\ &+ \big[-a^3 - ia^2b - a^2c - 3b^2c + 3ac^2 + ibc^2 + c^3\big] \end{aligned} \tag{2.7}$$

if and only if

$$bc \neq 0 \ \text{ and } \ \left(c = \frac{b}{2}(1-i) \ \text{ or } \ c = -\frac{b}{2}(1+i) \right); \tag{2.8}$$

(iii)

$$P_T(t) = t^3 + \big[-(3a+c)+ib\big]t^2 + \big[3(a^2-c^2)+2a(c-ib)\big]t \\ + \big[-a^3+ia^2b-a^2c-3b^2c+3ac^2-ibc^2+c^3\big] \tag{2.9}$$

if and only if

$$bc \neq 0 \ \ and \ \ \left(c = \frac{b}{2}(1+i) \ \ or \ \ c = -\frac{b}{2}(1-i)\right); \tag{2.10}$$

(iv)

$$P_T(t) = t^4 - 4at^3 + (6a^2-2c^2)t^2 + (-4a^3-4b^2c+4ac^2)t \\ + (a^2-c^2)^2 + b^2(4ac-b^2) \tag{2.11}$$

if and only if

$$\begin{cases} c \neq \frac{b}{2}(1-i), \\ c \neq -\frac{b}{2}(1+i), \\ c \neq \frac{b}{2}(1+i), \\ c \neq -\frac{b}{2}(1-i) \end{cases} \tag{2.12}$$

and $b \neq 0$.

Proof. We can write the operator T in the following form:

$$\begin{aligned} T &= a(P_0+P_1+P_2+P_3) + b(P_0 - iP_1 - P_2 + iP_3) \\ &\quad + c(P_0 - P1 + P_2 - P_3) \\ &= (a+c+b)P_0 + (a-c-ib)P_1 \\ &\quad + (a+c-b)P_2 + (a-c+ib)P_3 \\ &= (a+c+b; a-c-ib; a+c-b; a-c+ib). \end{aligned} \tag{2.13}$$

In order to determine the characteristic polynomial of the operator T, for each one of the cases, we may begin by considering a polynomial of order 2, that is, $P_T(t) = t^2 + mt + n$. In fact, a polynomial of order 1 is the characteristic polynomial of the operator T if and only if $b = 0$ and $c = 0$, but in this case, we obtain the trivial operator $T = aI$. That $P_T(t)$ is the characteristic polynomial of T if and only if $P_T(T) = 0$ and if there does not exist any polynomial Q with $\deg(Q) < 2$ such that $Q(T) = 0$.

Moreover, the condition $P_T(T) = 0$ is equivalent to

$$\begin{cases} (a+c+b)^2 + m(a+c+b) + n = 0, \\ (a-c-ib)^2 + m(a-c-ib) + n = 0, \\ (a+c-b)^2 + m(a+c-b) + n = 0, \\ (a-c+ib)^2 + m(a-c+ib) + n = 0. \end{cases}$$

The solution of this system is $b = 0$ and $c = 0$ (but in this case, we obtain the trivial operator $T = aI$) or that

$$\begin{cases} b = 0, \\ c \neq 0, \\ m = -2a, \\ n = a^2 - c^2. \end{cases}$$

So, if $b = 0$ and $c \neq 0$, then $P_T(t) = t^2 - 2at + a^2 - c^2$. Indeed, by using the operator T written in the above form (2.13), it is possible to verify that $P_T(T) = 0$:

$$\begin{aligned} &T^2 - 2aT + (a^2 - c^2)I \\ &= \big((a+c)^2; (a-c)^2; (a+c)^2; (a-c)^2\big) - 2a(a+c; a-c; a+c; a-c) \\ &\qquad\qquad + (a^2 - c^2)(1; 1; 1; 1) = (0; 0; 0; 0). \end{aligned}$$

Now, we will prove that there does not exist any polynomial Q with $\deg(Q) < 2$ such that $Q(T) = 0$.

Suppose that there exists a polynomial Q, defined by $Q(t) = t + m$, that satisfies $Q(T) = 0$. In this case, we would have the following system of equations:

$$\begin{cases} (a+c) + m = 0, \\ (a-c) + m = 0, \end{cases}$$

which is equivalent to $c = 0$, but this is not the case under the conditions imposed before.

Conversely, assume that $P_T(t) = t^2 - 2at + (a^2 - c^2)$ is the characteristic polynomial of T. Thus, $P_T(T) = 0$, which is equivalent to

$$\begin{aligned} 0 = T^2 &- 2aT + (a^2 - c^2)I \\ &= \big((a+c)^2; (a-c)^2; (a+c)^2; (a-c)^2\big) \\ &\quad - 2a(a+c; a-c; a+c; a-c) + (a^2 - c^2)(1; 1; 1; 1). \end{aligned}$$

This implies that $b = 0$ and $c = 0$ (which is the case of the trivial operator) or that $b = 0$. So, case (i) is proved.

To obtain the characteristic polynomial for the other cases, we have to consider polynomials with degree greater than 2. So, let us consider a polynomial $P_T(t) = t^3 + mt^2 + nt + p$ and repeat the same procedure. Thus, $P_T(T) = 0$ is equivalent to

$$\begin{cases} (a+c+b)^3 + m(a+c+b)^2 + n(a+c+b) + p = 0, \\ (a-c-ib)^3 + m(a-c-ib)^2 + n(a-c-ib) + p = 0, \\ (a+c-b)^3 + m(a+c-b)^2 + n(a+c-b) + p = 0, \\ (a-c+ib)^3 + m(a-c+ib)^2 + n(a-c+ib) + p = 0. \end{cases}$$

This system has as solutions $b = 0$ and $c = 0$ (in this case, we obtain the operator $T = aI$) or $b = 0$ and $c \neq 0$ (but for this case, the characteristic polynomial is of order 2 – case (i)) or

$$\begin{cases} b \neq 0, \\ c = \dfrac{b}{2}(1-i) \;\text{ or }\; c = -\dfrac{b}{2}(1+i), \\ m = -\big[(3a+c)+ib\big], \\ n = 3(a^2-c^2)+2a(c+ib), \\ p = -a^3 - ia^2b - a^2c - 3b^2c + 3ac^2 + ibc^2 + c^3 \end{cases}$$

or

$$\begin{cases} b \neq 0, \\ c = \dfrac{b}{2}(1+i) \;\text{ or }\; c = -\dfrac{b}{2}(1-i), \\ m = \big[-(3a+c)+ib\big], \\ n = 3(a^2-c^2)+2a(c-ib), \\ p = -a^3 + ia^2b - a^2c - 3b^2c + 3ac^2 - ibc^2 + c^3. \end{cases}$$

So,

- if $c = \frac{b}{2}(1-i)$ or $c = -\frac{b}{2}(1+i)$, then
$$\begin{aligned} P_T(t) = t^3 &- \big[(3a+c)+ib\big]t^2 + \big[3(a^2-c^2)+2a(c+ib)\big]t \\ &+ \big[-a^3 - ia^2b - a^2c - 3b^2c + 3ac^2 + ibc^2 + c^3\big]; \end{aligned}$$

- If $c = \frac{b}{2}(1+i)$ or $c = -\frac{b}{2}(1-i)$, then
$$\begin{aligned} P_T(t) = t^3&\big[-(3a+c)+ib\big]t^2 + \big[3(a^2-c^2)+2a(c-ib)\big]t \\ &+ \big[-a^3 + ia^2b - a^2c - 3b^2c + 3ac^2 - ibc^2 + c^3\big]. \end{aligned}$$

If we consider the case $c = \frac{b}{2}(1-i)$, by using the operator T written in the above form (2.13), we can prove that $P_T(T) = 0$. Indeed,

$$\begin{aligned} T^3 &- \big[(3a+c)+ib\big]T^2 + \big[3(a^2-c^2)+2a(c+ib)\big]T \\ &+ \big[-a^3 - ia^2b - a^2c - 3b^2c + 3ac^2 + ibc^2 + c^3\big]I \\ &= \big([a+c+b]^3; [a-c-ib]^3; [a+c-b]^3; [a-c+ib]^3\big) \\ &- \big[(3a+c)+ib\big]\big([a+c+b]^2; [a-c-ib]^2; [a+c-b]^2; [a-c+ib]^2\big) \\ &+ \big[3(a^2-c^2)+2a(c+ib)\big](a+c+b; a-c-ib; a+c-b; a-c+ib) \\ &+ \big[-a^3 - ia^2b - a^2c - 3b^2c + 3ac^2 + ibc^2 + c^3\big](1;1;1;1) \\ &= (0;0;0;0). \end{aligned}$$

Now we will prove that there does not exist any polynomial G with $\deg(G) < 3$ such that $G(T) = 0$.

Suppose that there exists a polynomial G, defined by $G(t) = t^2 + mt + n$, that satisfies $G(T) = 0$. In this case, we would have the following system of equations:

$$\begin{cases} (a+c+b)^2 + m(a+c+b) + n = 0, \\ (a-c-ib)^2 + m(a-c-ib) + n = 0, \\ (a+c-b)^2 + m(a+c-b) + n = 0, \\ (a-c+ib)^2 + m(a-c+ib) + n = 0. \end{cases}$$

For $c = \frac{b}{2}\,(1-i)$, we find that the second and third equations are equivalent. So, the last system is equivalent to

$$\begin{cases} (a+c+b)^2 + m(a+c+b) + n = 0, \\ (a-c-ib)^2 + m(a-c-ib) + n = 0, \\ (a-c+ib)^2 + m(a-c+ib) + n = 0, \end{cases}$$

which is equivalent to $b = 0$. This is a contradiction under the initial conditions of the theorem. In this way, we can say that there does not exist a polynomial G such that $\deg(G) < 3$ and this fulfills $G(T) = 0$.

So, we can conclude that under these conditions,

$$\begin{aligned} P_T(t) = t^3 &- \big[(3a+c)+ib\big]t^2 + \big[3(a^2-c^2)+2a(c+ib)\big]t \\ &+ \big[-a^3 - ia^2b - a^2c - 3b^2c + 3ac^2 + ibc^2 + c^3\big]. \end{aligned}$$

Conversely, suppose that $P_T(t)$ is the characteristic polynomial of T. In this case, we have $P_T(T) = 0$, which is equivalent to

$$\begin{aligned} 0 = T^3 &- \big[(3a+c)+ib\big]T^2 + \big[3(a^2-c^2)+2a(c+ib)\big]T \\ &+ \big[-a^3 - ia^2b - a^2c - 3b^2c + 3ac^2 + ibc^2 + c^3\big] \\ = &\big([a+c+b]^3; [a-c-ib]^3; [a+c-b]^3; [a-c+ib]^3\big) \\ &- \big[(3a+c)+ib\big]\big([a+c+b]^2; [a-c-ib]^2; [a+c-b]^2; [a-c+ib]^2\big) \\ &+ \big[3(a^2-c^2)+2a(c+ib)\big](a+c+b; a-c-ib; a+c-b; a-c+ib) \\ &+ \big[-a^3 - ia^2b - a^2c - 3b^2c + 3ac^2 + ibc^2 + c^3\big](1;1;1;1). \end{aligned}$$

This implies that $b = 0$ (which is the case (i)), $c = \frac{b}{2}\,(1-i)$ or $c = -\frac{b}{2}\,(1+i)$.

The remaining conditions in (2.8) and (2.10) can be proved in a similar way.

If

$$\begin{cases} c \neq \dfrac{b}{2}\,(1-i), \\ c \neq -\dfrac{b}{2}\,(1+i), \\ c \neq \dfrac{b}{2}\,(1+i), \\ c \neq -\dfrac{b}{2}\,(1-i), \end{cases}$$

then (2.7) and (2.9) are not anymore characteristic polynomials of T.

Additionally, if we consider a polynomial $P_T(t) = t^4 + mt^3 + nt^2 + pt + q$, such that $P_T(T) = 0$, we obtain the following system of equations:

$$\begin{cases} (a+c+b)^4 + m(a+c+b)^3 + n(a+c+b)^2 + p(a+c+b) + q = 0, \\ (a-c-ib)^4 + m(a-c-ib)^3 + n(a-c-ib)^2 + p(a+c+b) + q = 0, \\ (a+c-b)^4 + m(a+c-b)^3 + n(a+c-b)^2 + p(a+c+b) + q = 0, \\ (a-c+ib)^4 + m(a-c+ib)^3 + n(a-c+ib)^2 + p(a+c+b) + q = 0. \end{cases}$$

This is equivalent to $b = c = 0$ (which is the trivial case $T = aI$) or to $b = 0$ and $c \neq 0$ (which is the case (i)) or to the cases (ii) and (iii) or

$$\begin{cases} b \neq 0, \\ m = -4a, \\ n = 6a^2 - 2c^2, \\ p = -4a^3 - 4b^2c + 4ac^2, \\ q = (a^2 - c^2) + b^2(4ac - b^2). \end{cases}$$

In this case, we can say that if $b \neq 0$ and if (2.12) holds, then

$$\begin{aligned} P_T(t) = t^4 - 4at^3 + (6a^2 - 2c^2)t^2 + (-4a^3 - 4b^2c + 4ac^2)t \\ + (a^2 - c^2)^2 + b^2(4ac - b^2). \end{aligned}$$

On the other hand, with the use of operator T (written as in (2.13)), we can directly prove that $P_T(T) = 0$. Indeed,

$$\begin{aligned} T^4 - 4aT^3 + (6a^2 - 2c^2)T^2 + (-4a^3 - 4b^2c + 4ac^2)T \\ + \big[(a^2 - c^2)^2 + b^2(4ac - b^2)\big]I \\ = \big([a+c+b]^4; [a-c-ib]^4; [a+c-b]^4; [a-c+ib]^4\big) \\ - 4a\big([a+c+b]^3; [a-c-ib]^3; [a+c-b]^3; [a-c+ib]^3\big) \\ + (6a^2 - 2c^2)\big([a+c+b]^2; [a-c-ib]^2; [a+c-b]^2; [a-c+ib]^2\big) \\ + (-4a^3 - 4b^2c + 4ac^2)(a+c+b; a-c-ib; a+c-b; a-c+ib) \\ + \big[(a^2 - c^2)^2 + b^2(4ac - b^2)\big](1;1;1;1) = (0;0;0;0). \end{aligned}$$

Now, we will prove that there does not exist any polynomial G with $\deg(G) < 4$ that satisfies $G(T) = 0$ under these conditions. Towards this end, suppose that there exists a polynomial G, defined by $G(t) = t^3 + mt^2 + nt + p$, that satisfies $G(T) = 0$. In this case, we would have the following system of equations:

$$\begin{cases} (a+c+b)^3 + m(a+c+b)^2 + n(a+c+b) + p = 0, \\ (a-c-ib)^3 + m(a-c-ib)^2 + n(a-c-ib) + p = 0, \\ (a+c-b)^3 + m(a+c-b)^2 + n(a+c-b) + p = 0, \\ (a-c+ib)^3 + m(a-c+ib)^2 + n(a-c+ib) + p = 0, \end{cases}$$

which is equivalent to $b = 0$ or $c = \frac{b}{2}(1-i)$ or $c = -\frac{b}{2}(1+i)$ or $c = \frac{b}{2}(1+i)$ or $c = -\frac{b}{2}(1-i)$.

This is a contradiction under the conditions of part (iii) of the Theorem. In this way, we can say that there does not exist a polynomial G with $\deg(G) < 4$ that satisfies $G(T) = 0$.

So, we can conclude that under these conditions

$$\begin{aligned} P_T(t) = t^4 - 4at^3 + (6a^2 - 2c^2)t^2 + (-4a^3 - 4b^2c + 4ac^2)t \\ + (a^2 - c^2)^2 + b^2(4ac - b^2). \end{aligned}$$

Conversely, suppose that $P_T(t)$ is the characteristic polynomial of T. Consequently, we have $P_T(T) = 0$, which is equivalent to

$$\begin{aligned} 0 &= T^4 - 4aT^3 + (6a^2 - 2c^2)T^2 + (-4a^3 - 4b^2c + 4ac^2)T \\ &\quad + (a^2 - c^2)^2 + b^2(4ac - b^2) \\ &= \big([a+c+b]^4; [a-c-ib]^4; [a+c-b]^4; [a-c+ib]^4\big) \\ &\quad - 4a\big([a+c+b]^3; [a-c-ib]^3; [a+c-b]^3; [a-c+ib]^3\big) \\ &\quad + (6a^2 - 2c^2)\big([a+c+b]^2; [a-c-ib]^2; [a+c-b]^2; [a-c+ib]^2\big) \\ &\quad + (-4a^3 - 4b^2c + 4ac^2)(a+c+b; a-c-ib; a+c-b; a-c+ib) \\ &\quad + \big[(a^2 - c^2)^2 + b^2(4ac - b^2)\big](1;1;1;1). \end{aligned}$$

This condition is universal, and hence this case is proved. $\square$

3. Invertibility, Spectrum and Integral Equations

We will now investigate the operator T in view of invertibility, spectrum, convolutions and associated integral equations. This will be done in the next subsections, by separating different cases of the parameters a, b and c, due to their corresponding different nature. The case of $b = 0$ and $c \neq 0$ is here omitted simply because this is the easiest case (in the sense that for this case we even do not have an integral structure: T is just a combination of the reflection and the identity operators).

3.1. **Case $b \neq 0$ and $c = \frac{b}{2}(1-i)$.** In this subsection we will concentrate on the properties of the operator $T = aI + bF + cW$, $a, b, c \in \mathbb{C}$, $b, c \neq 0$, in the special case of $c = \frac{b}{2}(1-i)$ (whose importance is justified by the results of Section 2).

If we consider the following characteristic polynomial:

$$\begin{aligned} P_T(t) = t^3 - \big[(3a+c) + ib\big]t^2 + \big[3a^2 + 2ib(a+c) + 2ac - (b^2 + c^2)\big]t \\ + \big[-a^3 - ia^2b + ab^2 + ib^3 - a^2c - 2iabc - b^2c + ac^2 - ibc^2 + c^3\big] \end{aligned}$$

and if $c := \frac{b}{2}(1-i)$, we obtain that this polynomial is equivalent to

$$\begin{aligned} P_T(t) = t^3 &- \Big[3a + \frac{b}{2}(1+i)\Big]t^2 + \Big[3a^2 + ab(1+i) + \frac{3}{2}ib^2\Big]t \\ &+ \Big[-a^3 - \frac{1}{2}a^2b(1+i) - \frac{3}{2}iab^2 - \frac{5}{4}b^3(1-i)\Big]. \end{aligned}$$

3.1.1. *Invertibility and spectrum.* We will now present a characterization for the invertibility and the spectrum of the present T.

Theorem 3.1. *The operator T (with $c = \frac{b}{2}(1-i)$) is an invertible operator if and only if*

$$a + \Big(\frac{3}{2} - \frac{i}{2}\Big)b \neq 0, \quad a - \Big(\frac{1}{2} + \frac{i}{2}\Big)b \neq 0 \text{ and } a - \Big(\frac{1}{2} - \frac{3i}{2}\Big)b \neq 0. \tag{3.1}$$

In this case, the inverse operator is defined by

$$\begin{aligned} T^{-1} = &\frac{1}{a^3 + \frac{1}{2}a^2b(1+i) + \frac{3}{2}iab^2 + \frac{5}{4}b^3(1-i)} \\ &\times \Big[T^2 - \Big(3a + \frac{b}{2}(1+i)\Big)T + \Big(3a^2 + ab(1+i) + \frac{3}{2}ib^2\Big)I\Big]. \end{aligned} \tag{3.2}$$

Proof. Suppose that the operator T is invertible. Choosing the Hermite functions φ_k, we have:

- for $|k| \equiv 0 \pmod 4$, $(T\varphi_k)(x) = (a + \frac{3}{2}b - \frac{i}{2}b)\varphi_k(x)$, which implies that $a + (\frac{3}{2} - \frac{i}{2})b \neq 0$;
- for $|k| \equiv 1, 2 \pmod 4$, $(T\varphi_k)(x) = (a - \frac{b}{2} - \frac{i}{2}b)\varphi_k(x)$. So, $a - (\frac{1}{2} + \frac{i}{2})b \neq 0$;
- for $|k| \equiv 3 \pmod 4$, $(T\varphi_k)(x) = (a - \frac{b}{2} + \frac{3i}{2}b)\varphi_k(x)$, which implies that $a - (\frac{1}{2} - \frac{3i}{2})b \neq 0$.

Summarizing, we have:

$$(T\varphi_k)(x) = \begin{cases} \Big(a + \Big(\frac{3}{2} - \frac{i}{2}\Big)b\Big)\varphi_k(x) & \text{if } |k| \equiv 0 \pmod 4, \\ \Big(a - \Big(\frac{1}{2} + \frac{i}{2}\Big)b\Big)\varphi_k(x) & \text{if } |k| \equiv 1, 2 \pmod 4, \\ \Big(a - \Big(\frac{1}{2} - \frac{3i}{2}\Big)b\Big)\varphi_k(x) & \text{if } |k| \equiv 3 \pmod 4. \end{cases} \tag{3.3}$$

Conversely, suppose that we have (3.1). This implies that

$$a^3 + \frac{1}{2}a^2b(1+i) + \frac{3}{2}iab^2 + \frac{5}{4}b^3(1-i) \neq 0.$$

Hence, it is possible to consider the operator defined in (3.2) and, by a straightforward computation, verify that this is, indeed, the inverse of T. □

Remark 3.2.

(1) It is not difficult to see that

$$t_1 := a + \Big(\frac{3}{2} - \frac{i}{2}\Big)b, \quad t_2 := a - \Big(\frac{1}{2} + \frac{i}{2}\Big)b, \quad t_3 := a - \Big(\frac{1}{2} - \frac{3i}{2}\Big)b$$

are the roots of the polynomial $P_T(t)$. Consequently, t_1, t_2, t_3 are the characteristic roots of $P_T(t)$.

(2) T is not a unitary operator, unless $b = 0$ and $a = e^{i\alpha}$, $\alpha \in \mathbb{R}$, which is a somehow trivial case and is not under the conditions we have here imposed to this operator.

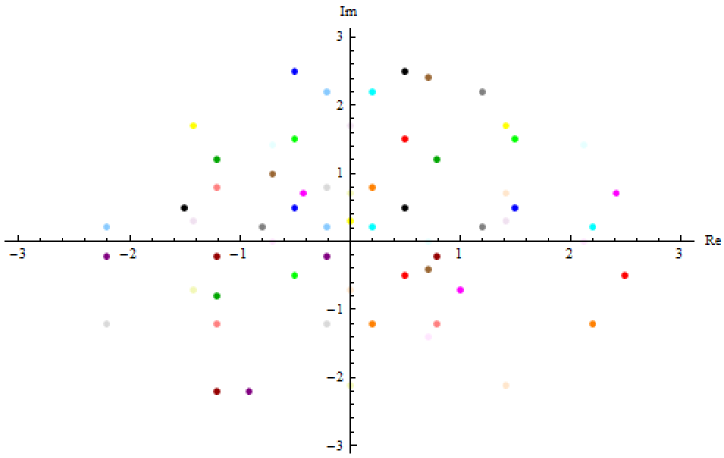

FIGURE 1. The spectrum of the operator T for different values of the parameters a and b.

Theorem 3.3. *The spectrum of the operator T is given by*

$$\sigma(T) = \Big\{a + \Big(\frac{3}{2} - \frac{i}{2}\Big)b, a - \Big(\frac{1}{2} + \frac{i}{2}\Big)b, a - \Big(\frac{1}{2} - \frac{3i}{2}\Big)b\Big\}$$

(*see Figure* 1).

Proof. For any $\lambda \in \mathbb{C}$, we have

$$\begin{aligned} t^3 - \Big[3a + \frac{b}{2}(1+i)\Big]t^2 + \Big[3a^2 + ab(1+i) + \frac{3}{2}ib^2\Big]t \\ + \Big[-a^3 - \frac{1}{2}a^2b(1+i) - \frac{3}{2}iab^2 - \frac{5}{4}b^3(1-i)\Big] \\ = (t-\lambda)\Big[t^2 + \Big(\lambda - 3a - \frac{b}{2}(1+i)\Big)t \\ + \Big(\lambda^2 - 3a\lambda - \frac{b}{2}(1+i) + 3a^2 + ab(1+i) + \frac{3}{2}ib^2\Big)\Big] + P_T(\lambda). \end{aligned}$$

Suppose that

$$\lambda \notin \left\{a + \left(\frac{3}{2} - \frac{i}{2}\right) b, a - \left(\frac{1}{2} + \frac{i}{2}\right) b, a - \left(\frac{1}{2} - \frac{3i}{2}\right) b\right\}.$$

This implies that

$$\begin{aligned} P_T(\lambda) = \lambda^3 &- \Big[3a + \frac{b}{2}(1+i)\Big]\lambda^2 + \Big[3a^2 + ab(1+i) + \frac{3}{2}ib^2\Big]\lambda \\ &+ \Big[- a^3 - \frac{1}{2}a^2b(1+i) - \frac{3}{2}iab^2 - \frac{5}{4}b^3(1-i)\Big] \neq 0. \end{aligned}$$

Then the operator $T - \lambda I$ is invertible, and its inverse operator is defined by

$$\begin{aligned} (T - \lambda I)^{-1} = -\frac{1}{P_T(\lambda)} &\Big[T^2 + \Big(\lambda - 3a - \frac{b}{2}(1+i)\Big)T \\ &+ \Big(\lambda^2 - 3a\lambda - \frac{b}{2}(1+i) + 3a^2 + ab(1+i) + \frac{3}{2}ib^2\Big)I\Big]. \end{aligned}$$

So, we have proved that if $T - \lambda I$ is not invertible, then $\lambda \in \sigma(T)$.

Conversely, if we choose $\lambda = t_1$, we obtain:

$$\begin{aligned} \Big[T - \Big(a + \Big(\frac{3}{2} - \frac{i}{2}\Big)b\Big)I\Big] &\Big[T^2 + (-2a + b(1-i))T \\ &+ \Big(a^2 - \frac{ab}{2}(1-3i) + 2b^2 - \frac{b}{2}(1+i)\Big)I\Big] = -P_T(\lambda)I. \end{aligned}$$

As $\lambda = a + (\frac{3}{2} - \frac{i}{2})b$, then $P_T(\lambda) = 0$. So, if $T - (a + (\frac{3}{2} - \frac{i}{2})b)I$ is invertible, then

$$T^2 + \big(- 2a + b(1-i)\big)T + \Big(a^2 - \frac{ab}{2}(1-3i) + 2b^2 - \frac{b}{2}(1+i)\Big)I = 0,$$

which implies that $b = 0$ and this is a contradiction. So, $T - (a + (\frac{3}{2} - \frac{i}{2})b)I$ is not invertible.

The same procedure can be repeated for $\lambda = t_2, t_3$, in which cases we obtain the same desired conclusion. □

Thanks to the identity (3.3), we obtain three types of eigenfunctions of T, represented as follows:

$$\Phi_I(x) = \sum_{|k|=0 \pmod 4}^{K} \alpha_k \varphi_k(x), \quad k \in \mathbb{C}, \tag{3.4}$$

$$\Phi_{II}(x) = \sum_{|k|=1,2 \pmod 4}^{K} \alpha_k \varphi_k(x), \quad k \in \mathbb{C}, \tag{3.5}$$

$$\Phi_{III}(x) = \sum_{|k|=3 \pmod 4}^{K} \alpha_k \varphi_k(x), \quad k \in \mathbb{C}. \tag{3.6}$$

3.1.2. *Parseval type identity.*

Theorem 3.4. *A Parseval type identity for T is given by*

$$\begin{aligned}\langle Tf, Tg\rangle_{L^2(\mathbb{R}^n)} &= \Big[\,|a|^2 + \frac{3}{2}\,|b|^2\Big]\langle f, g\rangle_{L^2(\mathbb{R}^n)} + 2\Re\{a\bar{b}\}\langle f, Fg\rangle_{L^2(\mathbb{R}^n)} \\ &\quad + \Re\big\{b(1-i)\overline{a}\big\}\langle f, Wg\rangle_{L^2(\mathbb{R}^n)} + |b|^2\langle f, F^{-1}g\rangle_{L^2(\mathbb{R}^n)}, \quad (3.7)\end{aligned}$$

for any $f, g \in L^2(\mathbb{R}^n)$.

Proof. For any $f, g \in L^2(\mathbb{R}^n)$, it is straightforward to verify the following identities:

$$\langle Wf, Wg\rangle_{L^2(\mathbb{R}^n)} = \langle f, g\rangle_{L^2(\mathbb{R}^n)}, \tag{3.8}$$

$$\langle f, Wg\rangle_{L^2(\mathbb{R}^n)} = \langle Wf, g\rangle_{L^2(\mathbb{R}^n)}. \tag{3.9}$$

If we have in mind (3.8)–(3.9) and as well that for any $f, g \in L^2(\mathbb{R}^n)$:

$$\begin{aligned}\langle Wf, Fg\rangle_{L^2(\mathbb{R}^n)} &= \langle f, F^{-1}g\rangle_{L^2(\mathbb{R}^n)},\\ \langle Ff, Wg\rangle_{L^2(\mathbb{R}^n)} &= \langle f, F^{-1}g\rangle_{L^2(\mathbb{R}^n)},\\ \langle Ff, Fg\rangle_{L^2(\mathbb{R}^n)} &= \langle f, g\rangle_{L^2(\mathbb{R}^n)},\\ \langle Ff, g\rangle_{L^2(\mathbb{R}^n)} &= \langle f, Fg\rangle_{L^2(\mathbb{R}^n)},\end{aligned} \tag{3.10}$$

then (3.7) directly appears by using (1.1). □

3.1.3. *Integral equations generated by T.* Now we will consider the operator equation, generated by the operator T (on $L^2(\mathbb{R}^n)$), of the following form

$$m\varphi + nT\varphi + pT^2\varphi = f, \tag{3.11}$$

where $m, n, p \in \mathbb{C}$ are given, $|m| + |n| + |p| \neq 0$, and f is predetermined.

As we proved previously, the polynomial $P_T(t)$ has the single roots $t_1 = a+(\frac{3}{2} - \frac{i}{2})b$, $t_2 = a-(\frac{1}{2}+\frac{i}{2})b$ and $t_3 = a-(\frac{1}{2} - \frac{3i}{2})b$. The projectors induced by T, in the sense of the Lagrange interpolation formula, are given by

$$P_1 = \frac{(T-t_2I)(T-t_3I)}{(t_1-t_2)(t_1-t_3)} = \frac{T^2-(t_2+t_3)T+t_2t_3}{(t_1-t_2)(t_1-t_3)}\,, \tag{3.12}$$

$$P_2 = \frac{(T-t_1I)(T-t_3I)}{(t_2-t_1)(t_2-t_3)} = \frac{T^2-(t_1+t_3)T+t_1t_3}{(t_2-t_1)(t_2-t_3)}\,, \tag{3.13}$$

$$P_3 = \frac{(T-t_1I)(T-t_2I)}{(t_3-t_1)(t_3-t_2)} = \frac{T^2-(t_1+t_2)T+t_1t_2}{(t_3-t_1)(t_3-t_2)}\,. \tag{3.14}$$

Then we have

$$P_jP_k = \delta_{jk}P_k, \quad T^\ell = t_1^\ell P_1 + t_2^\ell P_2 + t_3^\ell P_3, \tag{3.15}$$

for any $j, k = 1, 2, 3$, and $\ell = 0, 1, 2$. The equation (3.11) is equivalent to the equation

$$a_1P_1\varphi + a_2P_2\varphi + a_3P_3\varphi = f, \tag{3.16}$$

where $a_j = m + nt_j + pt_j^2$, $j = 1, 2, 3$.

Theorem 3.5.

(i) *The equation* (3.11) *has a unique solution for every* f *if and only if* $a_1a_2a_3 \neq 0$. *In this case, the solution of* (3.11) *is given by*

$$\varphi = a_1^{-1}P_1f + a_2^{-1}P_2f + a_3^{-1}P_3f. \tag{3.17}$$

(ii) *If* $a_j = 0$, *for some* $j = 1, 2, 3$, *then the equation* (3.11) *has a solution if and only if* $P_jf = 0$. *If this condition is satisfied, then the equation* (3.11) *has an infinite number of solutions given by*

$$\varphi = \sum_{\substack{j\leq 3\\ a_j\neq 0}} a_j^{-1}P_jf + z, \quad \textit{where} \;\; z \in \ker\Big(\sum_{\substack{j\leq 3\\ a_j\neq 0}} P_j\Big). \tag{3.18}$$

Proof. Suppose that the equation (3.11) has a solution $\varphi \in L^2(\mathbb{R}^n)$. Applying P_j to both sides of the equation (3.16), we obtain a system of three equations:

$$a_jP_j\varphi = P_jf, \;\; j = 1, 2, 3.$$

In this way, if $a_1a_2a_3 \neq 0$, then we have the following system of equations:

$$\begin{cases} P_1\varphi = a_1^{-1}P_1f, \\ P_2\varphi = a_2^{-1}P_2f, \\ P_3\varphi = a_3^{-1}P_3f. \end{cases} \tag{3.19}$$

Using the identity

$$P_1 + P_2 + P_3 = I,$$

we obtain (3.17). Conversely, we can verify that φ fulfills (3.16).

If $a_1a_2a_3 = 0$, then $a_j = 0$, for some $j \in \{1, 2, 3\}$. Therefore, it follows that $P_jf = 0$. Then, we have

$$\sum_{\substack{j\leq 3\\ a_j\neq 0}} P_j\varphi = \sum_{\substack{j\leq 3\\ a_j\neq 0}} a_j^{-1}P_jf.$$

Using the fact that $P_jP_k = \delta_{jk}P_k$, we get

$$\Big(\sum_{\substack{j\leq 3\\ a_j\neq 0}} P_j\Big)\varphi = \Big(\sum_{\substack{j\leq 3\\ a_j\neq 0}} P_j\Big)\Big[\sum_{\substack{j\leq 3\\ a_j\neq 0}} a_j^{-1}P_jf\Big]$$

or, equivalently,

$$\Big(\sum_{\substack{j\leq 3\\ a_j\neq 0}} P_j\Big)\Big[\varphi - \sum_{\substack{j\leq 3\\ a_j\neq 0}} a_j^{-1}P_jf\Big] = 0.$$

Therefore, we can obtain the solution (3.18).

Conversely, we can verify that φ fulfills (3.16). As the Hermite functions are the eigenfunctions of T, we can say that the cardinality of all functions φ in (3.18) is infinite. $\square$

3.1.4. *Convolution.* In this subsection we will focus on obtaining a new convolution $\overset{T}{*}$ for the operator T. We will perform it for the case $b \neq 0$ and $c = \frac{b}{2}(1-i)$, although the same procedure can be implemented for other cases of the parameters.

This means that we are identifying the operations that have a correspondent multiplication property for the operator T as the usual convolution has for the Fourier transform $(Tf)(Tg) = T(f \overset{T}{*} g)$.

Theorem 3.6. *For the operator $T = aI + bF + cW$, with $a, b, c \in \mathbb{C}$, $b \neq 0$ and $c = \frac{b}{2}(1-i)$, and $f, g \in L^2(\mathbb{R}^n)$, we have the following convolution:*

$$\begin{aligned} f \overset{T}{*} g = C\Big[& A_1 fg + A_2 (Wf)(Wg) + A_3 (f\,Wg + g\,Wf) \\ &+ A_4 (f\,Fg + g\,Ff) + A_5\big((Wf)(F^{-1}g) + (F^{-1}f)(Wg)\big) \\ &+ A_6\big((Wf)(Fg) + (Ff)(Wg)\big) + A_7 (g\,F^{-1}f + f\,F^{-1}g) \\ &+ A_8((Ff)(Fg)) + A_9\big((F^{-1}f)(F^{-1}g)\big) + A_{10}(F(fg)) \\ &+ A_{11}\big(F(f\,Wg) + F(g\,Wf)\big) + A_{12}(F^{-1}(fg)) \\ &+ A_{13}\big(F(f\,Fg) + F(g\,Ff)\big) + A_{14}\big(F^{-1}(f\,Fg) + F^{-1}(g\,Ff)\big) \\ &+ A_{15}\big(F((Ff)(Wg)) + F((Wf)(Fg))\big) \\ &+ A_{16}\big(F^{-1}((Ff)(Wg)) + F^{-1}((Wf)(Fg))\big) \\ &+ A_{17} F((Ff)(Fg)) + A_{18} F^{-1}((Ff)(Fg))\Big], \end{aligned} \tag{3.20}$$

where

$$\begin{aligned} C &= \frac{1}{a^3 + \frac{1}{2}\,a^2 b(1+i) + \frac{3}{2}\,iab^2 + \frac{5}{4}\,b^3(1-i)}, \\ A_1 &= a^4 + \frac{a^3 b}{2}\,(1+i) + ia^2b^2 + \frac{ab^3}{4}\,(1+i) + \frac{ib^4}{4}, \\ A_2 &= -\frac{a^2b^2}{2}\,(1+i) - \frac{ab^3}{2}\,(1-i) + \frac{b^4}{2} - \frac{a^3 b}{2}\,(1-i), \\ A_3 &= \frac{a^3 b}{2}\,(1+i) + \frac{a^2b^2}{2}\,(1+i) + \frac{ab^3}{2}\,(1+i) - \frac{ab^3}{4}\,(1-i), \\ A_4 &= a^3 b + \frac{a^2b^2}{2}\,(1+i) + iab^3, \quad A_5 = -\frac{a^2b^2}{2}\,(1-i) - \frac{ab^3}{2}, \\ A_6 &= \frac{a^2b^2}{2}\,(1-i) + \frac{ab^3}{2} + \frac{b^4}{2}\,(1+i), \quad A_7 = i\,\frac{ab^3}{2} - \frac{b^4}{4}\,(1-i), \\ A_8 &= a^2b^2 + \frac{ab^3}{2}\,(1+i) + ib^4, \quad A_9 = -\frac{ab^3}{2}\,(1-i) - \frac{b^4}{2}, \\ A_{10} &= -a^3 b - \frac{a^2b^2}{2}\,(1+i) - \frac{b^4}{2}\,(1+i), \\ A_{11} &= -\frac{a^2b^2}{2}\,(1-i) - \frac{ab^3}{2} - iab^3, \end{aligned}$$

$$A_{12} = i\,\frac{ab^3}{2} - \frac{b^4}{4}\,(1-i) + a^2b^2(1-i), \quad A_{13} = -a^2b^2 - \frac{ab^3}{2}\,(1+i),$$
$$A_{14} = ab^3(1-i), \quad A_{15} = -\frac{ab^3}{2}\,(1-i) - \frac{b^4}{2}\,,$$
$$A_{16} = -ib^4, \quad A_{17} = -ab^3 - \frac{b^4}{2}\,(1+i), \quad A_{18} = b^4(1-i).$$

Proof. Using the definition of T and a direct (but long) computation, we obtain the equivalence between (3.20) and

$$f \overset{T}{*} g = \frac{1}{a^3 + \frac{1}{2}\,a^2b(1+i) + \frac{3}{2}\,iab^2 + \frac{5}{4}\,b^3(1-i)}$$
$$\times \left[T^2 - \Big(3a + \frac{b}{2}\,(1+i)\Big)T + \Big(3a^2 + ab(1+i) + \frac{3}{2}\,ib^2\Big)I\right]\big[(Tf)(Tg)\big].$$

Thus, having in mind (3.2), we identify the last identity with

$$f \overset{T}{*} g = T^{-1}\big[(Tf)(Tg)\big],$$

which is equivalent to

$$(Tf)(Tg) = T\big(f \overset{T}{*} g\big),$$

as desired. □

3.2. **Case** $b \neq 0$ **and** $c \neq \pm\frac{b}{2}(1 \pm i)$**.** In the case of the operator $T := aI + bF + cW$, $a, b, c \in \mathbb{C}$, $b \neq 0$ and $c \neq \pm\frac{b}{2}\,(1 \pm i)$, whose characteristic polynomial is

$$P_T(t) = t^4 - 4at^3 + (6a^2 - 2c^2)t^2 + (-4a^3 - 4b^2c + 4ac^2)t$$
$$+ (a^2 - c^2)^2 + b^2(4ac - b^2),$$

we have the following properties.

3.2.1. *Invertibility and spectrum.*

Theorem 3.7. *T is an invertible operator if and only if*

$$a + c + b \neq 0, \quad a - c - ib \neq 0, \quad a + c - b \neq 0, \quad a - c + ib \neq 0. \tag{3.21}$$

In this case, the inverse operator is defined by the formula

$$T^{-1} = -\frac{1}{(a^2 - c^2)^2 + b^2(4ac - b^2)}$$
$$\times \Big[T^3 - 4aT^2 + (6a^2 - 2c^2)T - (-4a^3 - 4b^2c + 4ac^2)I\Big]. \tag{3.22}$$

Proof. Suppose that the operator T is invertible. Using the Hermite functions φ_k, we have:

$$(T\varphi_k)(x) = \begin{cases} (a + c + b)\varphi_k(x) & \text{if } |k| \equiv 0 \pmod 4, \\ (a - c - ib)\varphi_k(x) & \text{if } |k| \equiv 1 \pmod 4, \\ (a + c - b)\varphi_k(x) & \text{if } |k| \equiv 2 \pmod 4, \\ (a - c + ib)\varphi_k(x) & \text{if } |k| \equiv 3 \pmod 4. \end{cases} \tag{3.23}$$

Therefore,

- for $|k| \equiv 0 \pmod 4$, $(T\varphi_k)(x) = (a+b+c)\varphi_k(x)$, which implies that $a+c+b \neq 0$;
- for $|k| \equiv 1 \pmod 4$, $(T\varphi_k)(x) = (a-ib-c)\varphi_k(x)$, which implies that $a-c-ib \neq 0$;
- for $|k| \equiv 2 \pmod 4$, $(T\varphi_k)(x) = (a-b+c)\varphi_k(x)$, which implies that $a+c-b \neq 0$;
- for $|k| \equiv 3 \pmod 4$, $(T\varphi_k)(x) = (a+ib-c)\varphi_k(x)$, which implies that $a-c+ib \neq 0$.

Conversely, suppose that (3.21) holds. So,

$$(a^2-c^2)^2 + b^2(4ac-b^2) \neq 0.$$

Hence, it is easy to verify that the operator defined in (3.22) is the inverse of the operator T. □

Remark 3.8.

(1) The characteristic roots of the polynomial $P_T(t)$ are

$$t_1 = a+c+b, \quad t_2 = a-c-ib, \quad t_3 = a+c-b, \quad t_4 = a-c+ib.$$

(2) T is not a unitary operator, unless $a=0$, $b=e^{i\beta}$, $c=0$, $\beta \in \mathbb{R}$, (which is the operator $T = bF$, with $b \in \mathbb{C} \setminus \{0\}$) or $a = e^{i\alpha}$, $b=0$, $c=0$ or $a=0$, $b=0$, $c=e^{i\gamma}$, $\alpha, \varphi \in \mathbb{R}$, which are not under the conditions here considered for this operator.

Theorem 3.9. *The spectrum of the operator T is defined by*

$$\sigma(T) = \big\{a+c+b, a-c-ib, a+c-b, a-c+ib\big\}.$$

Proof. For any $\lambda \in \mathbb{C}$, we have

$$\begin{aligned} t^4 - 4at^3 + (6a^2-2c^2)t^2 + \big[-4a^3 - 4b^2c + 4ac^2\big]t + (a^2-c^2)^2 + b^2(4ac-b^2) \\ = (t-\lambda)\Big[t^3 + (\lambda-4a)t^2 + (\lambda^2 - 4a\lambda + 6a^2 - 2c^2)t \\ + \big(\lambda^3 - 4a\lambda^2 + (6a^2-2c^2)\lambda - 4a^3 - 4b^2c + 4ac^2\big)\Big] + P_T(\lambda). \end{aligned}$$

If $\lambda \notin \{a+c+b, a-c-ib, a+c-b, a-c+ib\}$, then

$$\begin{aligned} P_T(\lambda) = \lambda^4 - 4a\lambda^3 + (6a^2-2c^2)\lambda^2 \\ + [-4a^3 - 4b^2c + 4ac^2]\lambda + (a^2-c^2)^2 + b^2(4ac-b^2) \neq 0. \end{aligned}$$

In this way, the operator $T - \lambda I$ is invertible, and its inverse operator is defined by the following formula:

$$\begin{aligned} (T-\lambda I)^{-1} = -\frac{1}{P_T(\lambda)}\Big[T^3 + (\lambda-4a)T^2 + (\lambda^2 - 4a\lambda + 6a^2 - 2c^2)T \\ + \big(\lambda^3 - 4a\lambda^2 + (6a^2-2c^2)\lambda - 4a^3 - 4b^2c + 4ac^2\big)I\Big]. \end{aligned}$$

In this way, we have proved that if $T-\lambda I$ is not invertible, then $\lambda \in \sigma(T)$.

Conversely, if we choose $\lambda = t_1$, we obtain:

$$\begin{aligned}
&\big(T-(a+c+b)I\big)\Big[T^3+(-3a+b+c)T^2 \\
&\quad +(3a^2-2ab+b^2-2ac+2bc-c^2)T+(-a^3+a^2b-ab^2+b^3+4ac \\
&\qquad +a^2c-2abc-b^2c-3ac^2+bc^2-c^3)I\Big] = -P_T(\lambda)I.
\end{aligned}$$

As $\lambda = a+c+b$, $P_T(\lambda)=0$. So, if $T-(a+c+b)I$ is invertible, then

$$\begin{aligned}
&T^3+(-3a+b+c)T^2+(3a^2-2ab+b^2-2ac+2bc-c^2)T \\
&+(-a^3+a^2b-ab^2+b^3+4ac+a^2c-2abc-b^2c-3ac^2+bc^2-c^3)I=0,
\end{aligned}$$

which implies that $a=0$ and $b=0$ or that $b=0$ and $c=0$, which is not under the conditions imposed for this operator. So, we reach to a contradiction. Hence, $T-(a-c-b(1+i))I$ is not invertible.

Arguing in the same way for $\lambda = t_2, t_3, t_4$, we obtain a very similar conclusion. □

Thanks to the identity (3.23), we obtain four types of eigenfunctions of T, represented as follows:

$$\Phi_I(x) = \sum_{|k|\equiv 0 \pmod 4}^{K} \alpha_k \varphi_k(x), \quad k \in \mathbb{C}, \tag{3.24}$$

$$\Phi_{II}(x) = \sum_{|k|\equiv 1 \pmod 4}^{K} \alpha_k \varphi_k(x), \quad k \in \mathbb{C}, \tag{3.25}$$

$$\Phi_{III}(x) = \sum_{|k|\equiv 2 \pmod 4}^{K} \alpha_k \varphi_k(x), \quad k \in \mathbb{C}, \tag{3.26}$$

$$\Phi_{IV}(x) = \sum_{|k|\equiv 3 \pmod 4}^{K} \alpha_k \varphi_k(x), \quad k \in \mathbb{C}. \tag{3.27}$$

3.2.2. *Parseval type identity.* In the present case, a Parseval type identity takes the following form.

Theorem 3.10. *In the present case, a Parseval type identity for T is given by*

$$\begin{aligned}
\langle Tf, Tg\rangle_{L^2(\mathbb{R}^n)} &= \big[\,|a|^2+|b|^2+|c|^2\big]\langle f,g\rangle_{L^2(\mathbb{R}^n)} + 2\Re\{a\bar{b}\}\langle f, Fg\rangle_{L^2(\mathbb{R}^n)} \\
&\quad + 2\Re\{a\bar{c}\}\langle f, Wg\rangle_{L^2(\mathbb{R}^n)} + 2\Re\{b\bar{c}\}\langle f, F^{-1}g\rangle_{L^2(\mathbb{R}^n)}
\end{aligned} \tag{3.28}$$

for any $f,g \in L^2(\mathbb{R}^n)$.

Proof. The formula (3.28) is a direct consequence of (1.1), (3.8), (3.9) and (3.10). □

3.2.3. *Integral equations generated by T.* As before, we will now consider in the present case the following operator equation generated by the operator T, on $L^2(\mathbb{R}^n)$,

$$m\varphi + nT\varphi + pT^2\varphi = f, \tag{3.29}$$

where $m, n, p \in \mathbb{C}$ are given, $|m| + |n| + |p| \neq 0$, and f is predetermined.

The polynomial $P_T(t)$ has the single roots: $t_1 = a+c+b$, $t_2 = a-c-ib$, $t_3 = a+c-b$, $t_4 = a-c+ib$. Using the Lagrange interpolation structure, we construct the projectors induced by T:

$$\begin{aligned} P_1 &= \frac{(T-t_2I)(T-t_3I)(T-t_4I)}{(t_1-t_2)(t_1-t_3)(t_1-t_4)} \\ &= \frac{T^3-(t_2+t_3+t_4)T^2+(t_2t_3+t_2t_4+t_3t_4)T-t_2t_3t_4I}{(t_1-t_2)(t_1-t_3)(t_1-t_4)}, \end{aligned} \tag{3.30}$$

$$\begin{aligned} P_2 &= \frac{(T-t_1I)(T-t_3I)(T-t_4I)}{(t_2-t_1)(t_2-t_3)(t_2-t_4)} \\ &= \frac{T^3-(t_1+t_3+t_4)T^2+(t_1t_3+t_1t_4+t_3t_4)T-t_1t_3t_4I}{(t_2-t_1)(t_2-t_3)(t_1-t_4)}, \end{aligned} \tag{3.31}$$

$$\begin{aligned} P_3 &= \frac{(T-t_1I)(T-t_2I)(T-t_4I)}{(t_3-t_1)(t_3-t_2)(t_3-t_4)} \\ &= \frac{T^3-(t_1+t_2+t_4)T^2+(t_1t_2+t_1t_4+t_2t_4)T-t_1t_2t_4I}{(t_3-t_1)(t_3-t_2)(t_3-t_4)}, \end{aligned} \tag{3.32}$$

$$\begin{aligned} P_4 &= \frac{(T-t_1I)(T-t_2I)(T-t_3I)}{(t_4-t_1)(t_4-t_2)(t_4-t_3)} \\ &= \frac{T^3-(t_1+t_2+t_3)T^2+(t_1t_2+t_1t_3+t_2t_3)T-t_1t_2t_3I}{(t_4-t_1)(t_4-t_2)(t_4-t_3)}. \end{aligned} \tag{3.33}$$

Then, we have

$$P_jP_k = \delta_{jk}P_k; \;\; T^\ell = t_1^\ell P_1 + t_2^\ell P_2 + t_3^\ell P_3 + t_4^\ell P_4, \tag{3.34}$$

for any $j, k = 1, 2, 3, 4$, and $\ell = 0, 1, 2$. The equation (3.29) is equivalent to the equation

$$a_1P_1\varphi + a_2P_2\varphi + a_3P_3\varphi + a_4P_4\varphi = f, \tag{3.35}$$

where $a_j = m + nt_j + pt_j^2$, $j = 1, 2, 3, 4$.

Theorem 3.11.

(i) *Equation* (3.29) *has a unique solution for every f if and only if $a_1a_2a_3a_4 \neq 0$. In this case, the solution is given by*

$$\varphi = a_1^{-1}P_1f + a_2^{-1}P_2f + a_3^{-1}P_3f + a_4^{-1}P_4f. \tag{3.36}$$

(ii) *If $a_j = 0$, for some $j = 1, 2, 3, 4$, then the equation* (3.11) *has a solution if and only if $P_jf = 0$. If we have this, then the equation*

(3.11) *has an infinite number of solutions given by*

$$\varphi = \sum_{\substack{j\leq 4\\ a_j\neq 0}} a_j^{-1} P_j f + z, \quad \text{where } \ z \in \ker\Big(\sum_{\substack{j\leq 4\\ a_j\neq 0}} P_j\Big). \tag{3.37}$$

Proof. Suppose that the equation (3.29) has a solution $\varphi \in L^2(\mathbb{R}^n)$. Applying P_j to both sides of the equation (3.35), we obtain the system of four equations: $a_j P_j \varphi = P_j f, \ j = 1, 2, 3, 4$.

If $a_1a_2a_3a_4 \neq 0$, then we have the following system of equations:

$$\begin{cases} P_1\varphi = a_1^{-1} P_1 f, \\ P_2\varphi = a_2^{-1} P_2 f, \\ P_3\varphi = a_3^{-1} P_3 f, \\ P_4\varphi = a_4^{-1} P_4 f. \end{cases} \tag{3.38}$$

Using the identity

$$P_1 + P_2 + P_3 + P_4 = I,$$

we obtain (3.36). Conversely, we can verify that φ fulfills (3.35).

If $a_1a_2a_3a_4 = 0$, then $a_j = 0$ for some $j \in \{1, 2, 3, 4\}$. It follows that $P_j f = 0$. Then, we have

$$\sum_{\substack{j\leq 4\\ a_j\neq 0}} P_j\varphi = \sum_{\substack{j\leq 4\\ a_j\neq 0}} a_j^{-1} P_j f.$$

Using $P_j P_k = \delta_{jk} P_k$, we obtain

$$\Big(\sum_{\substack{j\leq 4\\ a_j\neq 0}} P_j\Big)\varphi = \Big(\sum_{\substack{j\leq 4\\ a_j\neq 0}} P_j\Big)\Big[\sum_{\substack{j\leq 4\\ a_j\neq 0}} a_j^{-1} P_j f\Big].$$

Equivalently,

$$\Big(\sum_{\substack{j\leq 4\\ a_j\neq 0}} P_j\Big)\Big[\varphi - \sum_{\substack{j\leq 4\\ a_j\neq 0}} a_j^{-1} P_j f\Big] = 0.$$

So, we obtain the solution (3.37).

Conversely, we can verify that φ fulfills (3.35). As the Hermite functions are the eigenfunctions of T, we can say that the cardinality of all functions φ in (3.37) is infinite. □

3.2.4. *Convolution.* In this subsection we will present a new convolution $\overset{T}{*}$ for the operator T. We will perform it for the case $b \neq 0$ and $c \neq \pm\frac{b}{2}(1 \pm i)$. This means that we are identifying the operations that have a correspondent multiplication property for the operator T as the usual convolution has for the Fourier transform $(Tf)(Tg) = T(f \overset{T}{*} g)$.

Theorem 3.12. *For the operator $T = aI + bF + cW$, with $a, b, c \in \mathbb{C}$, $b \neq 0$ and $c \neq \pm\frac{b}{2}(1 \pm i)$, and $f, g \in L^2(\mathbb{R}^n)$, we have the following convolution:*

$$\begin{aligned}
f \overset{T}{*} g = C\Big[&A_1 fg + A_2(Wf)(Wg) + A_3(f\,Wg + g\,Wf) \\
&+ A_4(f\,Fg + g\,Ff) + A_5\big((Wf)(F^{-1}g) + (F^{-1}f)(Wg)\big) \\
&+ A_6\big((Wf)(Fg) + (Ff)(Wg)\big) + A_7(g\,F^{-1}f + f\,F^{-1}g) \\
&+ A_8((Ff)(Fg)) + A_9\big((F^{-1}f)(F^{-1}g)\big) \\
&+ A_{10}(F(fg)) + A_{11}\big(F(f\,Wg) + F(g\,Wf)\big) + A_{12}(F^{-1}(fg)) \\
&+ A_{13}\big(F(f\,Fg) + F(g\,Ff)\big) + A_{14}\big(F^{-1}(f\,Fg) + F^{-1}(g\,Ff)\big) \\
&+ A_{15}\big(F\big((Ff)(Wg)\big) + F\big((Wf)(Fg)\big)\big) \\
&+ A_{16}\big(F^{-1}\big((Ff)(Wg)\big) + F^{-1}\big((Wf)(Fg)\big)\big) \\
&+ A_{17}F\big((Ff)(Fg)\big) + A_{18}F^{-1}\big((Ff)(Fg)\big)\Big],
\end{aligned} \tag{3.39}$$

where

$$\begin{gathered}
C = -\frac{1}{(a^2 - c^2)^2 + b^2(4ac - b^2)}, \\
A_1 = 7a^5 - 7a^3c^2 + 7a^2b^2c - ab^2c^2 + a^2c^3 - c^5, \\
A_2 = 7a^3c^2 - 7ac^4 + 7b^2c^3 - a^3b^2 + a^4c - a^2c^3, \\
A_3 = 7a^4c - 7a^2c^3 + 7ab^2c^2 - a^2b^2c + a^3c^2 - ac^4, \\
A_4 = 7a^4b - 7a^2bc^2 + 7ab^3c, \quad A_5 = -a^2b^3 + a^3bc - abc^3, \\
A_6 = 7a^3bc - 7abc^3 + 7b^3c^2, \quad A_7 = -ab^3c + a^2bc^2 - bc^4, \\
A_8 = 7a^3b^2 - 7ab^2c^2 + 7b^4c, \quad A_9 = -ab^4 + a^2b^2c - b^2c^3, \\
A_{10} = a^4b + a^2bc^2 + b^3c - 2abc^3, \\
A_{11} = a^3bc + abc^3 + ab^3c - 2a^2bc^2, \\
A_{12} = a^2bc^2 + bc^4 + a^2b^3 - 2a^3bc, \quad A_{13} = a^3b^2 + ab^2c^2, \\
A_{14} = ab^4 - 2a^2b^2c, \quad A_{15} = a^2b^2c + b^2c^3, \\
A_{16} = b^4c - 2ab^2c^2, \quad A_{17} = a^2b^3 + b^3c^2, \quad A_{18} = b^5 - 2ab^3c.
\end{gathered}$$

Proof. Using the definition of T, by computation we obtain the equivalence between (3.39) and

$$\begin{aligned}
f \overset{T}{*} g = &-\frac{1}{(a^2 - c^2)^2 + b^2(4ac - b^2)} \\
&\times \Big[T^3 - 4aT^2 + (6a^2 - 2c^2)T - (-4a^3 - 4b^2c + 4ac^2)I\Big]\big[(Tf)(Tg)\big].
\end{aligned}$$

Consequently, having in mind (3.22), we identify the last identity with

$$f \overset{T}{*} g = T^{-1}\big[(Tf)(Tg)\big],$$

which is equivalent to

$$(Tf)(Tg) = T\big(f \overset{T}{*} g\big),$$

as desired. □

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References

1. P. K Anh, N. M. Tuan, and P. D. Tuan, The finite Hartley new convolutions and solvability of the integral equations with Toeplitz plus Hankel kernels. *J. Math. Anal. Appl.* **397** (2013), No. 2, 537–549.
2. R. N. Bracewell, The Fourier transform and its applications. Third edition. McGraw-Hill Series in Electrical Engineering. Circuits and Systems. *McGraw-Hill Book Co., New York*, 1986.
3. R. N. Bracewell, The Hartley transform. Oxford Science Publications. Oxford Engineering Science Series, 19. *The Clarendon Press, Oxford University Press, New York*, 1986.
4. R. N. Bracewell, Aspects of the Hartley transform. *Proceedings of the IEEE* **82**(1994), No. 3, 381–387.
5. G. Bogveradze and L. P. Castro, Wiener–Hopf plus Hankel operators on the real line with unitary and sectorial symbols. *Operator theory, operator algebras, and applications*, 77–85, Contemp. Math., 414, *Amer. Math. Soc., Providence, RI*, 2006.
6. G. Bogveradze and L. P. Castro, Toeplitz plus Hankel operators with infinite index. *Integral Equations Operator Theory* **62** (2008), No. 1, 43–63.
7. L. P. Castro, Regularity of convolution type operators with PC symbols in Bessel potential spaces over two finite intervals. *Math. Nachr.* **261/262** (2003), 23–36.
8. L. P. Castro and D. Kapanadze, Dirichlet–Neumann-impedance boundary value problems arising in rectangular wedge diffraction problems. *Proc. Amer. Math. Soc.* **136** (2008), No. 6, 2113–2123.
9. L. P. Castro and D. Kapanadze, Exterior wedge diffraction problems with Dirichlet, Neumann and impedance boundary conditions. *Acta Appl. Math.* **110** (2010), No. 1, 289–311.
10. L. P. Castro and D. Kapanadze, Wave diffraction by wedges having arbitrary aperture angle. *J. Math. Anal. Appl.* **421** (2015), No. 2, 1295–1314.
11. L. P. Castro and A. S. Silva, Invertibility of matrix Wiener–Hopf plus Hankel operators with symbols producing a positive numerical range. *Z. Anal. Anwend.* **28** (2009), No. 1, 119–127.
12. L. P. Castro and E. M. Rojas, Explicit solutions of Cauchy singular integral equations with weighted Carleman shift. *J. Math. Anal. Appl.* **371** (2010), No. 1, 128–133.
13. L. P. Castro and E. M. Rojas, On the solvability of singular integral equations with reflection on the unit circle. *Integral Equations Operator Theory* **70** (2011), No. 1, 63–99.
14. L. P. Castro and E. M. Rojas, Bounds for the kernel dimension of singular integral operators with Carleman shift. *Application of mathematics in technical and natural sciences*, 68–76, AIP Conf. Proc., 1301, *Amer. Inst. Phys., Melville, NY*, 2010.

15. G. B. Folland and A. Sitaram, The uncertainty principle: a mathematical survey. *J. Fourier Anal. Appl.* **3** (1997), No. 3, 207–238.
16. H.-J. Glaeske and V. K. Tuấn, Mapping properties and composition structure of multidimensional integral transforms. *Math. Nachr.* **152** (1991), 179–190.
17. B. T. Giang, N. V. Mau, and N. M. Tuan, Operational properties of two integral transforms of Fourier type and their convolutions. *Integral Equations Operator Theory* **65** (2009), No. 3, 363–386.
18. K. B. Howell, Fourier Transforms. The Transforms and Applications Handbook (A. D. Poularikas, ed.). The Electrical Engineering Handbook Series (Third Ed.), *CRC Press with IEEE Press, Boca–Raton–London–New York*, 2010.
19. G. S. Litvinchuk, Solvability theory of boundary value problems and singular integral equations with shift. Mathematics and its Applications, 523. *Kluwer Academic Publishers, Dordrecht*, 2000.
20. V. Namias, The fractional order Fourier transform and its application to quantum mechanics. *J. Inst. Math. Appl.* **25** (1980), No. 3, 241–265.
21. A. P. Nolasco and L. P. Castro, A Duduchava–Saginashvili's type theory for Wiener-Hopf plus Hankel operators. *J. Math. Anal. Appl.* **331** (2007), No. 1, 329–341.
22. D. H. Phong and E. M. Stein, Models of degenerate Fourier integral operators and Radon transforms. *Ann. of Math. (2)* **140** (1994), No. 3, 703–722.
23. E. C. Titchmarsh, Introduction to the theory of Fourier integrals. Third edition. *Chelsea Publishing Co., New York*, 1986.
24. N. M. Tuan and N. T. T. Huyen, The solvability and explicit solutions of two integral equations via generalized convolutions. *J. Math. Anal. Appl.* **369** (2010), No. 2, 712–718.
25. N. M. Tuan and N. T. T. Huyen, Applications of generalized convolutions associated with the Fourier and Hartley transforms. *J. Integral Equations Appl.* **24** (2012), No. 1, 111–130.
26. N. M. Tuan and N. T. T. Huyen, The Hermite functions related to infinite series of generalized convolutions and applications. *Complex Anal. Oper. Theory* **6** (2012), No. 1, 219–236.

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Authors' addresses:

L. P. Castro, R. C. Guerra

CIDMA – Center for Research and Development in Mathematics and Applications, Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal.

E-mail: `castro@ua.pt`; `ritaguerra@ua.pt`

N. M. Tuan

Department of Mathematics, College of Education, Viet Nam National University, G7 Build., 144 Xuan Thuy Rd., Cau Giay Dist., Hanoi, Vietnam.

E-mail: `tuannm@hus.edu.vn`