Memoirs on Differential Equations and Mathematical Physics
Volume 66, 2015, 119–139

Vladimir Rabinovich

ON SPECTRAL PROPERTIES AND INVERTIBILITY OF SOME OPERATORS OF MATHEMATICAL PHYSICS

Dedicated to Boris Khvedelidze's 100-th birthday anniversary

Abstract. The main aim of the paper is to study the Fredholm property, essential spectrum, and invertibility of some operators of the Mathematical Physics, such that the Schrödinger and Dirac operators with complex electric potentials, and Maxwell operators in absorbing at infinity media. This investigation is based on the limit operators method, and the uniqueness continuation property for the operators under consideration.

2010 Mathematics Subject Classification. Primary 35J05; Secondary 35J10, 35K05, 35L05.

Key words and phrases. Strongly elliptic systems, limit operators, Schrödinger, Dirac, Maxwell operators.

რეზიუმე. სტატიის ძირითადი მიზანია მათემატიკური ფიზიკის ზოგიერთი ოპერატორის ფრედჰოლმის თვისებების შესწავლა, როგორიცაა შროდინგერის და დირაკის ოპერატორები კომპლექსური ელექტრული პოტენციალებით და მაქსველის ოპერატორები არეში, რომელსაც გააჩნია შთანთქმის თვისება უსასრულობის მიდამოში. ეს გამოკვლევა ეფუძნება ზღვრული ოპერატორების მეთოდს და განხილვის ქვეშ მყოფი ოპერატორების ერთადერთობის უწყვეტობის თვისებას.

1. Introduction

The main aim of the paper is the study of the Fredholm property, essential spectrum, and invertibility of some operators of the Mathematical Physics, such that the Schrödinger and Dirac operators with complex electric potentials, and Maxwell operators in absorbing at infinity media. This investigation is based on the limit operators method [23]. Earlier this method was applied to the investigation of the location of essential spectra of perturbed pseudodifferential operators with applications to electromagnetic Schrödinger operators, square-root Klein–Gordon, and Dirac operators under general assumptions with respect to the behavior of real valued magnetic and electric potentials at infinity. By means of this method a very simple and transparent proof of the well known Hunziker, van Winter, Zjislin theorem (HWZ-Theorem) for multi-particle Hamiltonians has been obtained [14, 15]. In the papers [19, 20, 22] the limit operators method was applied to the study of the location of the essential spectrum of discrete Schrödinger operators on $\mathbb{Z}^n$, and on periodic combinatorial graphs. We also note the recent papers [16–18] devoted to applications of the limit operators method to the investigation of the Fredholm properties of boundary and transmission problems, and the boundary equations for unbounded domains.

The paper is organized as follows. In Section 2 we give some notations and an auxiliary material. In Section 3 we consider the Fredholm property of strongly elliptic second order systems of differential operators of the form

$$Au(x) = \sum_{k,l=1}^{n} (i\partial_{x_k} - a_k(x)) b^{kl}(x)(i\partial_{x_l} - a_l(x))u(x) + W(x)u(x), \quad x \in \mathbb{R}^n, \tag{1.1}$$

where a_k are real-valued functions on $\mathbb{R}^n$ and b^{kl} are $N \times N$ Hermitian matrices, W is a complex-valued $N \times N$ matrix. We suppose that a_k, and the coefficients of the matrix b^{kl} belong to $C^1_{b,u}(\mathbb{R}^n)$, and the coefficients of the matrix W belong to $C_{b,u}(\mathbb{R}^n)$, where $C_{b,u}(\mathbb{R}^n)$ is the class of bounded uniformly continuous functions on $\mathbb{R}^n$, and $C^1_{b,u}(\mathbb{R}^n)$ is the class of functions a on $\mathbb{R}^n$ such that $\partial_{x_j} a \in C_{b,u}(\mathbb{R}^n)$, $j = 1, \ldots, n$. In this section we prove that if

$$\liminf_{x\to\infty} \inf_{\|h\|_{\mathbb{C}^N}=1} \Im(W(x)h, h) > 0,$$

then $A : H^2(\mathbb{R}^n, \mathbb{C}^N) \to L^2(\mathbb{R}^n, \mathbb{C}^N)$ is a Fredholm operator of the index 0. In Section 4, applying the results of Section 3, we study the spectra of electromagnetic Schrödinger operators on $\mathbb{R}^n$ with real magnetic and complex electric potentials Φ. We prove that if

$$\liminf_{x\to\infty} \Im(\Phi(x)) > 0, \tag{1.2}$$

where Φ is the electric potential, then the *essential spectrum* of the Schrödinger operator does not have intersections with the real line $\mathbb{R}$. If, in addition to (1.2),

$$\Im(\Phi(x)) \geq 0, \quad x \in \mathbb{R}^n, \tag{1.3}$$

then the *spectrum* of the Schrödinger operator does not intersect the real line $\mathbb{R}$. Under the proof of the last result we have used *the uniqueness of the continuation* for elliptic operators (see e.g. [4, 9, 10]). Note that there is an extensive literature devoted to the spectral properties of the Schrödinger operators (see e.g. [1, 5, 24–26]).

Section 5 is devoted to the investigation of spectra of the Dirac operators with real-valued magnetic and complex-valued electric potentials. We suppose here that the magnetic and electric potentials are slowly oscillating at infinity. We prove here that the conditions (1.2), (1.3) provide us with the spectrum of the Dirac operator which does not contain the real values. For the proof we use the results of Section 3 and the uniqueness of the continuation for some almost diagonal strongly elliptic systems of second order.

In Section 6, we consider the harmonic Maxwell system on $\mathbb{R}^3$ for isotropic nonhomogeneous media. We suppose that the electric and magnetic permittivities ε and μ are the slowly oscillating at infinity complex valued functions. We prove that the operator of Maxwell's system is invertible in admissible functional spaces if the electromagnetic medium is absorbing at infinity, that is,

$$\liminf_{x \to \infty} \Im(\varepsilon(x)\mu(x)) > 0.$$

The proof of this result is based on the realization of the Maxwell system in a quaternionic form (see e.g. [8, 11, 12]), applications of results of Section 3, and the uniqueness of the continuation for almost diagonal strongly elliptic systems of second order.

2. Auxiliary Material

2.1. **Notation.** We will use the following standard notation.

- Given Banach spaces X, Y, $\mathcal{L}(X, Y)$ is the space of all bounded linear operators from X into Y. We abbreviate $\mathcal{L}(X, X)$ to $\mathcal{L}(X)$. If X is a Hilbert spaces, then $(x, y)_X$ is a scalar product in X of x, y.
- $L^2(\mathbb{R}^n, \mathbb{C}^N)$ is the Hilbert space of all measurable functions on $\mathbb{R}^n$ with values in $\mathbb{C}^N$ provided with the norm

$$\|u\|_{L^2(\mathbb{R}^n, \mathbb{C}^N)} := \left(\int_{\mathbb{R}^n} \|u(x)\|^2_{\mathbb{C}^N} \, dx \right)^{1/2}.$$

- $H^s(\mathbb{R}^n, \mathbb{C}^N)$ is a Sobolev space of distributions with norm

$$\|u\|_{H^s(\mathbb{R}^n, \mathbb{C}^N)} := \left(\int_{\mathbb{R}^n} \left(1 + |\xi|^2\right)^s \|\widehat{u}(\xi)\|^2_{\mathbb{C}^N} \, d\xi \right)^{1/2},$$

where $\widehat{u}$ is the Fourier transform of u.

- We also use the standard multi-index notations. Thus, $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\alpha_j \in \mathbb{N} \cup \{0\}$ is a multi-index, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ is its length, and

$$\partial^\alpha := \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}; \quad D^\alpha := (-i\partial_{x_1})^{\alpha_1} \cdots (-i\partial_{x_n})^{\alpha_n}.$$

  Finally, $\langle \xi \rangle := (1 + |\xi|^2)^{1/2}$ for $\xi \in \mathbb{R}^n$.

- $C_b(\mathbb{R}^n)$ is the C^*-algebra of all bounded continuous functions on $\mathbb{R}^n$.
- $C_{b,u}(\mathbb{R}^n)$ is the C^*-subalgebra of $C_b(\mathbb{R}^n)$ of all uniformly continuous functions.
- $C_b^k(\mathbb{R}^n)$ is the C^*-subalgebra of $C_b(\mathbb{R}^n)$ of k-times differentiable functions such that $\partial_x^\alpha a \in C_b(\mathbb{R}^n)$ for $|\alpha| \leq k$, and $a \in C_{b,u}^k(\mathbb{R}^n)$ if $a \in C_b^k(\mathbb{R}^n)$ and $\partial_x^\alpha a \in C_{b,u}(\mathbb{R}^n)$ for $|\alpha| = k$.
- We say that $a \in C_0^k(\mathbb{R}^n)$ if $a \in C_b^k(\mathbb{R}^n)$ and $\lim_{x\to\infty} a(x) = 0$.
- We denote by $SO(\mathbb{R}^n)$ a C^*-subalgebra of $C_b(\mathbb{R}^n)$ which consists of all functions a, slowly oscillating at infinity in the sense that

$$\lim_{x\to\infty} \sup_{y\in K} |a(x+y) - a(x)| = 0$$

  for every compact subset K of $\mathbb{R}^n$.

- We denote by $SO^k(\mathbb{R}^n)$ the set of functions $a \in C_b^k(\mathbb{R}^n)$ such that

$$\lim_{x\to\infty} \frac{\partial a(x)}{\partial x_j} = 0, \quad j = 1, \dots, n.$$

  Evidently, $SO^k(\mathbb{R}^n) \subset SO(\mathbb{R}^n)$.

- If $\mathcal{A}(\mathbb{R}^n)$ is an algebra of functions on $\mathbb{R}^n$, then we set

$$\mathcal{A}(\mathbb{R}^n, \mathcal{L}(\mathbb{C}^N)) = \mathcal{A}(\mathbb{R}^n) \otimes \mathcal{L}(\mathbb{C}^N).$$

- $B_R = \{x \in \mathbb{R}^n : \ |x| < R\}$, and $B_R' = \{x \in \mathbb{R}^n : \ |x| > R\}$.

2.2. **Fredholm properties of matrix partial differential operators and limit operators.** We consider matrix partial differential operators of order m of the form

$$(Au)(x) = \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha u(x), \quad x \in \mathbb{R}^n, \tag{2.1}$$

under the assumption that the coefficients a_α belong to $C_{b,u}(\mathbb{R}^n, \mathcal{L}(\mathbb{C}^N))$. One can see that $A : H^m(\mathbb{R}^n, \mathbb{C}^N) \to L^2(\mathbb{R}^n, \mathbb{C}^N)$ is a bounded operator.

The operator A is said to be *elliptic* at the point $x \in \mathbb{R}^n$ if

$$\det a_0(x, \xi) \neq 0$$

for every point $\xi \neq 0$, where

$$a_0(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$$

is the main symbol of A, and A is called *uniformly elliptic* if

$$\inf_{x\in\mathbb{R}^n,\,\omega\in S^{n-1}} \left| \det \sum_{|\alpha|=m} a_\alpha(x)\omega^\alpha \right| > 0,$$

where S^{n-1} refers to the unit sphere in $\mathbb{R}^n$.

The Fredholm properties of the operator $A : H^s(\mathbb{R}^n,\mathbb{C}^N) \to H^{s-m}(\mathbb{R}^n,\mathbb{C}^N)$ can be expressed in terms of its limit operators which are defined as follows (see e.g. [21]). Let $h : \mathbb{N} \to \mathbb{R}^n$ be a sequence tending to infinity. Since $a_\alpha \in C_{b,u}(\mathbb{R}^n, \mathcal{L}(\mathbb{C}^N))$, the Arcelà–Ascoli's theorem combined with a Cantor diagonal argument implies that there exists a subsequence g of h such that the sequences of the functions $x \mapsto a_\alpha(x+g(k))$ converge as $k \to \infty$ to a limit function a_α^g uniformly on every compact set $K \subset \mathbb{R}^n$ for every multi-index α. The operator

$$A^g := \sum_{|\alpha|\le m} a_\alpha^g D^\alpha$$

is called the *limit operator of A defined by the sequence g*. We denote by $\mathrm{Lim}(A)$ the set of all limit operators of the differential operator A.

Theorem 2.1 ([21])**.** *Let A be a differential operator of the form* (2.1)*. Then $A : H^m(\mathbb{R}^n,\mathbb{C}^N) \to L^2(\mathbb{R}^n,\mathbb{C}^N)$ is a Fredholm operator if and only if:*

(i) *A is a uniformly elliptic operator on $\mathbb{R}^n$;*

(ii) *all limit operators of A are invertible as operators from $H^m(\mathbb{R}^n,\mathbb{C}^N)$ to $L^2(\mathbb{R}^n,\mathbb{C}^N)$.*

Note that the uniform ellipticity of the operator A implies the a priori estimate

$$\|u\|_{H^2(\mathbb{R}^n,\mathbb{C}^N)} \le C\left(\|Au\|_{L^2(\mathbb{R}^n,\mathbb{C}^N)} + \|u\|_{L^2(\mathbb{R}^n,\mathbb{C}^N)}\right). \tag{2.2}$$

This estimate allows one to consider the uniformly elliptic differential operator A as a closed unbounded operator on $L^2(\mathbb{R}^n,\mathbb{C}^N)$ with a dense domain $H^m(\mathbb{R}^n,\mathbb{C}^N)$. It turns out (see [2, p. 27–32]) that A, considered as an unbounded operator in this way, is an (unbounded) Fredholm operator if and only if A, considered as a bounded operator from $H^m(\mathbb{R}^n,\mathbb{C}^N)$ to $L^2(\mathbb{R}^n,\mathbb{C}^N)$, is a Fredholm operator.

We say that $\lambda \in \mathbb{C}$ belongs to the *essential spectrum* of A if the operator $A - \lambda I$ is not Fredholm as an unbounded differential operator. As above, we denote the essential spectrum of A by $\mathrm{sp}_{ess}\, A$ and the common spectrum of A (considered as an unbounded operator) by $\mathrm{sp}\, A$. Then the assertion of Theorem 2.1 can be stated as follows.

Theorem 2.2 ([21]). *Let A be a uniformly elliptic differential operator of the form* (2.1)*. Then*

$$\mathrm{sp}_{ess}\, A = \bigcup_{A^g \in \mathrm{Lim}(A)} \mathrm{sp}\, A^g. \tag{2.3}$$

3. Fredholm Property of Systems of Strongly Elliptic Partial Differential Operators on $\mathbb{R}^n$

We consider the system of partial differential equations of second order on $\mathbb{R}^n$ in the divergent form

$$\begin{aligned} Au(x) = \sum_{k,l=1}^{n} (i\partial_{x_k} - a_k(x)) b^{kl}(x)(i\partial_{x_l} - a_l(x))u(x) \\ + W(x)u(x), \quad x \in \mathbb{R}^n, \end{aligned} \tag{3.1}$$

where

$$a_k \in C^1_{b,u}(\mathbb{R}^n), \quad b^{kl} \in C^1_{b,u}(\mathbb{R}^n), \quad \mathcal{L}(\mathbb{R}^n)), \quad W \in C_{b,u}(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n)), \tag{3.2}$$

a_k are real-valued functions, b^{kl} are Hermitian matrices, that is, $b^{kl}(x)^* = b^{kl}(x)$, and W is a complex-valued matrix. The conditions (3.2) provide the boundedness of $A : H^2(\mathbb{R}^n, \mathbb{C}^n) \to L^2(\mathbb{R}^n, \mathbb{C}^n)$. We suppose that the operator A is strongly elliptic, that is there exists a constant $\gamma > 0$ such that for every $h \in \mathbb{C}^N$ and $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{R}^n$,

$$\sum_{k,l=1}^{n} (b^{kl}(x)h, h)_{\mathbb{C}^N} \nu_k \nu_l \geq \gamma \|h\|^2_{\mathbb{C}^N} \|\nu\|^2_{\mathbb{R}^n}. \tag{3.3}$$

Theorem 3.1. *Let the conditions* (3.2)*,* (3.3) *and*

$$\liminf_{x \to \infty} \inf_{\|h\|_{\mathbb{C}^N} = 1} \Im \langle W(x)h, h \rangle_{\mathbb{C}^N} > 0 \tag{3.4}$$

hold. Then $A : H^2(\mathbb{R}^n, \mathbb{C}^N) \to L^2(\mathbb{R}^n, \mathbb{C}^n)$ is a Fredholm operator of the index 0*.*

Proof. Since A is a uniformly elliptic operator, by the condition (3.3) we have to prove that all limit operators A^g of the operator A are invertible from $H^2(\mathbb{R}^n, \mathbb{C}^n) \to L^2(\mathbb{R}^n, \mathbb{C}^n)$. The limit operators A^g are of the form

$$\begin{aligned} A^g u(x) = \sum_{k,l=1}^{n} (i\partial_{x_k} - a^g_k(x))(b^{kl})^g(x)(i\partial_{x_l} - a^g_l(x))u(x) \\ + W^g(x)u(x), \quad x \in \mathbb{R}^n. \end{aligned} \tag{3.5}$$

The condition (3.4) implies that there exists $\epsilon > 0$ such that for every $x \in \mathbb{R}^n$,

$$\Im (W^g(x)h, h)_{\mathbb{C}^N} \geq \epsilon \|h\|^2_{\mathbb{C}^N}. \tag{3.6}$$

Then for every $u \in H^2(\mathbb{R}^n, \mathbb{C}^N)$,

$$\begin{aligned}\left|(A^g u, u)_{L^2(\mathbb{R}^n,\mathbb{C})}\right| &\geq \Im(A^g u, u)_{L^2(\mathbb{R}^n,\mathbb{C}^N)} \\ &= \int\limits_{\mathbb{R}^n} \Im(W^g u, u)_{\mathbb{C}^N}\, dx \geq \epsilon \|u\|^2_{L^2(\mathbb{R}^n,\mathbb{C}^N)}. \end{aligned} \tag{3.7}$$

This estimate yields that there exists an inverse in the algebraic sense operator $(A^g)^{-1}$, bounded in $L^2(\mathbb{R}^n, \mathbb{C}^N)$. Since A is a uniformly elliptic operator on $\mathbb{R}^n$, the following a priory estimate

$$\|u\|_{H^2(\mathbb{R}^n,\mathbb{C}^N)} \leq C \left(\|Au\|_{L^2(\mathbb{R}^n,\mathbb{C}^n)} + \|u\|_{L^2(\mathbb{R}^n,\mathbb{C}^N)}\right) \tag{3.8}$$

holds. The last estimate implies that all limit operators $A^g : H^2(\mathbb{R}^n, \mathbb{C}^N) \to L^2(\mathbb{R}^n, \mathbb{C}^n)$ are invertible. Then by Theorem 2.1, $A : H^2(\mathbb{R}^n, \mathbb{C}^N) \to L^2(\mathbb{R}^n, \mathbb{C}^n)$ is a Fredholm operator. Let us prove that $\operatorname{index} A = 0$. We consider the family of differential operators $A_\mu = A + \mu^2 I$, $\mu \geq 0$. As above, one can prove that $A_\mu : H^2(\mathbb{R}^n, \mathbb{C}^N) \to L^2(\mathbb{R}^n, \mathbb{C}^n)$ are Fredholm operators. Note that A_μ is an elliptic family depending on the parameter $\mu \geq 0$ (see e.g. [3]). Hence there exists $\mu_0 > 0$ such that A_μ is an invertible operator for $\mu > \mu_0$. Hence $\operatorname{index} A = 0$ because the family $A_\mu : H^2(\mathbb{R}^n, \mathbb{C}^N) \to L^2(\mathbb{R}^n, \mathbb{C}^N)$ is continuously depending on the parameter μ. □

4. Schrödinger Operators with a Complex Potential

We consider the Schrödinger operator

$$\begin{aligned}\mathcal{H}u(x) := \frac{1}{2m}\Big(D_j + \frac{e}{c}\, a_j(x)\Big)\rho^{jk}(x)\Big(D_j + \frac{e}{c}\, a_k(x)\Big)u(x) \\ + e\Phi(x)u(x), \quad x \in \mathbb{R}^n,\end{aligned}$$

where $D_j = \frac{\hbar}{i}\frac{\partial}{\partial x_j}$, $\hbar$ is a Planck constant, m is the electron mass, c is the light speed in the vacuum, $\mathbf{a} = (a_1, \ldots, a_n)$ is a magnetic potential, and Φ is an electrical potential on $\mathbb{R}^n$, the latter equipped with a Riemannian metric $\rho = (\rho_{jk})^n_{j,k=1}$ which is subject to the positivity condition

$$\inf_{x\in\mathbb{R}^n,\, \omega\in S^{n-1}} \rho_{jk}(x)\omega^j\omega^k > 0, \tag{4.1}$$

where $\rho_{jk}(x)$ refers to the matrix, inverse to $\rho^{jk}(x)$. Here and in what follows, we make use of Einstein's summation convention.

We suppose that ρ^{jk}, a_j are real-valued functions in $C^1_{b,u}(\mathbb{R}^n)$ and a complex valued electric potential $\Phi \in C_{b,u}(\mathbb{R}^n)$. Under these conditions, $\mathcal{H}$ can be considered as a closed unbounded operator on $L^2(\mathbb{R}^n)$ with domain $H^2(\mathbb{R}^n)$. If Φ is a real-valued function, then $\mathcal{H}$ is a self-adjoint operator and $\mathcal{H}$ has a real spectrum.

Theorem 4.1. (i) *Let*

$$\liminf_{x\to\infty} \Im\Phi(x) > 0. \tag{4.2}$$

Then the essential spectrum of the operator $\mathcal{H}$ does not contain real values.

(ii) *Let the condition* (4.2) *hold and*

$$\Im\Phi(x) \geq 0 \tag{4.3}$$

for every $x \in \mathbb{R}^n$. Then the spectrum of the operator $\mathcal{H}$ does not contain real values.

Proof. (i) According to formula (2.3),

$$\mathrm{sp}_{ess}\,\mathcal{H} = \bigcup_{A^g \in \mathrm{Lim}\,(\mathcal{H})} \mathrm{sp}\,\mathcal{H}^g, \tag{4.4}$$

where

$$\begin{aligned}\mathcal{H}^g u(x) := \frac{1}{2m}\Big(D_j + \frac{e}{c}\,a_j^g(x)\Big)(\rho^{jk})^g(x)\Big(D_k + \frac{e}{c}\,a_k^g(x)\Big)u(x)\\ + e\Phi^g(x)u(x),\quad x \in \mathbb{R}^n.\end{aligned}$$

We set $\Phi_\lambda = \Phi - \lambda I$, $\lambda \in \mathbb{R}$. The condition (4.2) implies that

$$\inf_{x\in\mathbb{R}^n} \Im\Phi_\lambda^g(x) > 0. \tag{4.5}$$

This condition implies that the operator $\mathcal{H}^g - \lambda I$, $\lambda \in \mathbb{R}$ is invertible with a bounded in $L^2(\mathbb{R}^n)$ inverse operator $(\mathcal{H}^g - \lambda I)^{-1}$. Hence $\mathbb{R} \ni \lambda \notin \mathrm{sp}\,\mathcal{H}^g$. Formula (4.4) implies that $(\mathrm{sp}_{ess}\,\mathcal{H}) \cap \mathbb{R} = \varnothing$.

(ii) As in the proof of Theorem 3.1, we obtain that $\mathcal{H}_\lambda = \mathcal{H} + \lambda I : H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ are Fredholm operators of the index zero. Let us prove that $\ker\mathcal{H}_\lambda = \{0\}$. Let $u \in \ker\mathcal{H}_\lambda$. Estimates (4.1), (4.2), and (4.3) imply that there exists ϵ and $R > 0$ such that

$$\begin{aligned}0 = \Im(\mathcal{H}_\lambda u, u)_{L^2(\mathbb{R}^n,\mathbb{C}^N)} &= \Im\int_{\mathbb{R}^n}(e\Phi(x)u(x), u(x))_{\mathbb{C}^N}\,dx\\ &= \Im\int_{|x|<R}(e\Phi(x)u(x), u(x))_{\mathbb{C}^N}\,dx + \Im\int_{|x|\geq R}(e\Phi(x)u(x), u(x))_{\mathbb{C}^N}\,dx\\ &\geq \epsilon\|u\|^2_{L^2(B'_R,\mathbb{C}^N)}.\end{aligned} \tag{4.6}$$

Since $\ker\mathcal{H}_\lambda \subset H^2(\mathbb{R}^n)$, the estimate (4.6) implies that

$$u\big|_{\partial B_R} = 0, \quad \frac{\partial u}{\partial\nu}\Big|_{\partial B_R} = 0, \tag{4.7}$$

where $\frac{\partial u}{\partial \nu}$ is a normal derivative to the sphere ∂B_R. By the uniqueness of a solution of the Cauchy problem, for elliptic equations with the oldest Lipschitz coefficients (see e.g. [4,7,9,10]), we obtain that the Cauchy problem

$$\begin{aligned}Au(x) = 0,\ \ x \in B_R,\\ u\big|_{\partial B_R} = 0, \quad \frac{\partial u}{\partial\nu}\,\big|_{\partial B_R} = 0\end{aligned}$$

has the trivial solution only. Hence $u = 0$ on $\mathbb{R}^n$. That is, $\ker \mathcal{H}_\lambda = \{0\}$ and $\mathcal{H}_\lambda : H^2(\mathbb{R}^n, \mathbb{C}^N) \to L^2(\mathbb{R}^n, \mathbb{C}^N)$ is an invertible operator. This implies that $\operatorname{sp} \mathcal{H} \cap \mathbb{R} = \varnothing$. □

5. Dirac Operators with Complex Electric Potentials

In this section we consider the Dirac operator on $\mathbb{R}^3$, equipped with the Riemannian metric tensor (ρ_{jk}) depending on $x \in \mathbb{R}^3$ (for a general account on Dirac operators see, for example, [28]). We suppose that there is a constant $C > 0$ such that

$$\rho_{jk}(x)\xi^j\xi^k \geq C|\xi|^2, \quad x \in \mathbb{R}^3, \tag{5.1}$$

where we use as above Einstein's summation convention. Let ρ^{jk} be the tensor, inverse to ρ_{jk}, and let $\phi^{jk}(x) = \sqrt{\rho^{jk}(x)}$ be the positive square root. The Dirac operator on $\mathbb{R}^3$ is the matrix operator defined as

$$\mathcal{D} := \frac{c}{2}\gamma_k(\phi^{jk}P_j + P_j\phi^{jk}) + c^2 m\gamma_0 + e\Phi E_4 \tag{5.2}$$

acting on vector functions on $\mathbb{R}^3$ with values in $\mathbb{C}^4$. In (5.2), the γ_k, $k = 0, 1, 2, 3$, are the 4×4 Dirac matrices, i.e., they satisfy

$$\gamma_j\gamma_k + \gamma_k\gamma_j = 2\delta_{jk}E_4 \tag{5.3}$$

for all choices of $j, k = 0, 1, 2, 3$, E_4 is the 4×4 unit matrix,

$$P_j = D_j + \frac{e}{c}\,a_j, \quad D_j = \frac{\hbar}{i}\,\frac{\partial}{\partial x_j}\,, \quad j = 1, 2, 3,$$

where $\hbar$ is the Planck constant, $\mathbf{a} = (a_1, a_2, a_3)$ is the vector potential of the magnetic field $\mathbf{H}$, that is, $\mathbf{H} = \nabla \times a$, Φ is the scalar potential of the electric field $\mathbf{E}$, that is, $\mathbf{E} = \nabla\Phi$, and m and e are the mass and the charge of the electron, c is a light speed in the vacuum.

We suppose that

$$\rho^{jk}, a_j \in SO^2(\mathbb{R}^3), \quad j, k = 1, 2, 3, \quad \Phi \in SO^1(\mathbb{R}^3), \tag{5.4}$$

and ρ^{jk}, A_j are real-valued functions, and electrical potential Φ can be a complex function. We consider the operator $\mathcal{D}$ as an unbounded operator on the Hilbert space $L^2(\mathbb{R}^3, \mathbb{C}^4)$ with domain $H^1(\mathbb{R}^3, \mathbb{C}^4)$.

Note that the main symbol of $\mathcal{D}$ is $\sigma_{\mathcal{D}}(x, \xi) = c\phi^{jk}(x)\xi_j\gamma_k$. Using (5.3) and the identity $\phi^{jk}\phi^{rt}\delta_{kt} = \rho^{jr}$, we obtain that

$$\begin{aligned}\sigma_{\mathcal{D}}(x, \xi)^2 &= c^2\hbar^2\phi^{jk}(x)\phi^{rt}(x)\xi_j\xi_r\gamma_k\gamma_t \\ &= c^2\hbar^2\phi^{jk}(x)\phi^{rt}(x)\delta_{kt}\xi_j\xi_r = (c^2\hbar^2\rho^{jr}(x)\xi_j\xi_r)E_4.\end{aligned}$$

Together with (5.1), this equality shows that $\mathcal{D}$ is a uniformly elliptic matrix differential operator on $\mathbb{R}^3$. Hence the following a priory estimate

$$\|u\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)} \leq C\left(\|\mathcal{D}u\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} + \|u\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\right)$$

holds which implies that $\mathcal{D}$ is a closed operator in $L^2(\mathbb{R}^3, \mathbb{C}^4)$ with domain $H^1(\mathbb{R}^3, \mathbb{C}^4)$. It follows from the conditions (5.4) that the limit operators $\mathcal{D}^g$

of $\mathcal{D}$ defined by sequences $g : \mathbb{Z} \to \mathbb{R}^3$ tending to infinity are the operators with the constant coefficients of the form

$$\mathcal{D}^g = c\gamma_k(\phi^{jk})^g\Big(D_j + \frac{e}{c}\,a_j^g\Big) + mc^2\gamma_0 - e\Phi^g E_4,$$

where

$$\begin{gathered}(\phi^{jk})^g := \lim_{m\to\infty} \phi^{jk}(g(m)),\\ a_j^g := \lim_{m\to\infty} a_j(g(m)), \quad \Phi^g := \lim_{m\to\infty} \Phi(g(m)).\end{gathered} \tag{5.5}$$

The operator $\mathcal{D}$ is unitarily equivalent to the operator

$$\mathcal{D}_1^g = c\gamma_k(\phi^{jk})^g D_j + \gamma_0 mc^2 + e\Phi^g,$$

and the equivalence is realized by the unitary operator $T_{a^g} : f \mapsto e^{i\,\frac{e}{c}\,a^g\cdot x} f$, $a^g := (a_1^g, a_2^g, a_3^g)$. Let $\Phi \in SO(\mathbb{R}^3)$, and $\Phi_\infty \subset \mathbb{C}$ be the set of all particular limits $\Phi^g = \lim\limits_{m\to\infty} \Phi(g(m))$ defined by sequences $\mathbb{R}^3 \ni g(m) \to \infty$.

Theorem 5.1. *Let the conditions* (5.1) *be fulfilled. Then the Dirac operator*

$$\mathcal{D} : H^1(\mathbb{R}^3, \mathbb{C}^4) \to L^2(\mathbb{R}^3, \mathbb{C}^4)$$

is a Fredholm operator if and only if

$$\Phi_\infty \cap (-\infty, -mc^2] = \varnothing, \quad \Phi_\infty \cap [mc^2, +\infty) = \varnothing. \tag{5.6}$$

Proof. Set

$$\widehat{\mathcal{D}}_0^g(\xi) := c\hbar\gamma_k(\phi^{jk})^g\xi_j + mc^2\gamma_0 \ \text{ and } \ (\rho^{jk})^g := \lim_{m\to\infty} \rho^{jk}(g(m)).$$

Then

$$\begin{aligned}&(\widehat{\mathcal{D}}_0^g(\xi) - e\Phi^g E_4)(\widehat{\mathcal{D}}_0^g(\xi) + e\Phi^g E_4)\\ &\qquad = \big(c^2\hbar^2(\rho^{jk})^g\xi_j\xi_k + m^2c^4 - (e\Phi^g)^2\big)E_4.\end{aligned} \tag{5.7}$$

The condition (5.6) and the identity (5.7) imply that

$$\det\big((\widehat{\mathcal{D}}_0^g(\xi) + e\Phi^g)E_4\big) \neq 0$$

for every $\xi \in \mathbb{R}^3$. Hence, the operator $\mathcal{D}_1^g : H^1(\mathbb{R}^3, \mathbb{C}^4) \to L^2(\mathbb{R}^3, \mathbb{C}^4)$ is invertible and, consequently, so is $\mathcal{D}^g$. By Theorem 2.1, $\mathcal{D}$ is a Fredholm operator. For the reverse implication, assume that the condition (5.6) is not fulfilled. Then there exist $\Phi^g \in \mathbb{C}$ and a vector $\xi^0 \in \mathbb{R}^3 \setminus \{0\}$ such that

$$c^2(\rho^{jk})^g\xi_j^0\xi_k^0 + m^2c^4 - (e\Phi^g)^2 = 0.$$

Given ξ^0, we find a vector $u \in \mathbb{C}^4$ such that $v := (\widehat{\mathcal{D}}_0^g(\xi^0) - (e\Phi^g)E)u \neq 0$. Then (5.7) implies that $(\mathcal{D}_0^g(\xi^0) + e\Phi^g E_4)v = 0$, whence $\det(\widehat{\mathcal{D}}_0^g(\xi^0) + e\Phi^g E_4) = 0$. Thus, the operator $\mathcal{D}^g$ is not invertible. By Theorem 2.1, $\mathcal{D}$ cannot be a Fredholm operator. □

Theorem 5.2. *If the condition* (5.1) *is satisfied, then*

$$\mathrm{sp}_{ess}\,\mathcal{D} = e\Phi_\infty + (-\infty, -mc^2] + [mc^2 + \infty),$$

where + denotes the algebraic sum of sets on the complex plane, and $e\Phi_\infty$ *is the set of particular limits of the function* $e\Phi$ *at infinity.*

Proof. Let $\lambda \in \mathbb{C}$. The symbol of the operator $\mathcal{D}^g - \lambda I$ is the function $\xi \mapsto \widehat{\mathcal{D}}_0^g(\xi) + (e\Phi^g - \lambda)E_4$. Invoking (5.7)), we obtain

$$(\widehat{\mathcal{D}}_0^g(\xi) - (e\Phi^g - \lambda)E_4)(\widehat{\mathcal{D}}_0^g(\xi) + (e\Phi^g - \lambda)E_4) \\ = \left[c^2\hbar^2(\rho^{jk})^g\xi_j\xi_k + m^2c^4 - (e\Phi^g - \lambda^2)\right]E_4. \quad (5.8)$$

Then eigenvalues $\lambda_\pm^g(\xi)$ of the matrix $\mathcal{D}_0^g(\xi) - e\Phi^g E_4$ are given by

$$\lambda_\pm^g(\xi) := e\Phi^g \pm (c^2\rho_g^{jk}\xi_j\xi_k + m^2c^4)^{1/2}. \quad (5.9)$$

This implies that

$$\operatorname{sp}\mathcal{D}^g = [e\Phi^g + mc^2, +\infty) \cup (-\infty, e\Phi^g - mc^2].$$

Hence,

$$\operatorname{sp}_{ess}\mathcal{D} = \cup_g \operatorname{sp}\mathcal{D}^g = e\Phi_\infty + [mc^2, +\infty) + [-\infty, -mc^2). \qquad \square$$

Theorem 5.3. *Let the condition* (5.3) *be satisfied and*

$$\inf \Im\Phi^2(x) \geq 0, \quad \liminf \Im\Phi^2(x) > 0. \quad (5.10)$$

Then the Dirac operator

$$\mathcal{D} : H^1(\mathbb{R}^3, \mathbb{C}^4) \to L^2(\mathbb{R}^3, \mathbb{C}^4)$$

is invertible.

Proof. Let $\mathcal{D}_\mu = \mathcal{D} + \mu I : H^1(\mathbb{R}^3, \mathbb{C}^4) \to L^2(\mathbb{R}^3, \mathbb{C}^4)$, $\mu \geq 0$. Then according to Theorem 5.1, $\mathcal{D}_\mu$ is the continuous family of Fredholm operators. Moreover, $\mathcal{D}_\mu$ is an elliptic family. This implies that there exists $\mu_0 > 0$ large enough such that $\mathcal{D}_\mu$ are invertible operators for $\mu \geq \mu_0$. Hence $\operatorname{index}\mathcal{D} = 0$. Let us prove that $\ker\mathcal{D} = \{0\}$. Note if $u \in \ker\mathcal{D}$, then $u \in \ker A$, where

$$A = (\mathcal{D}_0 - e\Phi E_4)(\mathcal{D}_0 + e\Phi E_4),$$

and

$$\mathcal{D}_0 = \frac{c}{2}\gamma_k(\phi^{jk}P_j + P_j\phi^{jk}) + c^2m\gamma_0.$$

Since $\rho^{jk} \in SO^2(\mathbb{R}^3)$ and $\Phi \in SO^1(\mathbb{R}^3)$, we obtain that

$$A = (\mathcal{D}_0 - e\Phi E_4)(\mathcal{D}_0 + e\Phi E_4) = L + \mathcal{R}, \quad (5.11)$$

where

$$L = \left[(c^2\hbar^2P_j\rho^{jk}P_k) + m^2c^4 - (e\Phi)^2\right]E_4$$

is the diagonal 4×4 matrix operator with strongly elliptic differential operators of second order on the main diagonal, and

$$\mathcal{R} = \sum_{j=1}^{3} r^j\partial_{x_j} + r^0$$

is a 4×4 matrix differential operator of the first order with coefficients $r^j \in C_0(\mathbb{R}^3, \mathcal{L}(\mathbb{C}^4))$, $j = 0,1,2,3$. Let $u \in \ker A$. Then we obtain

$$0 = (Au, u)_{L^2(\mathbb{R}^3,\mathbb{C}^4)} = c^2\hbar^2 \int\limits_{\mathbb{R}^3} \rho^{jk}(P_j u, P_k u)_{\mathbb{C}^4}\, dx$$
$$+ \int\limits_{\mathbb{R}^3} (m^2c^4 - (e\Phi(x))^2)\|u(x)\|^2_{\mathbb{C}^4}\, dx + \int\limits_{\mathbb{R}^3} (\mathcal{R}u(x), u(x))_{\mathbb{R}^4}\, dx. \quad (5.12)$$

Since $r^j \in C_0(\mathbb{R}^3, \mathcal{L}(\mathbb{C}^4))$ for every $\varepsilon > 0$, there exists $R_0 > 0$ such that

$$\|\mathcal{R}u\|_{L^2(B'_R,\mathbb{C}^4)} \le \varepsilon \|u\|_{L^2(B'_R,\mathbb{C}^4)} \quad (5.13)$$

for $R \ge R_0$. Let $R \ge R_0$ be such that

$$\inf_{\mathbb{B}_R} \Im(e\Phi(x))^2 \ge \epsilon - \varepsilon > 0.$$

The condition (5.10) and formulas (5.12)), (5.13) yield

$$0 = \Im(Au, u)_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \ge (\epsilon - \varepsilon) \int\limits_{B'_R} \|u(x)\|^2_{\mathbb{C}^4}\, dx. \quad (5.14)$$

Note that the operator of second order A is uniformly elliptic. This implies that $\ker A \subset H^2(\mathbb{R}^3, \mathbb{C}^4)$. Hence $u|_{B'_R} = 0$ implies that u is a solution of the homogeneous Cauchy problem

$$Au = 0, \quad x \in B_R, \quad (5.15)$$
$$u\big|_{\partial B_R} = 0, \quad \frac{\partial u}{\partial \nu}\Big|_{\partial B_R} = 0.$$

The matrix operator $A = L + \mathcal{R}$ is a perturbation of the diagonal elliptic operator L of second order by the first order operator $\mathcal{R}$ with bounded coefficients, conserving the Carleman estimates (see e.g. [27, Chapter 14], [6], [7]). Hence the Cauchy problem (5.15) has the trivial solution only, and $\ker \mathcal{D} = \{0\}$. Hence $\mathcal{D}$ is an invertible operator. □

Corollary 5.4. *Let the conditions* (5.3), (5.10) *be satisfied. Then the spectrum of $\mathcal{D}$ does not have real values.*

6. Maxwell's Equation with Complex Electric and Magnetic Permittivity

6.1. **Maxwell's system.** We consider the Maxwell's system describing the harmonic electromagnetic fields

$$\nabla \times \mathbf{H} = i\omega\mathbf{D} + \mathbf{j}, \quad (6.1)$$
$$\nabla \times \mathbf{E} = -i\omega\mathbf{B}, \quad (6.2)$$
$$\nabla \cdot \mathbf{D} = \rho, \quad (6.3)$$
$$\nabla \cdot \mathbf{B} = 0, \quad (6.4)$$

where $\omega > 0$ is a frequency of harmonic vibrations of the electromagnetic field,

$\rho = \rho(x)$ is the volume charge density,

$\mathbf{j} = \mathbf{j}(x)$ is the current density,

$\mathbf{E} = \mathbf{E}(x)$ is the electric field intensivity,

$\mathbf{H} = \mathbf{H}(x)$ is the magnetic field intensivity,

$\mathbf{D} = \mathbf{D}(x)$ is the electric induction vector,

$\mathbf{B} = \mathbf{B}(x)$ is the electric induction vector.

The Maxwell equations are provided by the constitutive relations connecting the vectors $\mathbf{E}$, $\mathbf{H}$ and $\mathbf{D}$, $\mathbf{B}$. We consider relations corresponding to isotropic nonhomogeneous media:

$$\mathbf{D}(x) = \varepsilon(x)\mathbf{E}(x), \tag{6.5}$$

$$\mathbf{B}(x) = \mu(x)\mathbf{H}(x), \tag{6.6}$$

where $\varepsilon = \varepsilon(x), \mu(x)$ are electric and magnetic permittivity given by complex-valued functions on $\mathbb{R}^3$ depending on the frequency ω, such that

$$\inf |\varepsilon(x)| > 0, \quad \inf |\mu(x)| > 0.$$

(In what follows, we will omit the dependence of these functions on ω).

The system (6.1)–(6.6) can be written as

$$\begin{aligned} \nabla \times \mathbf{H} &= i\omega\varepsilon\mathbf{H} + \mathbf{j}, \\ \nabla \times \mathbf{E} &= -i\omega\mu\mathbf{H}, \\ \nabla \cdot \varepsilon\,\mathbf{E} &= \rho, \\ \nabla \cdot \mu\,\mathbf{H} &= 0. \end{aligned} \tag{6.7}$$

We associate with the system (6.7) the operator $M : H^1(\mathbb{R}^3, \mathbb{C}^6) \to L^2(\mathbb{R}^3, \mathbb{C}^8)$.

6.2. **Quaternionic representation of Maxwell's system.** To study the Fredholm property and invertibility of the Maxwell's operators, it is convenient to consider their quaternionic realizations (see the book [11]). We let $\mathbb{H}$ denote the complex quaternionic algebra, which is the associative algebra over the field $\mathbb{C}$ generated by four elements 1, $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$ satisfying the conditions

$$\mathbf{e}^1\mathbf{e}^2 = \mathbf{e}^3, \quad \mathbf{e}^2\mathbf{e}^3 = \mathbf{e}^1, \quad \mathbf{e}^3\mathbf{e}^1 = \mathbf{e}^2$$

and

$$1^2 = 1, \quad (\mathbf{e}^k)^2 = -1, \quad 1\mathbf{e}^k = \mathbf{e}^k 1 = \mathbf{e}^k, \quad \mathbf{e}^k\mathbf{e}^j = -\mathbf{e}^j\mathbf{e}^k$$

for $j, k = 1, 2, 3$. Each of the elements $1, \mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$ commutes with the imaginary unit i. Hence, every element $\check{q} \in \mathbb{H}$ has a unique decomposition

$$\check{q} = q_0 + q_1\mathbf{e}^1 + q_2\mathbf{e}^2 + q_3\mathbf{e}^3 =: q_0 + \mathbf{q}$$

with $q_j \in \mathbb{C}$. The number q_0 is called the scalar part of the quaternion q, and $\mathbf{q}$ is its vector part. One can also think of $\mathbb{H}$ as a complex linear space of dimension 4 with usual linear operations.

With respect to the base $\{1, \mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3\}$ of this space, the operator of multiplication from the left and from the right by 1 has the unit 4×4 matrix E^0 as its matrix representation, whereas the matrix representations E_l^j and E_r^j of the operators of multiplication from the left and from the right by $\mathbf{e}_j$, $j = 1, 2, 3$, are real and skew-symmetric matrices. In what follows, if $\check{a}$ is a quaternion, we denote in a usual way the operator multiplication by $\check{a}$ from the left as $\mathbb{H} \ni \check{u} \to \check{a}\check{u} \in \mathbb{H}$, and we denote the operator multiplication by $\check{a}$ from the right as $\mathbb{H} \ni \check{u} \to \check{a}^r\check{u} = \check{u}\check{a} \in \mathbb{H}$. Let $\check{a} = a_0 + a_1\mathbf{e}^1 + a_2\mathbf{e}^2 + a_3\mathbf{e}^3$. Then the operators $\check{u} \to \check{a}\check{u}$ and $\check{u} \to \check{a}^r\check{u}$ have in the base $\{1, \mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3\}$

$$\text{the matrices } \mathfrak{M}_{\check{a}} \text{ and } \mathfrak{M}_{\check{a}^r}\colon\ \mathfrak{M}_{\check{a}} = \sum_{j=0}^{3} a_j E_l^j, \mathfrak{M}_{\check{a}^r} = \sum_{j=0}^{3} a_j E_r^j.$$

The space $\mathbb{H}$ carries also the structure of a complex Hilbert space via the scalar product

$$(\check{q}, \check{r})_{\mathbb{H}} := q_0\overline{r_0} + q_1\overline{r_1} + q_2\overline{r_2} + q_3\overline{r_3}.$$

By $L^2(\mathbb{R}^3, \mathbb{H})$ we denote the Hilbert space of all measurable and squared integrable quaternion valued functions $\check{u}(x) = u(x) + \mathbf{u}(x)$ on $\mathbb{R}^3$ which is provided with the scalar product

$$(\check{u}, \check{v})_{L^2(\mathbb{R}^3,\mathbb{H})} = \int_{\mathbb{R}^3} (\check{u}(x), \check{v}(x))_{\mathbb{H}}\, dx,$$

and by $H^s(\mathbb{R}^3, \mathbb{H})$ the Sobolev space of order $s \in \mathbb{R}$ with the norm

$$\|\check{u}\|_{H^s(\mathbb{R}^3,\mathbb{H})} = \left(\int_{\mathbb{R}^3} \|(1-\Delta)^{s/2}\check{u}(x)\|^2_{L^2(\mathbb{R}^3,\mathbb{H})}\, dx \right)^{1/2}.$$

It is clear that $L^2(\mathbb{R}^3, \mathbb{H})$ and $H^s(\mathbb{R}^3, \mathbb{H})$ are isometrically isomorphic to $L^2(\mathbb{R}^3, \mathbb{C}^4)$ and $H^s(\mathbb{R}^3, \mathbb{C}^4)$. Let

$$D\check{u}(x) = \mathbf{e}^j \partial_{x_j}\check{u}(x), \quad x \in \mathbb{R}^3,$$

be the Moisil–Teodorescu differential operator of the first order acting from $H^s(\mathbb{R}^3, \mathbb{H})$ into $H^{s-1}(\mathbb{R}^3, \mathbb{H})$. The operator D has remarkable properties:

$$D\check{u}(x) = Du_0(x) + D\mathbf{u}(x) = -\nabla \cdot u(x) + \nabla u_0(x) + \nabla \times \mathbf{u}(x) \tag{6.8}$$

for the quaternionic function $\check{u} = u_0 + \mathbf{u}$ and

$$D^2\check{u} = -\Delta\check{u}, \quad \check{u} \in H^2(\mathbb{R}^3, \mathbb{H})), \tag{6.9}$$

where $\Delta = \sum\limits_{j=1}^{3} \partial_{x_j^2}$ is the Laplacian. In what follows, we need the formula of differentiation of the product of a quaternion function $\check{f} \in C^1(\mathbb{R}^3, \mathbb{H})$ by a scalar function $a \in C^1(\mathbb{R}^3)$,

$$D(a\check{f}) = a(D\check{f}) + (\nabla a)\check{f}. \tag{6.10}$$

Properties (6.8), (6.10) allow us to write Maxwell's system (6.1)–(6.6) in the quaternionic form (see [11, p. 88]),

$$D\mathbf{E}(x) = \varepsilon^{-1}(x)\nabla\varepsilon(x)\cdot\mathbf{E} - i\omega\mu(x)\mathbf{H}(x) - \frac{\rho(x)}{\varepsilon(x)}\,, \tag{6.11}$$

$$D\mathbf{H}(x) = \mu^{-1}(x)\nabla\mu(x)\cdot\mathbf{H} + i\omega\varepsilon(x)\mathbf{E}(x) + \mathbf{j}(x), \tag{6.12}$$

where $\mathbf{a}\cdot\mathbf{b} = \sum_{j=1}^{3} a_j b_j$. Applying formula

$$\mathbf{a}\cdot\mathbf{b} = -\frac{1}{2}\,(\mathbf{ab} + \mathbf{ba}),$$

where **ab** and **ba** denote the product of the vectors as quaternions, we obtain

$$D\mathbf{E}(x) = -\frac{1}{2}\,\varepsilon^{-1}(x)(\nabla\varepsilon(x)) + (\nabla\varepsilon(x))^r\mathbf{E}(x) - i\omega\mu(x)\mathbf{H}(x) - \frac{\rho(x)}{\varepsilon(x)}\,,$$

$$D\mathbf{H}(x) = -\frac{1}{2}\,\mu^{-1}(x)(\nabla\mu(x) + (\nabla\mu(x))^r)\mathbf{H}(x) + i\omega\varepsilon(x)\mathbf{E}(x) + \mathbf{j}(x).$$

We associate with the system (6.11), (6.12) the quaternionic matrix operator

$$\mathcal{M}\begin{pmatrix}\mathbf{E}(x)\\ \mathbf{H}(x)\end{pmatrix} = \begin{pmatrix} D\mathbf{E}(x) + \frac{1}{2}\,(\varepsilon^{-1}(x)\nabla\varepsilon(x) + \nabla\varepsilon(x)^r\mathbf{E}(x)) + i\omega\mu(x)\mathbf{H}(x) \\ D\mathbf{H}(x) + \frac{1}{2}\left(\mu^{-1}(x)(\nabla\mu(x) + \nabla\mu(x)^r\mathbf{H}(x)) - i\omega\varepsilon(x)\mathbf{E}(x)\right)\end{pmatrix} \tag{6.13}$$

acting from $H^1(\mathbb{R}^3, \mathbb{H}^2)$ into $L^2(\mathbb{R}^3, \mathbb{H}^2), \mathbb{H}^2 = \mathbb{H}\times\mathbb{H}$.

Remark 6.1. Since a quaternionic system of equations can be written in the matrix-vectorial form, we can apply the limit operators approach for investigation of the Fredholm property of the operator $\mathcal{M}$.

6.3. **Fredholm property and invertibility.**

Theorem 6.2. *Let*

$$\liminf_{x\to\infty} \Im k^2(x) > 0, \tag{6.14}$$

where $k^2(x) = \omega^2\varepsilon(x)\mu(x)$ *is square of the wave number of Maxwell's system. Then* $\mathcal{M} : H^1(\mathbb{R}^3, \mathbb{H}^2) \to L^2(\mathbb{R}^3, \mathbb{H}^2)$ *is a Fredholm operator of the index* 0.

Proof. We follow to the above given scheme of the proof of the Fredholm properties. The main symbol of $\mathcal{M}$ is a quaternionic matrix function

$$\sigma_{\mathcal{M}}(\xi) = \begin{pmatrix} i\mathbf{e}^j\xi_j & 0 \\ 0 & i\mathbf{e}^j\xi_j\end{pmatrix},$$

and

$$\sigma^2_{\mathcal{M}}(\xi) = \begin{pmatrix} |\xi|^2E_4 & 0 \\ 0 & |\xi|^2E_4\end{pmatrix}, \quad |\xi|^2 = |\xi_1|^2 + |\xi_2|^2 + |\xi_3|^2. \tag{6.15}$$

Let

$$\widetilde{\sigma}_{\mathcal{M}}(\xi) = \begin{pmatrix} iE_l^j \xi_j & 0 \\ 0 & iE_l^j \xi_j \end{pmatrix}$$

be the main symbol of $\mathcal{M}$ in the matrix representation. Then (6.15) implies that $\mathcal{M}$ is a uniformly elliptic operator. The limit operators $\mathcal{M}^g$ are those with constant coefficients

$$\mathcal{M}^g \begin{pmatrix} \mathbf{E}(x) \\ \mathbf{H}(x) \end{pmatrix} = \begin{pmatrix} D\mathbf{E}(x) + i\omega\mu^g \mathbf{H}(x) \\ D\mathbf{H}(x) - i\omega\varepsilon^g \mathbf{E}(x) \end{pmatrix}, \tag{6.16}$$

where

$$\mu^g = \lim_{m\to\infty} \mu(g(m)), \quad \varepsilon^g = \lim_{m\to\infty} \varepsilon(g(m))$$

and

$$\lim_{x\to\infty} \nabla\mu(x) = \lim_{x\to\infty} \nabla\varepsilon(x) = 0,$$

since $\varepsilon, \mu \in SO^2(\mathbb{R}^n)$. We will prove that the condition (6.14) provides the invertibility of the operators $\mathcal{M}^g : H^1(\mathbb{R}^3, \mathbb{H}^2) \to L^2(\mathbb{R}^3, \mathbb{H}^2)$. Indeed, let

$$\mathcal{M}^g \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} D\mathbf{E} + i\omega\mu^g \mathbf{H} \\ D\mathbf{H} - i\omega\varepsilon^g \mathbf{E} \end{pmatrix} = \begin{pmatrix} \mathbf{F} \\ \mathbf{\Phi} \end{pmatrix}. \tag{6.17}$$

The system (6.17) is reduced to two independent equations

$$-(\Delta + (k^g)^2)\mathbf{E} = D\mathbf{F} - i\omega\mu^g \mathbf{\Phi}, \tag{6.18}$$

$$-(\Delta + (k^g)^2)\mathbf{H} = D\mathbf{\Phi} + i\omega\varepsilon^g \mathbf{F}, \tag{6.19}$$

where $(k^g)^2 = \omega^2 \varepsilon^g \mu^g$ is a square of the wave number of the limit operator. Since $\Im(k^g)^2 > 0$ the operators $(\Delta^2 + (k^g)^2) : H^2(\mathbb{R}^3, \mathbb{H}(\mathbb{C})) \to L^2(\mathbb{R}^3, \mathbb{H}(\mathbb{C}))$ are invertible, and we obtain

$$(\mathcal{M}^g)^{-1} \begin{pmatrix} \mathbf{F} \\ \mathbf{\Phi} \end{pmatrix} = \begin{pmatrix} -D(\Delta + (k^g)^2)^{-1}\mathbf{F} - i\omega\mu^g(\Delta + (k^g)^2)^{-1}\mathbf{\Phi} \\ -D(\Delta + (k^g)^2)^{-1}\mathbf{\Phi} + i\omega\varepsilon^g(\Delta + (k^g)^2)^{-1}\mathbf{F} \end{pmatrix}. \tag{6.20}$$

It follow from (6.20) that $(\mathcal{M}^g)^{-1}$ is a bounded operator from $L^2(\mathbb{R}^3, \mathbb{H}^2)$ into $H^1(\mathbb{R}^3, \mathbb{H}^2)$. Hence the limit operators

$$\mathcal{M}^g : H^1(\mathbb{R}^3, \mathbb{H}^2) \to L^2(\mathbb{R}^3, \mathbb{H}^2)$$

are invertible. Thus Theorem 2.1 implies that

$$\mathcal{M} : H^1(\mathbb{R}^3, \mathbb{H}^2) \to L^2(\mathbb{R}^3, \mathbb{H}^2)$$

is a Fredholm operator.

Let us consider the family of operators $\mathcal{M}_\lambda = \mathcal{M} + \lambda I$, $\lambda \geq 0$. It is easy to see that $\mathcal{M}_\lambda$ is the family of elliptic systems with a parameter. Moreover, as above, $\mathcal{M}_\lambda$ is a Fredholm family, continuously depending on the parameter $\lambda \geq 0$. Hence index $\mathcal{M} = 0$. □

Theorem 6.3. *Let $\varepsilon, \mu \in SO^2(\mathbb{R}^3)$, and*

$$\Im k^2(x) \geq 0, \tag{6.21}$$

and the condition (6.14) *be satisfied. Then the operator*

$$\mathcal{M} : H^1(\mathbb{R}^3, \mathbb{H}^2) \to L^2(\mathbb{R}^3, \mathbb{H}^2)$$

is invertible.

Proof. It remains to prove that $\ker \mathcal{M} = \{0\}$. Suppose that $\mathbf{u} = \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \in \ker \mathcal{M}$. Then $\mathbf{u}$ satisfies the homogeneous system of equations

$$D\mathbf{E}(x) = \varepsilon^{-1}(x)\nabla\varepsilon(x) \cdot \mathbf{E} - i\omega\mu(x)\mathbf{H}(x), \tag{6.22}$$

$$D\mathbf{H}(x) = \mu^{-1}(x)\nabla\mu(x) \cdot \mathbf{H} + i\omega\varepsilon(x)\mathbf{E}(x). \tag{6.23}$$

Applying differentiation formula (6.10)), we reduce this system to the following ones:

$$\begin{aligned} (D^2 - k^2(x))\mathbf{E}(x) - D(\varepsilon^{-1}(x)\nabla\varepsilon(x) \cdot \mathbf{E}) + i\omega\nabla\mu(x) \cdot \mathbf{H}(x) = 0, \\ (D^2 - k^2(x))\mathbf{H}(x) - D(\mu^{-1}(x)\nabla\mu(x) \cdot \mathbf{H}) - i\omega\nabla\varepsilon(x) \cdot \mathbf{E}(x) = 0. \end{aligned} \tag{6.24}$$

Hence $\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}$ satisfies the homogeneous system of quaternionic equations

$$\mathcal{B}\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} := -(\Delta + k^2(x))\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} + \mathcal{T}\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix},$$

where

$$\mathcal{T}\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} := \begin{pmatrix} -D(\varepsilon^{-1}(x)\nabla\varepsilon(x) \cdot \mathbf{E}) + i\omega\nabla\mu(x) \cdot \mathbf{H}(x) \\ -i\omega\nabla\mu(x) \cdot \mathbf{E}(x) - D(\mu^{-1}(x)\nabla\mu(x) \cdot \mathbf{H}) \end{pmatrix}.$$

Note that $\mathcal{T}$ is a matrix quaternionic differential operator of the first order with coefficients in the class $C_0^1(\mathbb{R}^n)$. This implies that

$$\begin{aligned} &\lim_{R\to\infty} \|\varphi_{B'_R}\mathcal{T}\|_{\mathcal{L}(H^2(\mathbb{R}^3,\mathbb{H}^2), L^2(\mathbb{R}^3,\mathbb{H}^2))} \\ &= \lim_{R\to\infty} \|\mathcal{T}\varphi_{B'_R}\|_{\mathcal{L}(H^2(\mathbb{R}^3,\mathbb{H}^2), L^2(\mathbb{R}^3,\mathbb{H}^2))} = 0, \end{aligned}$$

where $\varphi_{B'_R} \in C^\infty(\mathbb{R}^3)$, $0 \le \varphi_{B'_R} \le 1$, $\operatorname{supp}\varphi_{B'_R} \subset B'_R$, $\varphi_{B'_R}(x) = 1$ if $x \in B'_{2R}$. Note that $\ker \mathcal{B} \in H^2(\mathbb{R}^3, \mathbb{C}^6)$ because the operator $\mathcal{B}$ is uniformly elliptic on $\mathbb{R}^3$. Repeating the proof of triviality of the kernel of the Dirac operator, we obtain $\ker \mathcal{M} = \{0\}$. □

Theorems 6.2 and 6.3 imply the following result.

Theorem 6.4. *Let* $\varepsilon, \mu \in SO^2(\mathbb{R}^3)$. *Then:*

(i) *If the condition* (6.14) *is satisfied, then the operator*

$$M : H^1(\mathbb{R}^3, \mathbb{C}^6) \to L^2(\mathbb{R}^3, \mathbb{C}^8)$$

of the Maxwell system is a Fredholm one;

(ii) *If the conditions* (6.14) *and* (6.21) *are satisfied, then*

$$M : H^1(\mathbb{R}^3, \mathbb{C}^6) \to L^2(\mathbb{R}^3, \mathbb{C}^8)$$

is an invertible operator.

Note that the electric and magnetic permittivity in the *dispersive electromagnetic media* are complex-valued functions of the form (see e.g. [13]):

$$\begin{aligned} \varepsilon(x) &= \varepsilon_0\Big(1+\frac{i\sigma_\varepsilon(x)}{\omega}\Big),\\ \mu(x) &= \mu_0\Big(1+\frac{i\sigma_\mu(x)}{\omega}\Big), \end{aligned} \tag{6.25}$$

where ε_0, μ_0 are electric and magnetic permittivity in the vacuum, $\sigma_\varepsilon(x)$, $\sigma_\mu(x)$ are absorption coefficients for the electric and magnetic permittivity satisfying the conditions:

$$\sigma_\varepsilon(x) \geq 0, \quad \sigma_\mu(x) \geq 0. \tag{6.26}$$

This implies that

$$k^2(x) = \frac{\omega^2}{c_0^2}\Big(1+\frac{i\sigma_\varepsilon(x)}{\omega}\Big)\Big(1+\frac{i\sigma_\mu(x)}{\omega}\Big),$$

where c_0 is the light speed in the vacuum.

Thus Theorem 6.4 provides us with the following result.

Theorem 6.5. *Let* $\sigma_\varepsilon, \sigma_\mu \in SO^2(\mathbb{R}^3)$. *Then Maxwell's operator* $M : H^1(\mathbb{R}^3,\mathbb{C}^6) \to L^2(\mathbb{R}^3,\mathbb{C}^8)$ *is invertible if at least one of the conditions*

$$\liminf_{x\to\infty} \sigma_\varepsilon(x) > 0, \quad \liminf_{x\to\infty} \sigma_\mu(x) > 0$$

in (6.25) *holds.*

Acknowledgement

This work is partially supported by the National System of Investigators of Mexico (SNI), and Conacyt Project 000000000179872.

References

1. S. Agmon, Spectral properties of Schrödinger operators and scattering theory. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **2** (1975), No. 2, 151–218.
2. M. S. Agranovich, Elliptic operators on closed manifolds. (Russian) *Current problems in mathematics. Fundamental directions*, Vol. 63 (Russian), 5–129, *Itogi Nauki i Tekhniki*, *Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow*, 1990.
3. M. S. Agranovich and M. I. Vishik, Elliptic problems with a parameter and parabolic problems of general type. (Russian) *Uspehi Mat. Nauk* **19** (1964), No. 3 (117), 53–161.
4. A.-P. Calderón, Uniqueness in the Cauchy problem for partial differential equations. *Amer. J. Math.* **80** (1958), 16–36.
5. H. L. Cycon, R. G. Froese, W. Kirsch, and B. Simon, Schrödinger operators with application to quantum mechanics and global geometry. *Texts and Monographs in Physics. Springer Study Edition. Springer-Verlag, Berlin*, 1987.
6. L. De Carli and T. Ōkaji, Strong unique continuation property for the Dirac equation. *Publ. Res. Inst. Math. Sci.* **35** (1999), No. 6, 825–846.
7. R. Duduchava, Lions' lemma, Korn's inequalities and the Lamé operator on hypersurfaces. *Recent trends in Toeplitz and pseudodifferential operators*, 43–77, *Oper. Theory Adv. Appl.*, 210, *Birkhäuser Verlag, Basel*, 2010.

8. K. Gürlebeck and W. Sprössig, Quaternionic analysis and elliptic boundary value problems. *Mathematical Research*, 56. *Akademie-Verlag, Berlin*, 1989.
9. D. Jerison, Carleman inequalities for the Dirac and Laplace operators and unique continuation. *Adv. in Math.* **62** (1986), No. 2, 118–134.
10. D. Jerison and C. E. Kenig, Unique continuation and absence of positive eigenvalues for Schrödinger operators. With an appendix by E. M. Stein. *Ann. of Math. (2)* **121** (1985), No. 3, 463–494.
11. V. V. Kravchenko, Applied quaternionic analysis. Research and Exposition in Mathematics, 28. *Heldermann Verlag, Lemgo*, 2003.
12. V. V. Kravchenko and M. V. Shapiro, Integral representations for spatial models of mathematical physics. *Pitman Research Notes in Mathematics Series*, 351. *Longman, Harlow*, 1996.
13. K. E. Oughstun, Electromagnetic and optical pulse propagation 1. Spectral representations in temporally dispersive media. Springer Series in Optical Sciences, 125. *Springer, New York*, 2006.
14. V. S. Rabinovich, On the essential spectrum of electromagnetic Schrödinger operators. *Complex analysis and dynamical systems II*, 331–342, Contemp. Math., 382, *Amer. Math. Soc., Providence, RI*, 2005.
15. V. S. Rabinovich, Essential spectrum of perturbed pseudodifferential operators. Applications to the Schrödinger, Klein–Gordon, and Dirac operators. *Russ. J. Math. Phys.* **12** (2005), No. 1, 62–80.
16. V. S. Rabinovich, Boundary problems for domains with conical exits at infinity and limit operators. *Complex Var. Elliptic Equ.* **60** (2015), No. 3, 293–309.
17. V. S. Rabinovich, Transmission problems for conical and quasi-conical at infinity domains. *Appl. Anal.* **94** (2015), No. 10, 2077–2094.
18. V. S. Rabinovich, Integral equations of diffraction problems with unbounded smooth obstacles. *Integr. Equ. Oper. Theory, Springer Basel*, 2015; doi: `10.1007/s00020-015-2249-y`.
19. V. S. Rabinovich and S. Roch, The essential spectrum of Schrödinger operators on lattices. *J. Phys. A* **39** (2006), No. 26, 8377–8394.
20. V. S. Rabinovich and S. Roch, Essential spectra of difference operators on $\mathbb{Z}^n$-periodic graphs. *J. Phys. A* **40** (2007), No. 33, 10109–10128.
21. V. S. Rabinovich and S. Roch, Essential spectrum and exponential decay estimates of solutions of elliptic systems of partial differential equations. Applications to Schrödinger and Dirac operators. *Georgian Math. J.* **15** (2008), no. 2, 333–351.
22. V. S. Rabinovich and S. Roch, Essential spectra and exponential estimates of eigenfunctions of lattice operators of quantum mechanics. *J. Phys. A* **42** (2009), No. 38, 385207, 21 pp.
23. V. Rabinovich, S. Roch, and B. Silbermann, Limit operators and their applications in operator theory. *Operator Theory: Advances and Applications*, 150. *Birkhäuser Verlag, Basel*, 2004.
24. M. Reed and B. Simon, Methods of modern mathematical physics. I. Functional analysis. Second edition. *Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York*, 1980.
25. G. V. Rozenblyum, M. Z. Solomyak, and M. A. Shubin, Spectral theory of differential operators. (Russian) *Current problems in mathematics. Fundamental directions*, Vol. 64 (Russian), 5–248, *Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow*, 1989.
26. M. A. Shubin, Pseudodifferential operators and spectral theory. Translated from the 1978 Russian original by Stig I. Andersson. Second edition. *Springer-Verlag, Berlin*, 2001.
27. M. E. Taylor, Pseudodifferential operators. *Princeton Mathematical Series,* 34. *Princeton University Press, Princeton, N.J.*, 1981.

28. B. Thaller, Thaller, Bernd. The Dirac equation. *Texts and Monographs in Physics. Springer-Verlag, Berlin*, 1992.

(Received 03.03.2015)

Author's address:

National Polytechnic Institute of Mexico, ESIME Zacatenco, edif. 1, Av. Luis Enrique Erro S/N, C.P. 07738, Mexico, DF

E-mail: vladimir.rabinovich@gmail.com