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CERTAIN PROPERTIES OF GENERALIZED ANALYTIC FUNCTIONS FROM SMIRNOV CLASS WITH A VARIABLE EXPONENT

Abstract. Let D be a simply connected domain bounded by a simple, closed, rectifiable curve Γ, $p = p(t)$ be the given on Γ positive measurable function, and $z = z(\zeta)$, $\zeta = re^{i\vartheta}$ be conformal mapping of the circle $U = \{\zeta : \ |\zeta| < 1\}$ onto the domain D.

The function $W(z)$, generalized-analytical in I. Vekua's sense, belongs to the Smirnov class $E^{p(t)}(A; B; D)$, if

(1) $W \in U^{s,2}(A; B; D)$;

(2) $\sup\limits_{0<r<1} \int\limits_0^{2\pi} |W(z(re^{i\vartheta}))|^{p(z(e^{i\vartheta}))} |z'(re^{i\vartheta})| \, d\vartheta < \infty$

(see [15]).

When $p(t)$ is Log–Hölder function continuous in Γ and $\min p(t) = \underline{p} > 1$, we considers the problems of representability of functions from $E^{p(t)}(A; B; D)$ by the generalized Cauchy integral, show the connection between the generalized Cauchy type integral and the generalized singular integral; of special interest is the question of extendability of functions from those classes, and the symmetry principle is proved.

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რეზიუმე. ვთქვათ, D ცალადბმული არეა შემოსაზღვრული მარტივი, გაწრფევადი, შეკრული Γ წირით, $p = p(t)$ მასზე განსაზღვრული დადებითი ზომადი ფუნქციაა, ხოლო $z = z(\zeta)$, $\zeta = re^{i\vartheta}$ ერთეულოვანი $U = \{\zeta : \ |\zeta| < 1\}$ წრის D არეზე კონფორმულად ამსახველი ფუნქციაა.

D არეში ი. ვეკუას აზრით განზოგადებული ანალიზური $W(z)$ ფუნქცია ეკუთვნის სმირნოვის ცვლადმაჩვენებლიან $E^{p(t)}(A; B; D)$ კლასს, თუ:

(1) $W \in U^{s,2}(A; B; D)$;

(2) $\sup\limits_{0<r<1} \int\limits_0^{2\pi} |W(z(re^{i\vartheta}))|^{p(z(e^{i\vartheta}))} |z'(re^{i\vartheta})| \, d\vartheta < \infty$

(იხ. [15]).

ნაშრომში განიხილება $E^{p(t)}(A; B; D)$ კლასის ფუნქციათა კოშის ინტეგრალით წარმოდგენადობის საკითხი, კოშის ტიპის განზოგადებული ინტეგრალით წარმოდგენილი ფუნქციის სმირნოვის კლასისადმი მიკუთვნების საკითხი, შესწავლილია სმირნოვის კლასის ფუნქციათა გაგრძელებადობის საკითხი; დამტკიცებულია სიმეტრიის პრინციპი, როცა $p(t)$ Log–Hölder-ის აზრით უწყვეტი ფუნქციაა Γ-ზე და $\min p(t) = \underline{p} > 1$.

1. Introduction

The Hardy classes H^p of analytic in a unit circle U functions and their generalizations, i.e., Smirnov classes $E^p(D)$, $p > 0$, are the main objects of investigation of mathematical analysis (see [2, 3, 10, 16], etc.). They have a great number of applications in the boundary value problems of the theory of analytic functions.

Recently, the Lebesgue spaces with a variable exponent $L^{p(t)}$ and their applications attract attention of many mathematicians. This tendency has touched upon certain questions of the theory of analytic functions. The notions of Hardy and Smirnov classes (with a variable exponent) of analytic functions have been introduced in [5] and [6] and successfully applied to the boundary value problems; a part of those applications are reflected in [7].

For a constant p, the analogues of Smirnov classes for generalized analytic functions are presented in [4, 11–14] and some boundary value problems in these classes are studied therein.

The perspective to investigate the boundary value problems for generalized analytic functions more thoroughly made it necessary to introduce Smirnov classes with a variable exponent. But towards this end, one has, first of all, to know the properties of generalized Cauchy type integrals and generalized singular integrals with densities from the class $L^{p(t)}$. These questions have been studied in [9]. In particular, the validity of analogues of Sokhotski-Plemelj's formulas in the case of arbitrary, simple, rectifiable curves and summable densities has been proved, and the continuity in the space $L^{p(t)}(\Gamma)$ (with weight) of the operator $\widetilde{S}_\Gamma$ generated by a generalized singular integral when Γ is the Carleson curve has been proved, as well. All that made it possible to introduce the notion of Smirnov classes with a variable exponent for generalized analytic functions and to establish a series of their properties [15]; some of them we will frequently refer to in this work, are cited below, in Subsection 3.1. It should be noted here that in [15] the questions of extension and the symmetry principle for the introduced classes were left unconsidered; the case of unbounded domains was‘ considered superficially; the belonging of the generalized Cauchy type integrals with density from $L^{p(t)}$ to Smirnov classes was not considered in detail.

The present paper, being the continuation of our previous work [15], deals with the problems just mentioned and provides us with many new properties of the generalized Cauchy type integrals and Smirnov classes (with a variable exponent) of generalized analytic functions.

Relying mainly on the results obtained in [9, 15], we have succeeded in investigating the Riemann problem for generalized analytic functions from the introduced Smirnov classes with a variable exponent [8].

2. Preliminaries

2.1. **Generalized analytic functions in I. N. Vekua's sense.** Let D be a simply connected domain bounded by a simple, closed, rectifiable curve Γ and $A(z)$, $B(z)$ be the functions given on D. We extend them by zero on the set $E \setminus D$ when E is the complex plane, retaining the same notation for the obtained functions.

Let $s > 0$ and $L^s(D)$ be a set of functions f, summable on D, of degree s. If $D = E$, then we put $f_\nu(z) \equiv z^\nu f(\frac{1}{z})$, $\nu \in (-\infty, +\infty)$. The set of functions f for which

$$f \in L^s(U), \quad f_\nu(z) \in L^s(U), \ \ s \geq 1, \quad U = \{z: \ |z| < 1\},$$

we denote by $L^{s,\nu}(E)$.

A solution $W(z)$ of the equation

$$LW = \partial_{\overline{z}} W + A(z)W + B(z)\overline{W} = 0 \tag{2.1}$$

is said to be regular in the domain D, if every point $z_0 \in D$ possesses the neighborhood $D(z_0) \subset D$, where W has a generalized in Sobolev sense derivative $\partial_{\overline{z}} W \equiv \frac{1}{2}\,(\frac{\partial W}{\partial x} + i\,\frac{\partial W}{\partial y})$.

If $A, B \in L^{s,2}(D)$, then we denote by $U^{s,2}(A; B; D)$ the set of all regular solutions of the equation (2.1). For $s > 2$, the equation (2.1) has regular solutions and each solution $W(z)$ is representable in the form

$$W(z) = \Phi_{W}(z) \exp \omega_{W}(z) \ \ (= \Phi \exp \omega), \tag{2.2}$$

where Φ_W is analytic in D function, and

$$\omega_W(z) = \frac{1}{2\pi i} \iint\limits_D \Big(A(\zeta) + B(\zeta) \frac{\overline{W(\zeta)}}{W(\zeta)} \Big) \frac{ds}{\zeta - z}\,.$$

The function ω_W belongs to the Hölder class $H_{\frac{s-2}{s}}(E)$ [17, pp. 156, 163]. The function $\Phi_W(z)$ is called a normal analytic divisor of the generalized analytic function $W(z)$ [17, p. 160].

2.2. **Principal kernels of the class $U^{s,2}(A;B;D)$.** Let

$$\phi_1(z) = \frac{1}{2(t-z)}\,, \quad \phi_2(z) = \frac{1}{2i(t-z)}\,,$$

where t is a fixed point of the plane E. Then there exist the functions $X_j(z)$, $j = 1, 2$ (solutions of the equation (2.1)), such that:

(1) $X_{j,0}(z) = \frac{X_j(z)}{\phi_j(z)} \in H_{\frac{s-2}{s}}(E)$;

(2) the functions $X_{j,0}(z)$ are continuous in D and continuously extendable on E;

(3) $X_{j,0}(z) \neq 0$;

(4) $X_{j,0}(t) = 1$.

The functions

$$\Omega_1(z,t) = X_1(z,t) + iX_2(z,t), \quad \Omega_2(z,t) = X_1(z,t) - iX_2(z,t)$$

are called principal normalized kernels of the class $U^{s,2}(A;B;D)$, $s > 2$ [17, p. 193]. There exist bounded functions $m_1(z,t)$, $m_2(z,t)$ such that

$$\Omega_1(z,t) = \frac{1}{t-z} + \frac{m_1(z,t)}{|t-z|^\alpha}\,, \quad \Omega_2(z,t) = \frac{m_2(z,t)}{|t-z|^\alpha}\,, \quad \alpha = \frac{2}{s} \tag{2.3}$$

(see [17, p. 179]).

2.3. **The generalized Cauchy type integral and generalized singular integral.** Let

$$\Gamma = \big\{ t \in E : \ t = t(\sigma),\ 0 \le \sigma \le \ell \big\},$$

where σ is the arc coordinate of the point t.

If Ω_1, Ω_2 are the principal normalized kernels of the class $U^{s,2}(A;B;D)$ and $f \in L(\Gamma)$, then the function

$$W(z) = (\widetilde{K}_\Gamma f)(z) = \frac{1}{2\pi i} \int\limits_\Gamma \Omega_1(z,\tau) f(\tau)\, d\tau - \Omega_2(z,\tau) \overline{f}(\tau)\, d\overline{\tau}$$

is a regular solution of the equation (2.1) of the class $U^{s,2}(A;B;D)$ [17, pp. 156, 168].

The function $(\widetilde{K}_\Gamma f)(z)$ is called the generalized Cauchy type integral. The corresponding singular integral is defined by the equality

$$(\widetilde{S}_\Gamma f)(t) = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int\limits_{\Gamma - \Gamma_\varepsilon(t)} \Omega_1(\tau, z) f(\tau)\, d\tau - \Omega_2(z,\tau) \overline{f}(\tau)\, d\overline{\tau},$$

where $\Gamma_\varepsilon(t)$ is a small in length arc lying on Γ with the ends $t(\sigma - \varepsilon)$ and $t(\sigma + \varepsilon)$.

Under different assumptions for Γ and f, the integrals $(\widetilde{K}_\Gamma f)(z)$ and $(\widetilde{S}_\Gamma f)(t)$ and their interconnections have been studied in [11–14] (for details see [9]). In particular, analogues of Sokhotski–Plemelj's formulas have been obtained. Here we cite the most general results stated in [9].

If Γ is a simple rectifiable curve and $f \in L(\Gamma)$, then the generalized Cauchy type integral $(\widetilde{K}_\Gamma f)(z)$ for almost all $t \in \Gamma$ has angular boundary values $(\widetilde{K}_\Gamma f)^+(t)$ and $(\widetilde{K}_\Gamma f)^-(t)$, and the equalities

$$(\widetilde{K}_\Gamma f)^\pm(t) = \pm \frac{1}{2} f(t) + \frac{1}{2} (\widetilde{S}_\Gamma f)(t) \tag{2.4}$$

are valid.

2.4. **The space $L^{p(t)}(\Gamma)$.** Let $p = p(t)$ be a measurable positive function on Γ. Assume

$$\|f\|_{p(t)} = \inf\Big\{\lambda > 0 : \int\limits_0^\ell \Big|\frac{f(t(\sigma))}{\lambda}\Big|^{p(t(\sigma))} d\sigma \le 1\Big\}$$

and

$$L^{p(t)}(\Gamma) = \big\{f : \|f\|_{p(t)} < \infty\big\}.$$

2.5. **The class of exponents $\mathcal{P}(\Gamma)$.** By $\mathcal{P}(\Gamma)$ we denote a union of those measurable on Γ positive functions $p(t)$ for which:

(1) there exists a constant $c(p)$ such that for any $t_1, t_2 \in \Gamma$ we have

$$|p(t_1) - p(t_2)| < c(p)\big|\ln|t_1 - t_2|\big|^{-1};$$

(2) $\underline{p} = \inf\limits_{t\in\Gamma} p(t) > 1$.

2.6. **On the continuity of the operator $\widetilde{S}_\Gamma$ in the space $L^{p(t)}(\Gamma)$.** Not touching upon the questions dealing with the investigation of that operator for constant p, we will cite here the most general result for the variable $p(t)$ [9].

If Γ is the Carleson curve (in the sequel, we will write $\Gamma \in \mathbb{R}$) and $p(t) \in \mathcal{P}(\Gamma)$, then the operator $\widetilde{S}_\Gamma : f(t) \to (\widetilde{S}_\Gamma f)(t)$ is continuous in $L^{p(t)}(\Gamma; \omega)$, where ω belongs to the definite class of weighted functions, inclusive all admissible power functions of the type

$$\omega = |t - a|^\alpha, \quad -\frac{1}{p(a)} < \alpha < \frac{1}{p'(a)} \quad a \in \Gamma, \quad p'(t) = \frac{p(t)}{p(t) - 1}\,.$$

3. The Variable Smirnov Classes of Generalized Analytic Functions

3.1. **The case of a bounded domain.** Let D be a finite domain bounded by a simple rectifiable curve Γ and μ be a measurable function different from zero almost everywhere on Γ.

We say that the generalized analytic function $W(z)$ belongs to the Smirnov class $E^{p(t)}(A; B; \mu; D)$ if:

(1) $W \in U^{s,2}(A; B; D)$, $s > 2$;

(2)

$$\sup_{0<r<1} \int\limits_0^{2\pi} \big|W(z(re^{i\vartheta}))\mu(z(re^{i\vartheta}))\big|^{p(z(e^{i\vartheta}))} |z'(re^{i\vartheta})|\, d\vartheta < \infty, \tag{3.1}$$

where $z = z(re^{i\vartheta})$ is conformal mapping of U onto D.

Assume $E^{p(t)}(A; B; D) = E^{p(t)}(A; B; 1; D)$.

This class of functions has been considered in [15]. Here we present the results from [15] which we will need in the sequel.

Statement 3.1. *The function $W \in U^{s,2}(A; B; D)$, $s > 2$, belongs to $E^{p(t)}(A; B; D)$ if and only if its normal analytic divisor Φ_W (see Subsection 2.1) belongs to $E^{p(t)}(D)$, i.e.,*

$$\sup_{0<r<1} \int\limits_0^{2\pi} \big|\Phi_W(z(re^{i\vartheta}))\big|^{p(z(e^{i\vartheta}))} |z'(re^{i\vartheta})|\, d\vartheta < \infty. \tag{3.2}$$

Statement 3.2. *The function $W(z) \in E^{p(t)}(A; B; D)$, $\underline{p} > 0$, has angular boundary values $W^+(t)$ for almost all $t \in \Gamma$ and, moreover, $W^+(t) \in L^{p(t)}(\Gamma)$. If $p \in \mathcal{P}(\Gamma)$, then*

$$(\widetilde{K}_\Gamma W^+)(z) = \begin{cases} W(z), & z \in D, \\ 0, & z \in E \setminus D. \end{cases} \tag{3.3}$$

Remark 3.1. It follows from Statement 3.1 that if $W \in E^{p(t)}(A; B; D)$, $\underline{p} > 0$, and $W^+(t) = 0$, $t \in \mathcal{E}$, $\mathcal{E} \subset \Gamma$, $\operatorname{mes}\mathcal{E} > 0$, then $W(z) \equiv 0$, $z \in D$.

Statement 3.3. *If $W \in U^{s,2}(A;B;D)$, $s>2$, and it belongs to $E^1(\widetilde{A};\widetilde{B};D)$, where*

$$\widetilde{A}(z)=\begin{cases} A(z), & z\in D,\\ 0, & z\in E\setminus D,\end{cases}\qquad \widetilde{B}(z)=\begin{cases} B(z), & z\in D,\\ 0, & z\in E\setminus D,\end{cases}$$

then it is representable by the formula

$$W(z)=\frac{1}{2\pi i}\int\limits_{\Gamma}\Omega_1(z,t)W^+(t)\,dt-\Omega_2(z,t)\overline{W^+}(t)\,d\bar{t},$$

when $\Omega_k(z,t)$, $k=1,2$, are the principal normalized kernels of the class $U^{s,2}(\widetilde{A};\widetilde{B};E)$.

Statement 3.4. *If $A,B\in L^{s,2}(D)$, $\Gamma\in\mathbb{R}$, $p\in\mathcal{P}(\Gamma)$, $\overline{p}\,'=\sup\limits_{t\in\Gamma}p'(t)$, $\frac{s}{2}>\overline{p}\,'$, $f\in L^{p(t)}(\Gamma)$, then $\widetilde{K}_\Gamma f$ belongs to $E^{p(t)}(A;B;D)$.*

Corollary 3.1. *If $A,B\in L^\infty(D)$, $\Gamma\in\mathbb{R}$, $p\in\mathcal{P}(\Gamma)$, $f\in L^{p(t)}(\Gamma)$, then $(\widetilde{K}_\Gamma f)(z)\in E^{p(t)}(A;B;D)$.*

3.2. **The case of an unbounded domain.** We will consider only those unbounded domains D whose boundary is a simple, closed, rectifiable curve. For the sake of simplicity, we consider only conformal mappings $z=z(s)$ of the circle U onto the domain D (which we denote by D^-) for which $z(0)=\infty$ and assume that $W\in E^{p(t)}(A;B;D^-)$ if the conditions (3.1) are fulfilled.

From the definition it follows that if $W\in E^1(A;B;D)$, then $W(\infty)=0$. If $p\in\mathcal{P}(\Gamma)$, then this is likewise valid when $W\in E^{p(t)}(A;B;D^-)$ (since $E^{p(t)}(A;B;D^-)\in E^1(A;B;D^-)$).

Theorem 3.1. *If D^- is an outer domain bounded by a simple, closed, rectifiable curve Γ, and $W\in E^1(A;B;D^-)$, then*

$$W(z)=(\widetilde{K}_\Gamma W^-)(z),\ \ z\in D^-, \tag{3.4}$$

where Γ denotes the curve oriented so that moving around it leaves D^- on the left.

Proof. Denote by Γ_ρ the image of the circumference $\{\zeta:\ |\zeta|=\rho<1\}$ under the conformal mapping of the circle U onto the domain D^-. Further, let Γ_R be the circumference $\{z:\ |z|=R>1\}$. Then for ρ, close to unity, and for sufficiently large R, the curve Γ_ρ lies inside of the circle $\{z:\ |z|<R\}$. The function $W(z)$ defined in a doubly-connected domain $\mathcal{E}$ with the boundary $\Gamma_\rho\cup\Gamma_R$ is representable by the Cauchy integral [17, p. 186], that is,

$$W(z)=(\widetilde{K}_{\Gamma_\rho}W)(z)+(\widetilde{K}_{\Gamma_R}W)(z). \tag{3.5}$$

We have

$$W(z(\rho e^{i\vartheta}))=\Phi_W(z(\rho e^{i\vartheta}))\exp\omega_W(z(\rho e^{i\vartheta})).$$

Assume

$$\varphi_\rho(\vartheta)=\Phi_W(z(\rho e^{i\vartheta}))z'(\rho e^{i\vartheta})i\rho e^{i\vartheta},$$

then

$$W(z(\rho e^{i\vartheta}))z'(\rho e^{i\vartheta})i\rho e^{i\vartheta}=\varphi_\rho(\vartheta)\exp\omega_W(z(\rho e^{i\vartheta})).$$

Therefore

$$(\widetilde{K}_{\Gamma_\rho}W)(z(re^{i\beta}))=\frac{1}{2\pi i}\int\limits_0^{2\pi}\Omega_1\big(z(re^{i\beta}),z(\rho e^{i\vartheta})\big)\varphi_\rho(\vartheta)\exp\big(\omega_W(z(\rho e^{i\vartheta}))\big)\,d\vartheta$$
$$-\,\Omega_2\big(z(re^{i\beta}),z(\rho e^{i\vartheta})\big)\overline{\varphi}_\rho(\vartheta)\exp\big(\omega_W(z(\rho e^{i\vartheta}))\big)\,d\vartheta. \tag{3.6}$$

Since $W(z)\in E^1(A;B;D^-)$, Φ_W belongs to the class $E^1(D^-)$ (see Statement 3.1). Consequently, the sequence $\{\varphi_\rho(\vartheta)\}$ for $\vartheta\to 1$ converges in the space $L([0,2\pi])$ to the function $\varphi_1(\vartheta)$ [16, p. 89].

Since $\exp(\omega_W(z(\zeta)))$ is continuous in $\overline{U}$, from the above-said it follows that the sequence $\{W(z(\rho e^{i\vartheta}))\}$ for $\rho\to 1$ converges in $L([0,2\pi])$ to $W(z(e^{i\vartheta}))$.

Let $\rho_0\in(0,1)$ and $\varepsilon>0$ be a small number such that $\rho_0(1+\varepsilon)=\rho_1<1$. We take the point $z(re^{i\beta})$, $r\in(0,\rho_0)$. If $\rho\in(\rho_1,1)$, then

$$\big|z(re^{i\beta})-z(\rho e^{i\vartheta})\big|\geq \operatorname{dist}\big(z(re^{i\beta}),\Gamma_{\rho_0}\cup\Gamma_{\rho_1}\big)=m_0>0.$$

By the equality (2.3), there exists a number c such that

$$|\Omega_1(z,t)| < \frac{c}{|z-t|} = \frac{c}{|z(re^{i\beta}) - z(\rho e^{i\vartheta})|} \le \frac{c}{m_0}\,.$$

Owing to this fact, if we put

$$g_\rho(\vartheta) = \Omega_1\big(z(re^{i\beta}), z(\rho e^{i\vartheta})\big)\varphi_\rho(\vartheta),$$

then

$$|g_\rho(\vartheta)| < \frac{c}{m_0}\,|\varphi_\rho(\vartheta)|.$$

From the convergence of $\{\varphi_\rho\}$ to φ_1 in $L([0,2\pi])$ it follows that for any set $\mathcal{E} \subset [0,2\pi]$ the sequence $\{\varphi_\rho\}$ converges to φ_1 in $L(\mathcal{E})$ (see, e.g., [17]). According to the Hahn–Banach theorem [1, p. 255], we can conclude that the family $\{\varphi_\rho\}$ has absolutely continuous integrals of the same degree. Moreover, as $\rho \to 1$, the sequence $|g_\rho(\vartheta)|$ converges almost everywhere to $g_1(\vartheta)$.

Now, owing to the Vitali theorem [1, p. 255], we can conclude that in (3.6) the limiting passage under the integral sign is admissible and hence

$$\lim_{\rho\to 1}(\widetilde{K}_{\Gamma_\rho}W)(z(re^{i\beta})) = \frac{1}{2\pi i}\int\limits_0^{2\pi} \Omega_1\big(z(re^{i\beta}), z(e^{i\vartheta})\big)\Phi_W(z(re^{i\vartheta}))ie^{i\vartheta}\exp\omega_W(z(e^{i\vartheta}))\,d\vartheta$$
$$-\,\Omega_2\big(z(re^{i\beta}), z(e^{i\vartheta})\big)\overline{\Phi_W(z(re^{i\vartheta}))z'(e^{i\vartheta})ie^{i\vartheta}\exp\omega_W(z(e^{i\vartheta}))}\,d\vartheta = (\widetilde{K}_\Gamma W)(z(re^{i\beta})). \quad (3.7)$$

Let us prove that

$$\lim_{R\to\infty}(\widetilde{K}_{\Gamma_R}W)(z) = 0.$$

Let $|z| = R$ and $t \in \Gamma_R$. Then $|t| = R$ and it can be easily verified that $|\Omega_j(z,t)| < \frac{M}{R-|z|}$. Therefore

$$|(\widetilde{K}_{\Gamma_R}W)| < 2M\int\limits_0^{2\pi}\frac{|W(Re^{i\vartheta})|}{(R-|z|)^\alpha}\,d\vartheta, \quad \alpha = \frac{2}{s}\,.$$

Since $\lim\limits_{R\to\infty}|W(Re^{i\vartheta})| = 0$ for large R, we have $|W(Re^{i\vartheta})| \le M_0$ and hence

$$|(\widetilde{K}_{\Gamma_R}W)| \le \frac{2\pi MM_0}{(R-|z|)^\alpha} \longrightarrow 0.$$

This, together with (3.5) and (3.7), results in the equality (3.4). □

Remark 3.2. If orientation on Γ is chosen such that when moving around in this direction the domain D^+ leaves to the left, then the formula (3.4) takes the form

$$W(z) = -(\widetilde{K}_\Gamma W^-)(z), \quad z \in D^-.$$

3.3. **On the belonging of the function** $(\widetilde{K}_\Gamma f)(z))$ **to Smirnov class.** First, let us prove an analogue of Statement 3.4 for an unbounded domain. Towards this end, we will need the following

Lemma 3.1. *Let*

(1) Γ *be a simple, closed, rectifiable curve bounding the finite* D^+ *and the infinite* D^- *domains;*
(2) $p \in \mathcal{P}(\Gamma)$*;*
(3) $\zeta = \zeta(z)$ *be conformal mapping of* U^+ *onto* D^-*;*
(4) $\omega(\zeta) = \frac{k}{\zeta - a}$, $a \in D^+$, $\zeta \in D^-$, *and* k *be the constant such that* $k \le [\operatorname{dist}(a,\Gamma)]^2 = d^2$, *hence* $\widetilde{\Gamma} = \partial\widetilde{D}$, $\omega : D^- \to \widetilde{D}$, *where* $\widetilde{D}$ *is the bounded domain;*
(5) *the function* $\tau = \frac{k}{t-a}$ *map* Γ *onto* $\widetilde{\Gamma}$.

Assume $\widetilde{p}(\tau) = p(\frac{k}{\tau} + a)$. *Then*

$$\widetilde{p}(\tau) \in \mathcal{P}(\widetilde{\Gamma}). \quad (3.8)$$

Proof. Let $|\tau_1-\tau_2|<\frac{1}{2}$. We have

$$\big|\widetilde{p}(\tau_1)-\widetilde{p}(\tau_2)\big|=\Big|p\Big(\frac{k}{\tau_1}+a\Big)-p\Big(\frac{k}{\tau_2}+a\Big)\Big|\leq\frac{c(p)}{\big|\ln\frac{k|\tau_2-\tau_1|}{|\tau_1\tau_2|}\big|}\,. \tag{3.9}$$

Since $|\tau_1|\geq d$, $|\tau_2|\geq d$, owing to the condition (4), we obtain $\frac{k}{|\tau_1\tau_2|}\leq\frac{k}{d^2}\leq 1$. Therefore $\frac{k|\tau_1-\tau_2|}{|\tau_1\tau_2|}\leq|\tau_1-\tau_2|<\frac{1}{2}$, which implies that

$$\Big|\ln\frac{k|\tau_1-\tau_2|}{|\tau_1\tau_2|}\Big|>\big|\ln|\tau_1-\tau_2|\big|,$$

and from (3.9) we can conclude that $|\widetilde{p}(\tau_1)-\widetilde{p}(\tau_2)|<\frac{c(p)}{|\ln|\tau_1-\tau_2||}$. Moreover, it is obvious that $\min\widetilde{p}(\tau)=\min\limits_{t\in\gamma}p(t)=\underline{p}>1$. Thus the inclusion (3.8) is proved. □

Theorem 3.2. *Let Γ be the simple, closed, rectifiable curve bounding the domain D^-, and let the conditions*

$$A(z),B(z)\in L^\infty(D^-),\ \ \Gamma\in\mathbb{R},\ \ f\in L^{p(t)}(\Gamma),\ \ p\in\mathcal{P}(\Gamma), \tag{3.10}$$

be fulfilled. Then the function

$$W(z)=(\widetilde{K}_\Gamma f)(z),\ \ z\in D^-,$$

belongs to the class $E^{p(t)}(A;B;D^-)$.

Proof. We choose a point a from D^+ and assume $\zeta=\frac{k}{z-a}$, where k is chosen as in Lemma 3.1. Then $z=a+\frac{k}{\zeta}$ and

$$W\Big(\frac{k}{\zeta}+a\Big)=(\widetilde{K}_{\widetilde{\Gamma}}f)\Big(\frac{k}{\zeta}+a\Big). \tag{3.11}$$

We replace the integral variable in the right-hand side of (3.11) by the equality $t=\frac{k}{\tau}+a$. As a result, we obtain

$$\widetilde{W}(\zeta)=\frac{1}{2\pi i}\int\limits_{\Gamma}\widetilde{\Omega}_1(\zeta,\tau)F(\tau)\,d\tau-\widetilde{\Omega}_2(\zeta,\tau)\overline{F}(\tau)\,d\overline{\tau}, \tag{3.12}$$

where

$$\widetilde{W}(\zeta)=W\Big(\frac{k}{\zeta}+a\Big),\quad \widetilde{\Omega}_j(\zeta,\tau)=\Omega_j\Big(\frac{k}{\zeta}+a,\frac{k}{\tau}+a\Big),\ \ j=1,2,\quad F(\tau)=-\frac{f(\frac{k}{\tau}+a)}{\tau^2}\,k. \tag{3.13}$$

Since $f\in L^{p(t)}(\Gamma)$, we have $F\in L^{\widetilde{p}(\tau)}(\widetilde{\gamma})$, $\widetilde{p}(\tau)=p(\frac{k}{\tau}+a)$. In our assumptions Lemma 3.1 is applicable by virtue of which we have $\widetilde{p}(\tau)\in\mathcal{P}(\widetilde{\Gamma})$.

It can be easily verified that $\widetilde{\Omega}_k(\zeta,\tau)$, $k=1,2$, are the kernels of the type of principal normal kernels. Therefore following the proof of Statement 3.4 (see Theorem 3 of [15]), we find that $\widetilde{W}(\zeta)\in E^{\widetilde{p}(\tau)}(\widetilde{A};\widetilde{B};\widetilde{D})$. It is not difficult to show that $W\in E^{p(t)}(A;B;D)$. □

From Statement 3.4 and Theorem 3.2 follows one statement on the generalized Cauchy type integral which we formulate in the form of the following

Lemma 3.2. *Let Γ be the simple, closed, rectifiable curve dividing the plane E into the domains D^+ and D^-; next, let*

$$A(z),B(z)\in L^\infty(E),\ \ \Gamma\in\mathbb{R},\ \ f\in L^{p(t)}(\Gamma),\ \ p\in\mathcal{P}(\Gamma). \tag{3.14}$$

Then the narrowings on D^+ and D^- of the function $W(z)=(\widetilde{K}_\Gamma f)(z)$ belong to the classes $E^{p(t)}(A;B;D^+)$ and $E^{p(t)}(A;B;D^-)$, respectively, vice versa, if $W_1(z)\in E^{p(t)}(A;B;D^+)$ and $W_2(z)\in E^{p(t)}(A;B;D^-)$, then the function

$$W(z)=\begin{cases}W_1(z), & z\in D^+,\\ W_2(z), & z\in D^-\end{cases}$$

is representable by the generalized Cauchy type integral with density from $L^{p(t)}(\Gamma)$.

Proof. First, we note that if $W \in E^{p(t)}(A;B;D^+)$, then according to Statement 3.2 we have

$$(\widetilde{K}_\Gamma W^+)(z) = \begin{cases} W(z), & z \in D^+, \\ 0, & z \in D^- \end{cases} \tag{3.15}$$

(see (3.3)).

Relying on Remark 3.2, it is not difficult to establish that if $W \in E^{p(t)}(A;B;D^-)$, then

$$(\widetilde{K}_\Gamma W^-)(z) = \begin{cases} 0, & z \in D^+, \\ -W(z), & z \in D^-. \end{cases} \tag{3.16}$$

Let now $W(z) = (\widetilde{K}_\Gamma f)(z)$; if we consider it in the domain D^+, then according to Statement 3.4 we find that $W \in E^{p(t)}(A;B;D^+)$, but if we consider W in the domain D^-, then it belongs to $W \in E^{p(t)}(A;B;D^-)$, by Theorem 3.2.

The formulas (2.4) result in $W^+ - W^- = f$, hence $W = \widetilde{K}_\Gamma(W^+ - W^-)$.

Since for W_1 and W_2 respectively the relations (3.15) and (3.16) are valid, we have

$$\big[\widetilde{K}_\Gamma(W_1^+ - W_2^-)\big](z) = \begin{cases} W_1(z), & z \in D^+, \\ -W_2(z), & z \in D^-. \end{cases} \tag{3.17}$$

Obviously, $[W_1^+(t) - W_2^-(t)] \in L^{p(\cdot)}(\Gamma)$, hence $W(z) \in \widetilde{K}^{p(\cdot)}(\Gamma)$. □

4. Certain Properties of Integrals $\widetilde{K}_\Gamma f$ and $\widetilde{S}_\Gamma f$

Theorem 4.1. *In order for the function $W(z) \in U^{s,2}(A;B;D)$, $s > 2$, the equality*

$$W(z) = (\widetilde{K}_\Gamma W^+)(z) \tag{4.1}$$

to take place, it is necessary and sufficient that for almost all $t \in \Gamma$ the equality

$$(\widetilde{S}_\Gamma W^+)(t) = W^+(t) \tag{4.2}$$

to hold.

Proof. The necessity. It follows from the representation (4.1) that $W^+ \in L(\Gamma)$. By the equalities (2.4) we have

$$W^+(t) = \frac{1}{2}\, W^+(t) + \frac{1}{2}\, (\widetilde{S}_\Gamma W^+)(t),$$

and hence the equality (4.2) is valid.

Sufficiency. Let the equality (4.2) hold. Let us show that the equality (4.1) is likewise valid.

Consider the function

$$M(z) = W(z) - (\widetilde{K}_\Gamma W^+)(z), \ \ z \in D.$$

We have

$$M^+ = W^+ - \frac{1}{2}\, (W^+ + \widetilde{S}_\Gamma W^+) = \frac{1}{2}\, (W^+ - \widetilde{S}_\Gamma W^+). \tag{4.3}$$

By virtue of (4.2), we can conclude that $M^+(t) = 0$.

Since $W \in U^{s,2}(A;B;D)$, $s > 2$, we have $\widetilde{K}_\Gamma W^+ \in U^{s,2}(A;B;D)$ (see Subsection 2.3); consequently, $M(z) \in U^{s,2}(A;B;D)$. Therefore we have the representation

$$M(z) = \Phi_M(z)\omega_M(z), \ \ z \in D,$$

(see Subsection 2.1, the equality (2.2)). Here $\omega_M(z) \neq 0$ everywhere on $E \setminus \Gamma$.

Consequently, $\omega_M^+ \neq 0$, and from the equality $M^+ = 0$ we conclude that $\Phi_M^+(t) = 0$ almost everywhere on Γ. From the theorem on the uniqueness of analytic functions we find that $\Phi_M(z) = 0$; hence $M(z) = 0$, and from (4.3) follows (4.1). □

Remark 4.1. If D is an unbounded domain, then for the equality $W(z) = -(\widetilde{K}_\Gamma W^-)(z)$ it is necessary and sufficient that the equality

$$(\widetilde{S}_\Gamma W^-)(t) = -W^-(t)$$

to be fulfilled.

Theorem 4.2. *Let*

$$A, B \in L^{\infty}(D), \quad \Gamma \in \mathbb{R}, \quad p \in \mathcal{P}(\Gamma). \tag{4.4}$$

For the generalized analytic function $W(z)$ to have the boundary function $W^+(z)$ of the class $L^{p(t)}(\Gamma)$ and the equality

$$W(z) = (\widetilde{K}_\Gamma W^+)(\tau) \tag{4.5}$$

to hold, it is necessary and sufficient that $W(z)$ belong to the class $E^{p(t)}(A;B;D)$.

Proof. The necessity. Let the conditions (4.4) be fulfilled and there exist $W^+(t)$ and $W^+ \in L^{p(t)}(\Gamma)$, then by Corollary 3.1 we conclude that $(\widetilde{K}_\Gamma W^+)(z) \in E^{p(t)}(A;B;D)$.

Sufficiency. Let $W \in E^{p(t)}(A;B;D)$ and $p \in \mathcal{P}(\Gamma)$, then $W \in E^1(A;B;D)$. According to Statement 3.3 and Theorem 3.1, the equality (4.5) holds. This allows us to conclude that $W^+ \in L^{p(t)}(\Gamma)$, by virtue of Statement 3.2. □

Remark 4.2. Theorem 4.2 is a certain analogue of the Fichtenholz theorem [9, p. 97].

Theorem 4.3. *If the assumptions* (4.4) *holds and $f \in L^{p(t)}(\Gamma)$, then*

$$\widetilde{S}_\Gamma^2 f = f \tag{4.6}$$

holds.

Proof. By virtue of Corollary 3.1, the function $W(z) = (\widetilde{K}_\Gamma f)(z)$ belongs to $E^{p(t)}(A;B;D)$. Then by Statement 3.2 we have $(\widetilde{K}_\Gamma W^+)(z) = W(z)$. Now, by Theorem 4.1 we can conclude that $W^+(t) = (\widetilde{S}_\Gamma W^+(t)$. Using the first of the formulas (2.4), we write the last equality in the form

$$\frac{1}{2}\,(f + \widetilde{S}_\Gamma f) = \frac{1}{2}\,\widetilde{S}_\Gamma (f + \widetilde{S}_\Gamma f)$$

from which follows the equality (4.6). □

Tracing the proof of the theorem, we easily find that the following assertion is valid.

Lemma 4.1. *Let $W = \Phi_W \exp \omega_W$ be the function of the class $U^{s,2}(A;B;D)$, $s > 2$, and φ be analytic function in D, then*

$$\varphi W = \Phi_{\varphi W} \exp \omega_{\varphi W} \in U^{s,2}\Big(A; B\,\frac{\varphi}{\overline{\varphi}}\,; D\Big),$$

where

$$\Phi_{\varphi W} = \varphi \Phi_W \quad \textit{and} \quad \omega_{\varphi W} = \omega_W .$$

Proof. Since $\partial_{\overline{z}} \varphi = 0$, we have $\partial_{\overline{z}}(\varphi W) = \varphi \partial_{\overline{z}} W$. Moreover, $\partial_{\overline{z}} W + AW + B\overline{W} = 0$, hence

$$\partial_{\overline{z}} \varphi W + A\varphi W + B\,\frac{\varphi}{\overline{\varphi}}\,\overline{\varphi W} = 0.$$

This implies that $\varphi W = U^{s,2}(A; B\,\frac{\varphi}{\overline{\varphi}}\,; D)$.

Find the function $\omega_{\varphi W}$. We have [17, p. 192]

$$\omega_{\varphi W}(\zeta) = \frac{1}{\pi} \iint\limits_D \Big(A(t) + B(t)\,\frac{\varphi(t)}{\overline{\varphi}(t)}\,\frac{\overline{\varphi W}}{\varphi W}\Big)\,\frac{d\xi\,d\eta}{t-\zeta} = \frac{1}{\pi} \iint\limits_D \Big(A(t) + B(t)\,\frac{\overline{W}(t)}{W(t)}\Big)\,\frac{d\xi\,d\eta}{t-\zeta} = \varphi_W(\zeta).$$

Next, taking into account the above equality, we obtain

$$\varphi W = \varphi \Phi_W \exp \omega_W = \{\varphi \Phi_W\} \exp \omega_W ,$$

from which we get both provable equalities. □

5. Extensions of Generalized Smirnov Class Analytic Functions

Theorem 5.1. *Let D_1 and D_2 be the domains lying outside of each other, bounded with simple rectifiable curves of the class $\mathbb{R}$, and:*

(1) *boundaries of the domains D_1 and D_2 have common arc Γ, so that $\partial D_1 = \Gamma_1 \cup \Gamma$, $\partial D_2 = \Gamma_2 \cup \Gamma$;*

(2) *$p_1(t) \in \mathcal{P}(\Gamma_1)$, $p_2(t) \in \mathcal{P}(\Gamma_2)$;*

(3) *$A_1, B_1 \in L^\infty(D_1)$, $A_2, B_2 \in L^\infty(D_2)$ and $W_1 \in E^{p_1(t)}(A_1; B_1; D_1)$, $W_2 \in E^{p_2(t)}(A_2; B_2; D_2)$;*

(4) *$p_1(a) = p_2(a)$, $p_1(b) = p_2(b)$, where a and b are the ends of the arc Γ;*

(5) *$W_1(t) = W_2(t)$, $t \in \Gamma$.*

Then the function

$$W(z) = \begin{cases} W_1(z), & z \in D_1, \\ W_2(z), & z \in D_2, \\ W_1(t) = W_2(t), & t \in \Gamma, \end{cases} \tag{5.1}$$

belongs to the Smirnov class $E^{p(t)}(A; B; D)$, where $D = D_1 \cup D_2 \cup \Gamma$,

$$p(t) = \begin{cases} p_1(t), & t \in \Gamma_1, \\ p_2(t), & t \in \Gamma_2, \end{cases},$$

and

$$A(z) = \begin{cases} A_1(z), & z \in D_1, \\ A_2(z), & z \in D_2, \end{cases} \qquad B(z) = \begin{cases} B_1(z), & z \in D_1, \\ B_2(z), & z \in D_2. \end{cases}$$

Proof. Assume

$$\widetilde{A}_k(z) = \begin{cases} A_k(z), & z \in D_k, \\ 0, & z \in E \setminus D_k, \end{cases} \qquad \widetilde{B}_k(z) = \begin{cases} B_k(z), & z \in D_k, \\ 0, & z \in E \setminus D_k, \end{cases} \qquad k = 1, 2.$$

Then $A = \widetilde{A}_1 + \widetilde{A}_2$, $\widetilde{B} = \widetilde{B}_1 + \widetilde{B}_2$. By virtue of the assumption (3), we have $A, B \in L^\infty(D)$. Further, owing to (3.3) and assumption (3),

$$(\widetilde{K}_{\Gamma_1 \cup \Gamma} W_1)(z) = 0, \;\; z \in D_2, \quad (\widetilde{K}_{\Gamma_2 \cup \Gamma} W_2)(z) = 0, \;\; z \in D_1. \tag{5.2}$$

In these integrals, the integration sets are $\Gamma_1 \cup \Gamma$ and $\Gamma_2 \cup \Gamma$. In addition, the curve $\Gamma_1 \cup \Gamma$ is oriented so that moving in this direction, the domain D_1 leaves to the left, analogously, $\Gamma_2 \cup \Gamma$ is oriented so that moving in this direction, the domain D_2 leaves to the left. These orientations on Γ generate on Γ opposite directions. Therefore, if we denote the oriented arc of Γ on the boundary ∂D_1 of the domain D_1 by Γ^+, then on ∂D_2 it will be Γ^-.

In the domain D, let us consider the function

$$\begin{aligned} F(z) &= (\widetilde{K}_{\Gamma_1 \cup \Gamma^+} W_1)(z) + (\widetilde{K}_{\Gamma_1 \cup \Gamma^-} W_2)(z) = F_1(z) + F_2(z) = (\widetilde{K}_{\Gamma_1} W_1)(z) + (\widetilde{K}_{\Gamma_2} W_2)(z) \\ &= \frac{1}{2\pi i} \int\limits_{\Gamma^+} \Omega_1(z,t) W_1^+(t)\, dt - \Omega_2(z,t) \overline{W}_1(t)\, d\bar{t} + \frac{1}{2\pi i} \int\limits_{\Gamma^-} \Omega_1(z,t) W_2(t)\, dt - \Omega_2(z,t) \overline{W}_2(t)\, d\bar{t}, \end{aligned}$$

where Ω_1, Ω_2 are the principal kernels of the class $U^\infty(A; B; E)$.

We write $F(z)$ in the form

$$\begin{aligned} F(z) &= (\widetilde{K}_{\Gamma_1} W_1)(z) + (\widetilde{K}_{\Gamma_2} W_2)(z) \\ &\quad + \frac{1}{2\pi i} \int\limits_{\Gamma^+} \Omega_1(z,t)(W_1(t) - W_2(t))\, dt - \frac{1}{2\pi i} \int\limits_{\Gamma^-} \Omega_2(z,t)(\overline{W}_1(t) - \overline{W}_2(t))\, d\bar{t} \\ &= (K_{\Gamma_1 \cup \Gamma_2} W)(z) \end{aligned} \tag{5.3}$$

(we have taken into account that $W_1(t) = W_2(t)$, $t \in \Gamma$).

In view of the equality (5.2) we have

$$(\widetilde{K}_{\Gamma_1\cup\Gamma_2}W)(z)=\begin{cases}W_1(z), & z\in D_1,\\ W_2(z), & z\in D_2,\end{cases}$$

that is, $F(z)=W(z)$, $z\in D_1\cup D_2$. Moreover, for $t\in\Gamma$ we have

$$\lim_{z\widehat{\to}t,\, z\in D_k} F(z)=W_k(t),$$

that is, $F(t)=W_1(t)=W_2(t)$, $t\in\Gamma$. Consequently, almost everywhere on D, we get

$$F(z)=W(z). \tag{5.4}$$

The function $p(t)$ given on $\Gamma_1\cup\Gamma_2$ is, by assumption (4), of the class $\mathcal{P}(\Gamma_1\cup\Gamma_2)$. Therefore, it can be easily seen from (5.3) that $F(z)$ is the generalized Cauchy type integral with density from $L^{p(t)}(\Gamma_1\cup\Gamma_2)$. In view of Statement 3.4, we can conclude that $F(z)\in E^{p(t)}(A;B;D)$, and hence owing to (5.4), $W(z)\in E^{p(t)}(A;B;D)$, as well. □

6. The Symmetry Principle for Smirnov Class Functions

Before we proceed to formulating and proving the above-mentioned principle, we will prove below the following Lemmas 6.1 and 6.2. We denote $U^+=U$, $U^-=E\setminus\overline{U^+}$.

Lemma 6.1. *Let the domain D lie in U^+ and a part of its boundary lie on γ. Assume $D_*=\{\zeta:\zeta=\frac{1}{\overline{z}},\ z\in D\}$, and let $A(z),B(z)\in L^{s,2}(D)$, $s>2$. Then the functions*

$$A_0(\zeta)=\begin{cases}A(\zeta), & \zeta\in D,\\ -\dfrac{1}{\overline{\zeta}^2}\,\overline{A}\Big(\dfrac{1}{\overline{\zeta}}\Big), & \zeta\in D_*,\end{cases}\qquad B_0(\zeta)=\begin{cases}B(\zeta), & \zeta\in D,\\ -\dfrac{1}{\overline{\zeta}^2}\,\overline{B}\Big(\dfrac{1}{\overline{\zeta}}\Big), & \zeta\in D_*\end{cases} \tag{6.1}$$

belong to the class $L^{s,2}(D\cup D_*)$.

Proof. Show that $A_0\in L^{s,2}(D\cup D_*)$. Let $\zeta=x+iy$ and

$$J=\iint\limits_{D_*}|A_0(\zeta)|^s\,dx\,dy=\iint\limits_{D}\Big|-\frac{1}{\overline{\zeta}\,2}\,A\Big(\frac{1}{\overline{\zeta}}\Big)\Big|^s\,dx\,dy.$$

Assume $\tau=\alpha+i\beta$ and transform the variable ζ by the equality $\zeta=\frac{1}{\overline{\tau}}$, i.e., $x=\frac{\alpha}{\alpha^2+\beta^2}$, $y=\frac{\beta}{\alpha^2+\beta^2}$. Then

$$J=\iint\limits_{D}|\tau^2A(\tau)|^s|I|\,d\alpha\,d\beta,$$

where

$$I=\begin{vmatrix}x'_\alpha & x'_\beta\\ y'_\alpha & y'_\beta\end{vmatrix}=\begin{vmatrix}(\beta^2-\alpha^2)(\alpha^2+\beta^2)^{-2} & -2\alpha\beta(\alpha^2+\beta^2)^{-2}\\ -2\alpha\beta(\alpha^2+\beta^2)^{-2} & (\alpha^2-\beta^2)(\alpha^2+\beta^2)^{-2}\end{vmatrix}=-\frac{1}{(\alpha^2+\beta^2)^2}=-\frac{1}{|\tau|^4}\,.$$

Therefore

$$I=\iint\limits_{D}|\tau^2A(\tau)|^s\,\frac{d\alpha\,d\beta}{|\tau|^4}=\iint\limits_{D}|A(\tau)|^s|\tau|^{2(s-2)}\,d\alpha\,d\beta=\iint\limits_{D}|A(\tau)|^s\,d\alpha\,d\beta<\infty.$$

(We have taken into account that $s>2$, $|\tau|<1$ and $A,B\in L^\infty(D)$.) This implies that $A_0\in L^{s,2}(D\cup D_*)$.

In the same manner we can prove that $B_0\in L^{s,2}(D\cup D_*)$. □

Assume that the domain D is bounded by a simple, rectifiable, closed curve, $D\subset U^+$ and a part of the boundary D is the arc lying on γ.

Given $W(z)$ on D, we put

$$W_*(z)=\begin{cases}W(z), & z\in D,\\ -\overline{W}\Big(\dfrac{1}{\overline{z}}\Big), & z\in D_*.\end{cases}$$

Lemma 6.2. *Let $W(z) \in E^1(A;B;D)$ and either $z=0 \notin D$, or $z=0 \in D$, $W(0)=0$. Then $W_*(z) \in E^1(A_0;B_0;D_*)$, where A_0, B_0 are defined by the equality* (6.1).

Proof. According to the definition of the class $E^1(A_0,B_0,D_*)$, we have to establish that

$$W_*(z) \in U^{s,2}(A_0;B_0;D_*) \tag{6.2}$$

and

$$\sup_{0<r<1} \int_0^{2\pi} \big|W_*(z(re^{i\vartheta}))\big|\,|z'(re^{i\vartheta}|^r\,d\vartheta < \infty, \tag{6.3}$$

where $z=z(re^{i\vartheta})$ is conformal mapping of U^+ onto D_*.

We start from the first one. By Lemma 6.1, $A_0,B_0 \in L^{s,2}(D\cup D_*)$. Therefore we have to prove that for $z \in D_*$ we have the equality

$$\partial_{\overline{z}}W_* + A_0(z)W_*(z) + B_0(z)\overline{W}_*(z) = 0. \tag{6.4}$$

Assuming $W(z)=u(z)+iv(z)$, we have

$$\partial_{\overline{z}}W_* = -\partial_{\overline{z}}\overline{W}\Big(\frac{1}{\overline{z}}\Big) = -\Big[\partial_{\overline{z}}\Big(u\Big(\frac{1}{\overline{z}}\Big) - iv\Big(\frac{1}{\overline{z}}\Big)\Big]\Big(-\frac{1}{\overline{z}^2}\Big). \tag{6.5}$$

But

$$u_{\overline{z}}\Big(\frac{1}{\overline{z}}\Big) + iv_{\overline{z}}\Big(\frac{1}{\overline{z}}\Big) = -\overline{A}\Big(\frac{1}{\overline{z}}\Big)\Big(u\Big(\frac{1}{\overline{z}}\Big)+iv\Big(\frac{1}{\overline{z}}\Big)\Big) - \overline{B}\Big(\frac{1}{\overline{z}}\Big)\Big(u\Big(\frac{1}{\overline{z}}\Big)+iv\Big(\frac{1}{\overline{z}}\Big)\Big),$$

and from (6.5), we get

$$\begin{aligned}-\partial_{\overline{z}}W_*(z) &= \Big(-\frac{1}{\overline{z}^2}\Big)\Big[\overline{A}\Big(\frac{1}{\overline{z}}\Big)\overline{W}\Big(\frac{1}{\overline{z}}\Big) - \overline{B}\Big(\frac{1}{\overline{z}}\Big)W\Big(\frac{1}{\overline{z}}\Big)\Big]\\ &= -\frac{1}{\overline{z}^2}\,\overline{A}\Big(\frac{1}{\overline{z}}\Big)W_*(z) - \frac{1}{\overline{z}^2}\,\overline{B}\Big(\frac{1}{\overline{z}}\Big)\overline{W}_*(z) = A_0(z)W_*(z) + B_0(z)\overline{W}_*(z),\end{aligned}$$

that is,

$$\partial_{\overline{z}}W_*(z) + A_0(z)W_*(z) + B_0(z)\overline{W}_*(z) = 0, \ \ z \in D_*.$$

Let now $W \in E^1(A;B;D)$. This implies that

$$\sup_{0<r<1} \int_0^{2\pi} \big|W(\zeta(re^{i\vartheta}))\zeta'(re^{i\vartheta})\big|r\,d\vartheta = M < \infty, \tag{6.6}$$

where the function $\zeta=\zeta(re^{i\vartheta})$ is conformal mapping of U^+ onto D and if $0\in D$, then $\zeta(0)=0$.

The function $z=\frac{1}{\zeta(re^{i\vartheta})}$ is the conformal mapping of U^+ onto D_*.

We need to prove that

$$\sup_{0<r<1} \int_0^{2\pi} \Big|W_*\Big(\frac{1}{\zeta(re^{i\vartheta})}\Big)\,\frac{\zeta'(re^{i\vartheta})}{\zeta^2(re^{i\vartheta})}\Big|r\,d\vartheta < \infty$$

We have

$$\begin{aligned}J_r &= \int_0^{2\pi} \Big|W_*\Big(\frac{1}{\zeta(re^{i\vartheta})}\Big)\,\frac{\zeta'(re^{i\vartheta})}{\zeta^2(re^{i\vartheta})}\Big|r\,d\vartheta\\ &= \int_0^{2\pi} \Big|\overline{W}(\overline{\zeta})\,\frac{\zeta'}{\zeta^2}\Big|r\,d\vartheta = \int_0^{2\pi} |W(\overline{\zeta})\,\frac{\overline{\zeta}\,'}{\zeta^2}\Big|r\,d\vartheta\\ &= \int_0^{2\pi} |W(\overline{\zeta})\,\frac{(\overline{\zeta})'}{\zeta^2}\,\frac{\overline{\zeta}\,'}{(\overline{\zeta})'}\Big|r\,d\vartheta = \int_0^{2\pi} |W(\overline{\zeta})\,\frac{(\overline{\zeta})'}{\zeta^2}\Big|r\,d\vartheta.\end{aligned} \tag{6.7}$$

If $0 \notin D$, then

$$J_r < \frac{M}{[\mathrm{dist}(0;D)]^2} = \frac{M}{m^2}\,. \tag{6.8}$$

If $0 \in D$, then $\zeta(0) = 0$, $W(0) = 0$, hence for small r (say, for $0 < r < r_0$) we have $|W(re^{-i\vartheta})| \sim r$, $|\zeta(re^{i\vartheta})| < cr$. Owing to that facts, there exists the constant c such that $|W(re^{i\vartheta})| < cr$, $|\zeta(r(e^{i\vartheta})| \sim cr$. Therefore, for small r we get

$$J_r < \int\limits_0^{2\pi} \frac{cr}{r^2} \, |\zeta'(re^{i\vartheta})| r \, d\vartheta = \frac{c}{c_1} \int\limits_0^{2\pi} |\zeta'(re^{i\vartheta})| \, d\vartheta = d < \infty. \tag{6.9}$$

Now, from (6.8), (6.9), when $r \in (0,1)$, we have $J_r < (\frac{M}{m^2} + d)$. This implies that the inequality (6.3) is valid, and since (6.2) is already proved, we have $W_* \in E^1(A_0; B_0; D_*)$. □

Corollary 6.1. *If $W \in E^{p(t)}(A; B; D)$, $p \in \mathcal{P}(\Gamma)$ and either $0 \notin D$, or $0 \in D$ and $W(0) = 0$, then $W \in E^{\ell(\tau)}(A; B; D_*)$, $\ell(\tau) = p(z(\frac{1}{\overline{\tau}}) \equiv p(z(\tau))$.*

Indeed, since $E^{p(t)}(A; B; D) \subset E^1(A; B; D)$, we have $W_* = \widetilde{K}_{\Gamma_*} W_*^+$, where Γ_* is the boundary of the domain D_*. In addition,

$$\int\limits_{\Gamma_*} |W_*(\zeta)|^{p(\zeta)} |z'(\zeta)| \, |d\zeta| = \int\limits_{\gamma} \Big| W\Big(\frac{1}{\overline{\tau}}\Big)\Big|^{p(z(\frac{1}{\overline{\tau}}))} \Big|\frac{1}{\tau^2}\Big| \, d\tau = \int\limits_{\gamma} |W(\tau)|^{p(z(\tau))} \, |d\tau| < \infty.$$

(We have taken into account that if $\tau \in \gamma$, then $\frac{1}{\overline{\tau}} = \tau$.)

Theorem 6.1 (The symmetry principle for the Smirnov class functions)**.** *Let:*

(1) *D be the simply connected domain bounded by a simple, closed, rectifiable curve $\gamma_2 \cup \gamma_1 \in \mathbb{R}$, lying inside of U^+, and the arc γ_1 lying on γ;*

(2) *$A, B \in L^\infty(D)$;*

(3) *D_* be a mirror image of D with respect to γ;*

(4) *$W \in E^{p(t)}(A; B; D)$;*

(5) *$W^+(t) + \overline{W^+}(t) = 0$, $t \in \gamma_1$;*

(6) *A_0 and B_0 are defined by the equalities* (6.1)*, $D_0 = D \cup D_* \cup \gamma_1$ and $p_0(t) = \begin{cases} p(t), & t \in \gamma_2, \\ p(\frac{1}{\overline{t}}), & t \in (\gamma_2)_*. \end{cases}$*

Then, if either $z = 0 \notin D$, or $0 \in D$ and $W(0) = 0$, then there exists a function $F \in E^{p_0(t)}(A_0; B_0; D_0)$ which for $z \in D$ coincides with $W(z)$, and for $z \in D_*$ with $W_*(z)$, but if $t \in \gamma_1$, then $F(t) = W^+(t) = -\overline{W^+}(t)$.

Proof. Assume $W_1(z) = W(z)$, $z \in D$, and $W_2(z) = W_*(z)$, $z \in D_*$. For the points t lying on γ_1, we have

$$W_1(t) = \lim_{z \widehat{\to} t, \, z \in D} W_1(z) = W(t), \quad W_2(t) = \lim_{z \widehat{\to} t, \, z \in D_*} \Big[-\overline{W}\Big(\frac{1}{\overline{z}}\Big)\Big] = -\overline{W}\Big(\frac{1}{\overline{t}}\Big) = -\overline{W}(t).$$

Due to the condition $W^+(t) + \overline{W^+}(t) = 0$, $t \in \gamma_1$, we have

$$W_1(t) = W_2(t), \quad t \in \gamma_1.$$

We have the right to apply Theorem 5.1 due to which the function $F(z)$ given by the equality (5.3) coincides with the function W given by the equality (5.1). Thus the proof of theorem is complete. □

Corollary 6.2. *If $A(z), B(z) \in L^\infty(D)$, $W(z) \in E^{p(t)}(A; B; U^+)$, $W(0) = 0$, and $W^+(t) + \overline{W^+}(t) = 0$, $t \in \gamma$, then $W_*(z) \in E^{p_*(t)}(A_0, B_0; U^-)$, where $p_*(t) = p(\frac{1}{\overline{t}}) = p(t)$.*

Indeed, if we take $D = U^+$, $\gamma_1 = \gamma$, then $D_* = U^-$, and hence the validity of Corollary 6.2 follows from Theorem 6.1.

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References

1. N. Danford and T. Shvartz, Linear operators. Part I: General theory. (Russian) *Izdat. Inostran. Lit., Moscow*, 1962.
2. P. L. Duren, Theory of H^p spaces. *Pure and Applied Mathematics*, Vol. 38. *Academic Press, New York–London*, 1970.
3. G. M. Goluzin, Geometric theory of functions of a complex variable. (Russian) *Nauka, Moscow*, 1966.
4. S. B. Klimentov, The Riemann–Hilbert problem for generalized analytic functions in Smirnov classes. (Russian) *Vladikavkaz. Mat. Zh.* **14** (2012), no. 3, 63–73.
5. V. Kokilashvili and V. Paatashvili, On Hardy classes of analytic functions with a variable exponent. *Proc. A. Razmadze Math. Inst.* **142** (2006), 134–137.
6. V. Kokilashvili and V. Paatashvili, On variable Hardy and Smirnov classes of analytic functions. *Georgian Internat. J. Sci., Nova Sci. Publ., Inc* **1** (2008), 67–81.
7. V. Kokilashvili and V. Paatashvili, Boundary value problems for analytic and harmonic functions in nonstandard Banach function spaces. *Nova Science Publishers, New York, NY*, 2012.
8. V. Kokilashvili and V. Paatashvili, The Riemann boundary value problem in variable exponent Smirnov class of generalized analytic functions. *Proc. A. Razmadze Math. Inst.* **169** (2015), 105–118.
9. V. Kokilashvili and S. Samko, Vekua's generalized singular integral on Carleson curves in weighted variable Lebesgue spaces. *Operator algebras, operator theory and applications*, 283–293, *Oper. Theory Adv. Appl.*, 181, *Birkhäuser Verlag, Basel*, 2008.
10. P. Koosis, Introduction to H^p spaces. With an appendix on Wolff's proof of the corona theorem. *London Mathematical Society Lecture Note Series*, 40. *Cambridge University Press, Cambridge–New York*, 1980.
11. G. F. Manjavidze, Boundary value problems for conjugation with shift for analytic and generalized analytic functions. (Russian) *Tbilis. Gos. Univ., Tbilisi*, 1990.
12. G. Manjavidze and G. Akhalaia, Boundary value problems of the theory of generalized analytic vectors. *Complex methods for partial differential equations (Ankara, 1998)*, 57–95, *Int. Soc. Anal. Appl. Comput.*, 6, *Kluwer Acad. Publ., Dordrecht*, 1999.
13. K. M. Musaev, Some boundary properties of generalized analytic functions. (Russian) *Dokl. Akad. Nauk SSSR* **181** (1968), 1335–1338.
14. K. M. Musaev and T. Kh. Gasanova, Riemann boundary value problem in a class of generalized analytic functions. *Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb.* **22** (2005), 93–98.
15. V. Paatashvili, Variable exponent Smirnov classes of generalized analytic functions. *Proc. A. Razmadze Math. Inst.* **163** (2013), 93–110.
16. I. I. Privalov, Boundary properties of analytic functions. (Russian) *Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow–Leningrad*, 1950.
17. I. N. Vekua, Generalized analytic functions. (Russian) *Fizmatgiz, Moscow*, 1958.

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