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GRAND CONFLUENT HYPERGEOMETRIC FUNCTION APPLYING REVERSIBLE THREE-TERM RECURRENCE FORMULA

Abstract. In this paper, by applying a reversible three-term recurrence formula (R3TRF) (see [13, Chapter 1]), we construct:

(1) power series expansions in closed forms of the grand confluent hypergeometric (GCH) equation,

(2) its integral forms for an infinite series and a polynomial which makes the leading non-constant coefficient on the RHS of the recurrence relation terminated,

(3) generating functions for GCH polynomials which makes the leading coefficient on the RHS terminated.

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Key words and phrases. Biconfluent Heun equation, generating function, integral form, reversible three-term recurrence formula.

რეზიუმე. ნაშრომში შექცევადი სამწევრა რეკურენტული ფორმულის (R3TRF) (იხ. [13, Chapter 1]) გამოყენებით აგებულია:

(1) GCH განტოლების წარმოდგენა ხარისხოვანი მწკრივის სახით;

(2) ინტეგრალური წარმოდგენები;

(3) GCH პოლინომთა მაწარმოებელი ფუნქციები.

1 Introduction

The equation

$$x \frac{d^2y}{dx^2} + (\mu x^2 + \varepsilon x + \nu) \frac{dy}{dx} + (\Omega x + \varepsilon \omega) y = 0 \quad (1.1)$$

is the grand confluent hypergeometric (GCH) differential equation where μ, ε, ν, Ω and ω are real or complex parameters [9, 11]. The GCH ordinary differential equation is of Fuchsian types with two singular points: one regular singular point which is zero with exponents $\{0, 1-\nu\}$, and another irregular singular point which is infinity with an exponent Ω/μ. In contrast, the Heun equation of Fuchsian types has four regular singularities. The Heun equation has four kinds of confluent forms [20]: (1) confluent Heun (two regular and one irregular singularities), (2) doubly confluent Heun (two irregular singularities), (3) biconfluent Heun (one regular and one irregular singularities), (4) triconfluent Heun equations (one irregular singularity).

The BCH equation is derived from the GCH equation by changing all coefficients* [36]. The GCH (or BCH) equation is applicable in the modern physics [1,21,22,35,37]. The BCH equation appears in the radial Schrödinger equation with those potentials such as the rotating harmonic oscillator [30], the doubly anharmonic oscillator [6,7,23], a three-dimensional anharmonic oscillator [17,18,23], Coulomb potential with a linear confining potential [23,34] and other kinds of potentials [24,25].

The fundamental solutions of the BCH equation for an infinite series and the BCH spectral polynomials about $x = 0$ in the canonical form were obtained by applying the power series expansion [2,15,19,39]. For the case of the irregular singular point $x = \infty$, the three-term recurrence of the power series in the BCH equation was derived [26,31], and the analytic solution of the BCH equation was left as solutions of recurrences due to a 3-term recursive relation between successive coefficients in its power series expansion of the BCH equation.† In comparison with the two term recursion relation of the power series in a linear differential equation, analytic solutions in closed forms on the three-term recurrence relation of the power series are unknown currently because of their complex mathematical calculations.

As is known, there are no examples for analytic solutions of the BCH equation about $x = 0$ and $x = \infty$ in the form of definite or contour integrals containing the well-known special functions such as ${}_2F_1$ or ${}_1F_1$, consisting of two-term recursion relation in their power series of linear differential equations. In place of describing the integral representation of the BCH equation involving only simple functions, especially for confluent hypergeometric functions, the BCH equation is obtained by means of Fredholm-type integral equations; such integral relationships express one analytic solution in terms of another analytic solution [3–5, 8, 27–29].

2 The GCH equation about a regular singular point at zero

Assume that the solution of (1.1) is

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+\lambda}, \quad (2.1)$$

where λ is an indicial root. Substitute (2.1) into (1.1). We obtain a three-term recurrence relation for the coefficients c_n:

$$c_{n+1} = A_n c_n + B_n c_{n-1}, \quad n \geq 1, \quad (2.2)$$

*For the canonical form of the BCH equation [36], replace μ, ε, ν, Ω and ω by -2, $-\beta$, $1+\alpha$, $\gamma-\alpha-2$ and $1/2(\delta/\beta+1+\alpha)$ in (1.1). For DLFM version ([32] or [38]), replace μ and ω by 1 and $-q/\varepsilon$ in (1.1).

†For the special case, the explicit solutions of the BCH equation in the canonical form was constructed when one of the coefficients $\beta = 0$ [16].

where

$$A_n = \frac{-\varepsilon(n+\omega+\lambda)}{(n+1+\lambda)(n+\nu+\lambda)}\,, \tag{2.3a}$$

$$B_n = -\frac{\Omega+\mu(n-1+\lambda)}{(n+1+\lambda)(n+\nu+\lambda)}\,, \tag{2.3b}$$

$$c_1 = A_0 c_0. \tag{2.3c}$$

We have two indicial roots which are $\lambda = 0$ and $1-\nu$.

2.1 Power series

2.1.1 Polynomial of type 2

By putting a power series $y(x) = \sum_{n=0}^{\infty} c_n x^{n+\lambda}$ into a linear ordinary differential equation (ODE), the recurrence relation between successive coefficients starts to appear. In general, the recurrence relation for a 3-term is given by (2.2) where $c_1 = A_0 c_0$ and $c_0 \neq 0$. As is known, there are two types of power series expansions for the two-term recurrence relation in a linear ODE such as a polynomial and an infinite series. In contrast, there are an infinite series and three types of polynomials in the three term recurrence relation of a linear ODE:

(1) polynomial which makes B_n term terminated: A_n term is not terminated, designated as 'a polynomial of type 1',

(2) polynomial which makes A_n term terminated: B_n term is not terminated, denominated as 'a polynomial of type 2',

(3) polynomial which makes A_n and B_n terms terminated simultaneously.

For $n = 0, 1, 2, 3, \ldots$ in (2.2), the sequence c_n is expanded to combinations of A_n and B_n terms. It is suggested that a sub-power series $y_l(x)$, where $l \in \mathbb{N}_0$, is constructed by observing the term of sequence c_n which includes l terms of $A_n's$ [10]. The power series solution is described by sums of each $y_l(x)$ such as $y(x) = \sum_{n=0}^{\infty} y_n(x)$. By allowing for A_n in the sequence c_n to be the leading term of each sub-power series $y_l(x)$, the general summation formulas of the 3-term recurrence relation in a linear ODE are constructed for an infinite series and a polynomial of type 1, designated as 'three-term recurrence formula (3TRF)'.

Similarly, by allowing for B_n in the sequence c_n to be the leading term of each sub-power series in a function $y(x)$ [13, Chapter 1], we have obtained the general summation formulas of the 3-term recurrence relation in a linear ODE for an infinite series and a polynomial of type 2: the term of the sequence c_n which includes zero term of B_n's, one term of B_n's, two terms of B_n's, three terms of B_n's, etc. is observed. These general summation expressions are denominated as 'reversible three-term recurrence formula (R3TRF)'.

In general, the GCH polynomial is defined as type 3 polynomial where A_n and B_n terms terminated. For the type 3 GCH polynomial about $x = 0$, it has a fixed integer value of Ω, just as it has a fixed value of ω. In the three-term recurrence relation, a polynomial of type 3 is categorized as a complete polynomial. In Chapters 9 and 10 of [14], general solutions in series for the GCH polynomial of type 3 around $x = 0$ and $x = \infty$ are constructed.

For type 1, the GCH polynomial about $x = 0$, μ, ε, ν and ω are treated as free variables and Ω as a fixed value. In [11,12], the analytic solutions of the GCH equation about the regular singular point at $x = 0$ are constructed by applying the three-term recurrence formula (3TRF) [10]:

(1) power series expansions in closed forms for an infinite series and a polynomial of type 1,

(2) their integral forms,

(3) generating functions for GCH polynomials of type 1.

Four examples of the analytic wave functions and their eigenvalues in the radial Schrödinger equation with certain potentials are presented:

(1) Schrödinger equation with the rotating harmonic oscillator and a class of confinement potentials,

(2) the spin free Hamiltonian involving only scalar potential for the $q-\bar{q}$ system,

(3) the radial Schrödinger equation with confinement potentials,

(4) two interacting electrons in a uniform magnetic field and a parabolic potential.

The Frobenius solutions in closed forms and their combined definite and contour integrals of these four quantum mechanical wave functions are derived analytically.

For the GCH polynomial of type 2 about $x=0$, μ, ε, ν and Ω are treated as free variables and ω as a fixed value. In this paper, by applying R3TRF in Chapter 1 of [13], the power series expansions are constructed in closed forms of the GCH equation about the regular singular point at $x=0$ for an infinite series and a polynomial of type 2. The integral forms of the GCH equation and their generating functions for GCH polynomials of type 2 are derived analytically. Also, the Frobenius solutions of the GCH equation about the irregular singular point at $x=\infty$ by applying 3TRF [10] are obtained analytically including their integral representations and generating functions for the GCH polynomials of type 1.

In Chapter 1 of [13], the general expression of a power series of $y(x)$ for a polynomial of type 2 is defined by

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} y_n(x) = y_0(x)+y_1(x)+y_2(x)+y_3(x)+\cdots \\ &= c_0 x^{\lambda}\left\{ \sum_{i_0=0}^{\alpha_0}\left(\prod_{i_1=0}^{i_0-1} A_{i_1}\right) x^{i_0} + \sum_{i_0=0}^{\alpha_0}\left\{ B_{i_0+1}\prod_{i_1=0}^{i_0-1} A_{i_1} \sum_{i_2=i_0}^{\alpha_1}\left(\prod_{i_3=i_0}^{i_2-1} A_{i_3+2}\right)\right\} x^{i_2+2} \right. \\ &\quad + \sum_{N=2}^{\infty}\left\{ \sum_{i_0=0}^{\alpha_0}\left\{ B_{i_0+1}\prod_{i_1=0}^{i_0-1} A_{i_1} \prod_{k=1}^{N-1}\left(\sum_{i_{2k}=i_{2(k-1)}}^{\alpha_k} B_{i_{2k}+2k+1} \prod_{i_{2k+1}=i_{2(k-1)}}^{i_{2k}-1} A_{i_{2k+1}+2k}\right) \right.\right. \\ &\quad \left.\left.\left. \times \sum_{i_{2N}=i_{2(N-1)}}^{\alpha_N}\left(\prod_{i_{2N+1}=i_{2(N-1)}}^{i_{2N}-1} A_{i_{2N+1}+2N}\right)\right\}\right\} x^{i_{2N}+2N}\right\}. \end{aligned} \tag{2.4}$$

Here $\alpha_i \leq \alpha_j$ only if $i \leq j$, where $i, j, \alpha_i, \alpha_j \in \mathbb{N}_0$.

For a polynomial, we need the condition

$$A_{\alpha_i+2i} = 0 \quad \text{where} \quad i, \alpha_i = 0, 1, 2, \ldots\,. \tag{2.5}$$

In this paper, the Pochhammer symbol $(x)_n$ is used to represent the rising factorial: $(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}$. In the above, α_i is an eigenvalue that makes A_n term terminated at certain value of the index n. (2.5) makes each $y_i(x)$ where $i = 0, 1, 2, \ldots$ as the polynomial in (2.4). Replace α_i by ω_i in (2.5) and put $n = \omega_i + 2i$ in (2.3a) with the condition $A_{\omega_i+2i} = 0$. Then we obtain eigenvalues ω such that

$$\omega = -(\omega_i + 2i + \lambda).$$

In (2.3a), we replace ω by $-(\omega_i + 2i + \lambda)$ and insert it and (2.3b) in (2.4), where the index α_i is replaced by ω_i. After the replacement process, the general expression of a power series of the GCH equation for a polynomial of type 2 is given by

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} y_n(x) = y_0(x)+y_1(x)+y_2(x)+y_3(x)+\cdots \\ &= c_0 x^{\lambda}\left\{ \sum_{i_0=0}^{\omega_0} \frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}} \eta^{i_0} + \left\{ \sum_{i_0=0}^{\omega_0} \frac{(i_0+\Omega/\mu+\lambda)}{(i_0+2+\lambda)(i_0+1+\nu+\lambda)} \frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}} \right.\right. \end{aligned}$$

$$
\begin{aligned}
&\times \sum_{i_1=i_0}^{\omega_1} \frac{(-\omega_1)_{i_1}(3+\lambda)_{i_0}(2+\nu+\lambda)_{i_0}}{(-\omega_1)_{i_0}(3+\lambda)_{i_1}(2+\nu+\lambda)_{i_1}}\,\eta^{i_1}\Bigg\}\rho \\
&+\sum_{n=2}^{\infty}\Bigg\{\sum_{i_0=0}^{\omega_0}\frac{(i_0+\Omega/\mu+\lambda)}{(i_0+2+\lambda)(i_0+1+\nu+\lambda)}\,\frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}} \\
&\times\prod_{k=1}^{n-1}\Bigg\{\sum_{i_k=i_{k-1}}^{\omega_k}\frac{(i_k+2k+\Omega/\mu+\lambda)}{(i_k+2k+2+\lambda)(i_k+2k+1+\nu+\lambda)} \\
&\times\frac{(-\omega_k)_{i_k}(2k+1+\lambda)_{i_{k-1}}(2k+\nu+\lambda)_{i_{k-1}}}{(-\omega_k)_{i_{k-1}}(2k+1+\lambda)_{i_k}(2k+\nu+\lambda)_{i_k}}\Bigg\} \\
&\times\sum_{i_n=i_{n-1}}^{\omega_n}\frac{(-\omega_n)_{i_n}(2n+1+\lambda)_{i_{n-1}}(2n+\nu+\lambda)_{i_{n-1}}}{(-\omega_n)_{i_{n-1}}(2n+1+\lambda)_{i_n}(2n+\nu+\lambda)_{i_n}}\,\eta^{i_n}\Bigg\}\rho^n\Bigg\}, \qquad (2.6)
\end{aligned}
$$

where

$$
\begin{cases}
\eta=-\varepsilon x,\\
\rho=-\mu x^2,\\
\omega=-(\omega_j+2j+\lambda)\ \text{ as } j,\omega_j\in\mathbb{N}_0,\\
\omega_i\le\omega_j\ \text{ only if }\ i\le j\ \text{ where }\ i,j\in\mathbb{N}_0.
\end{cases}
$$

Put $c_0=1$ as $\lambda=0$ for the first kind of independent solution of the GCH equation and as $\lambda=1-\nu$ for the second one in (2.6).

Remark 2.1. The power series expansion of the first kind GCH equation for a polynomial of type 2 about $x=0$ as $\omega=-(\omega_j+2j)$, where $j,\omega_j\in\mathbb{N}_0$, is

$$
\begin{aligned}
y(x)&=QW^R_{\omega_j}\Big(\mu,\varepsilon,\nu,\Omega,\omega=-(\omega_j+2j);\rho=-\mu x^2,\eta=-\varepsilon x\Big)\\
&=\sum_{i_0=0}^{\omega_0}\frac{(-\omega_0)_{i_0}}{(1)_{i_0}(\nu)_{i_0}}\eta^{i_0}+\Bigg\{\sum_{i_0=0}^{\omega_0}\frac{(i_0+\Omega/\mu)}{(i_0+2)(i_0+1+\nu)}\,\frac{(-\omega_0)_{i_0}}{(1)_{i_0}(\nu)_{i_0}}\sum_{i_1=i_0}^{\omega_1}\frac{(-\omega_1)_{i_1}(3)_{i_0}(2+\nu)_{i_0}}{(-\omega_1)_{i_0}(3)_{i_1}(2+\nu)_{i_1}}\,\eta^{i_1}\Bigg\}\rho\\
&+\sum_{n=2}^{\infty}\Bigg\{\sum_{i_0=0}^{\omega_0}\frac{(i_0+\Omega/\mu)}{(i_0+2)(i_0+1+\nu)}\,\frac{(-\omega_0)_{i_0}}{(1)_{i_0}(\nu)_{i_0}}\\
&\times\prod_{k=1}^{n-1}\Bigg\{\sum_{i_k=i_{k-1}}^{\omega_k}\frac{(i_k+2k+\Omega/\mu)}{(i_k+2k+2)(i_k+2k+1+\nu)}\,\frac{(-\omega_k)_{i_k}(2k+1)_{i_{k-1}}(2k+\nu)_{i_{k-1}}}{(-\omega_k)_{i_{k-1}}(2k+1)_{i_k}(2k+\nu)_{i_k}}\Bigg\}\\
&\times\sum_{i_n=i_{n-1}}^{\omega_n}\frac{(-\omega_n)_{i_n}(2n+1)_{i_{n-1}}(2n+\nu)_{i_{n-1}}}{(-\omega_n)_{i_{n-1}}(2n+1)_{i_n}(2n+\nu)_{i_n}}\,\eta^{i_n}\Bigg\}\rho^n. \qquad (2.7)
\end{aligned}
$$

For the minimum value of the first kind GCH equation for a polynomial of type 2 around $x=0$, we put $\omega_0=\omega_1=\omega_2=\cdots=0$ in (2.7).

$$
\begin{aligned}
y(x)&=QW^R_0\Big(\mu,\varepsilon,\nu,\Omega,\omega=-2j;\rho=-\mu x^2,\eta=-\varepsilon x\Big)\\
&={}_1F_1\Big(\frac{\Omega}{2\mu},\frac{\nu}{2}+\frac{1}{2},-\frac{1}{2}\mu x^2\Big),\ \text{ where }\ -\infty<x<\infty.
\end{aligned}
$$

As in the above, ${}_1F_1(a,b,x)=\sum_{n=0}^{\infty}\frac{(a)_n}{(b)_n}\frac{x^n}{n!}$.

Remark 2.2. The power series expansion of the second kind GCH equation for a polynomial of type 2 about $x=0$ as $\omega=-(\omega_j+2j+1-\nu)$, where $j,\omega_j\in\mathbb{N}_0$, is

$$
y(x)=RW^R_{\omega_j}\Big(\mu,\varepsilon,\nu,\Omega,\omega=-(\omega_j+2j+1-\nu);\rho=-\mu x^2,\eta=-\varepsilon x\Big)
$$

$$
\begin{aligned}
&= x^{1-\nu}\Bigg\{ \sum_{i_0=0}^{\omega_0} \frac{(-\omega_0)_{i_0}}{(2-\nu)_{i_0}(1)_{i_0}}\,\eta^{i_0} \\
&\quad + \Bigg\{ \sum_{i_0=0}^{\omega_0} \frac{(i_0+1+\Omega/\mu-\nu)}{(i_0+3-\nu)(i_0+2)}\,\frac{(-\omega_0)_{i_0}}{(2-\nu)_{i_0}(1)_{i_0}} \sum_{i_1=i_0}^{\omega_1} \frac{(-\omega_1)_{i_1}(4-\nu)_{i_0}(3)_{i_0}}{(-\omega_1)_{i_0}(4-\nu)_{i_1}(3)_{i_1}}\,\eta^{i_1}\Bigg\}\rho \\
&\quad + \sum_{n=2}^{\infty}\Bigg\{ \sum_{i_0=0}^{\omega_0} \frac{(i_0+1+\Omega/\mu-\nu)}{(i_0+3-\nu)(i_0+2)}\,\frac{(-\omega_0)_{i_0}}{(2-\nu)_{i_0}(1)_{i_0}} \\
&\quad\times \prod_{k=1}^{n-1}\Bigg\{ \sum_{i_k=i_{k-1}}^{\omega_k} \frac{(i_k+2k+1+\Omega/\mu-\nu)}{(i_k+2k+3-\nu)(i_k+2k+2)}\,\frac{(-\omega_k)_{i_k}(2k+2-\nu)_{i_{k-1}}(2k+1)_{i_{k-1}}}{(-\omega_k)_{i_{k-1}}(2k+2-\nu)_{i_k}(2k+1)_{i_k}}\Bigg\} \\
&\quad\times \sum_{i_n=i_{n-1}}^{\omega_n} \frac{(-\omega_n)_{i_n}(2n+2-\nu)_{i_{n-1}}(2n+1)_{i_{n-1}}}{(-\omega_n)_{i_{n-1}}(2n+2-\nu)_{i_n}(2n+1)_{i_n}}\,\eta^{i_n}\Bigg\}\rho^n\Bigg\}. \qquad (2.8)
\end{aligned}
$$

For the minimum value of the second kind GCH equation, for a polynomial of type 2 about $x=0$, we put $\omega_0=\omega_1=\omega_2=\cdots=0$ in (2.8).

$$
\begin{aligned}
y(x) &= RW_0^R\Big(\mu,\varepsilon,\nu,\Omega,\omega=-(2j+1-\nu);\rho=-\mu x^2,\eta=-\varepsilon x\Big) \\
&= x^{1-\nu}\ {}_1F_1\Big(\frac{\Omega}{2\mu}-\frac{\nu}{2}+\frac{1}{2},-\frac{\nu}{2}+\frac{3}{2},-\frac{1}{2}\mu x^2\Big),\quad \text{where } -\infty<x<\infty.
\end{aligned}
$$

In [11,12], Ω is treated as a fixed value and μ, ε, ν, ω are treated as free variables to construct the GCH polynomials of type 1 around $x=0$: (1) if $\Omega=-\mu(2\beta_j+j)$, where $j,\beta_j\in\mathbb{N}_0$, an analytic solution of the GCH equation turns to be the first kind of independent solution of the GCH polynomial of type 1; (2) if $\Omega=-\mu(2\psi_j+j+1-\nu)$ where $j,\psi_j\in\mathbb{N}_0$, an analytic solution of the GCH equation turns to be the second kind of independent solution of the GCH polynomial of type 1.

In this paper, ω is treated as a fixed value and μ, ε, ν, Ω are treated as free variables to construct the GCH polynomials of type 2 around $x=0$: (1) if $\omega=-(\omega_j+2j)$, where $j,\omega_j\in\mathbb{N}_0$, an analytic solution of the GCH equation turns to be the first kind of independent solution of the GCH polynomial of type 2; (2) if $\omega=-(\omega_j+2j+1-\nu)$, the analytic solution of the GCH equation turns to be the second kind of independent solution of the GCH polynomial of type 2.

2.1.2 Infinite series

In Chapter 1 of [13], the general expression of a power series of $y(x)$ for an infinite series is defined by

$$
\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} y_n(x) = y_0(x)+y_1(x)+y_2(x)+y_3(x)+\cdots \\
&= c_0x^{\lambda}\Bigg\{ \sum_{i_0=0}^{\infty}\Big(\prod_{i_1=0}^{i_0-1}A_{i_1}\Big)x^{i_0} + \sum_{i_0=0}^{\infty}\Bigg\{B_{i_0+1}\prod_{i_1=0}^{i_0-1}A_{i_1}\sum_{i_2=i_0}^{\infty}\Big(\prod_{i_3=i_0}^{i_2-1}A_{i_3+2}\Big)\Bigg\}x^{i_2+2} \\
&\quad + \sum_{N=2}^{\infty}\Bigg\{\sum_{i_0=0}^{\infty}\Bigg\{B_{i_0+1}\prod_{i_1=0}^{i_0-1}A_{i_1}\prod_{k=1}^{N-1}\Big(\sum_{i_{2k}=i_{2(k-1)}}^{\infty} B_{i_{2k}+2k+1}\prod_{i_{2k+1}=i_{2(k-1)}}^{i_{2k}-1}A_{i_{2k+1}+2k}\Big) \\
&\qquad \times \sum_{i_{2N}=i_{2(N-1)}}^{\infty}\Big(\prod_{i_{2N+1}=i_{2(N-1)}}^{i_{2N}-1}A_{i_{2N+1}+2N}\Big)\Bigg\}\Bigg\}x^{i_{2N}+2N}\Bigg\}. \qquad (2.9)
\end{aligned}
$$

Substitute (2.3a)–(2.3c) into (2.9). The general expression of a power series of the GCH equation for an infinite series about $x=0$ is given by

$$
y(x) = \sum_{n=0}^{\infty} y_n(x) = y_0(x)+y_1(x)+y_2(x)+y_3(x)+\cdots
$$

$$
\begin{aligned}
&= c_0 x^{\lambda}\Bigg\{ \sum_{i_0=0}^{\infty} \frac{(\omega+\lambda)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}}\, \eta^{i_0} + \Bigg\{ \sum_{i_0=0}^{\infty} \frac{\Xi^{(i_0)}(\omega+\lambda)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}} \\
&\quad \times \sum_{i_1=i_0}^{\infty} \frac{(\omega+2+\lambda)_{i_1}(3+\lambda)_{i_0}(2+\nu+\lambda)_{i_0}}{(\omega+2+\lambda)_{i_0}(3+\lambda)_{i_1}(2+\nu+\lambda)_{i_1}}\, \eta^{i_1} \Bigg\}\rho \\
&+ \sum_{n=2}^{\infty} \Bigg\{ \sum_{i_0=0}^{\infty} \frac{\Xi^{(i_0)}(\omega+\lambda)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}} \\
&\quad \times \prod_{k=1}^{n-1} \Bigg\{ \sum_{i_k=i_{k-1}}^{\infty} \frac{\Xi^{(i_k)}(\omega+2k+\lambda)_{i_k}(2k+1+\lambda)_{i_{k-1}}(2k+\nu+\lambda)_{i_{k-1}}}{(\omega+2k+\lambda)_{i_{k-1}}(2k+1+\lambda)_{i_{k-1}}(2k+\nu+\lambda)_{i_k}} \Bigg\} \\
&\quad \times \sum_{i_n=i_{n-1}}^{\infty} \frac{(\omega+2n+\lambda)_{i_n}(2n+1+\lambda)_{i_{n-1}}(2n+\nu+\lambda)_{i_{n-1}}}{(\omega+2n+\lambda)_{i_{n-1}}(2n+1+\lambda)_{i_{n-1}}(2n+\nu+\lambda)_{i_n}}\, \eta^{i_n} \Bigg\}\rho^n \Bigg\}, \qquad (2.10)
\end{aligned}
$$

where

$$
\begin{cases}
\Xi^{(i_0)} = \dfrac{(i_0+\Omega/\mu+\lambda)}{(i_0+2+\lambda)(i_0+1+\nu+\lambda)}\,, \\
\Xi^{(i_k)} = \dfrac{(i_k+2k+\Omega/\mu+\lambda)}{(i_k+2k+2+\lambda)(i_k+2k+1+\nu+\lambda)}\,.
\end{cases}
$$

Put $c_0= 1$ as $\lambda = 0$ for the first kind of independent solution of the GCH equation and as $\lambda = 1-\nu$ for the second one in (2.10).

Remark 2.3. The power series expansion of the GCH equation of the first kind for an infinite series about $x = 0$ using R3TRF is

$$
\begin{aligned}
&y(x) = QW^R\big(\mu,\varepsilon,\nu,\Omega,\omega;\rho=-\mu x^2,\eta=-\varepsilon x\big) \\
&\quad = \sum_{i_0=0}^{\infty} \frac{(\omega)_{i_0}}{(1)_{i_0}(\nu)_{i_0}}\eta^{i_0} + \Bigg\{ \sum_{i_0=0}^{\infty} \frac{(i_0+\Omega/\mu)}{(i_0+2)(i_0+1+\nu)}\, \frac{(\omega)_{i_0}}{(1)_{i_0}(\nu)_{i_0}} \sum_{i_1=i_0}^{\infty} \frac{(\omega+2)_{i_1}(3)_{i_0}(2+\nu)_{i_0}}{(\omega+2)_{i_0}(3)_{i_1}(2+\nu)_{i_1}}\, \eta^{i_1} \Bigg\}\rho \\
&\qquad + \sum_{n=2}^{\infty} \Bigg\{ \sum_{i_0=0}^{\infty} \frac{(i_0+\Omega/\mu)}{(i_0+2)(i_0+1+\nu)} \frac{(\omega)_{i_0}}{(1)_{i_0}(\nu)_{i_0}} \\
&\qquad \times \prod_{k=1}^{n-1} \Bigg\{ \sum_{i_k=i_{k-1}}^{\infty} \frac{(i_k+2k+\Omega/\mu)}{(i_k+2k+2)(i_k+2k+1+\nu)}\, \frac{(\omega+2k)_{i_k}(2k+1)_{i_{k-1}}(2k+\nu)_{i_{k-1}}}{(\omega+2k)_{i_{k-1}}(2k+1)_{i_{k-1}}(2k+\nu)_{i_k}} \Bigg\} \\
&\qquad \times \sum_{i_n=i_{n-1}}^{\infty} \frac{(\omega+2n)_{i_n}(2n+1)_{i_{n-1}}(2n+\nu)_{i_{n-1}}}{(\omega+2n)_{i_{n-1}}(2n+1)_{i_{n-1}}(2n+\nu)_{i_n}}\, \eta^{i_n} \Bigg\}\rho^n. \qquad (2.11)
\end{aligned}
$$

Remark 2.4. The power series expansion of the GCH equation of the second kind for an infinite series about $x = 0$ using R3TRF is

$$
\begin{aligned}
&y(x) = RW^R\big(\mu,\varepsilon,\nu,\Omega,\omega;\rho=-\mu x^2,\eta=-\varepsilon x\big) \\
&\quad = x^{1-\nu}\Bigg\{ \sum_{i_0=0}^{\infty} \frac{(\omega+1-\nu)_{i_0}}{(2-\nu)_{i_0}(1)_{i_0}}\, \eta^{i_0} \\
&\qquad + \Bigg\{ \sum_{i_0=0}^{\infty} \frac{(i_0+1+\Omega/\mu-\nu)}{(i_0+3-\nu)(i_0+2)}\, \frac{(\omega+1-\nu)_{i_0}}{(2-\nu)_{i_0}(1)_{i_0}} \sum_{i_1=i_0}^{\infty} \frac{(\omega+3-\nu)_{i_1}(4-\nu)_{i_0}(3)_{i_0}}{(\omega+3-\nu)_{i_0}(4-\nu)_{i_1}(3)_{i_1}}\, \eta^{i_1} \Bigg\}\rho \\
&\qquad + \sum_{n=2}^{\infty} \Bigg\{ \sum_{i_0=0}^{\infty} \frac{(i_0+1+\Omega/\mu-\nu)}{(i_0+3-\nu)(i_0+2)}\, \frac{(\omega+1-\nu)_{i_0}}{(2-\nu)_{i_0}(1)_{i_0}} \\
&\qquad\quad \times \prod_{k=1}^{n-1} \Bigg\{ \sum_{i_k=i_{k-1}}^{\infty} \frac{(i_k+2k+1+\Omega/\mu-\nu)}{(i_k+2k+3-\nu)(i_k+2k+2)}
\end{aligned}
$$

$$\times \frac{(\omega+2k+1-\nu)_{i_k}(2k+2-\nu)_{i_{k-1}}(2k+1)_{i_{k-1}}}{(\omega+2k+1-\nu)_{i_{k-1}}(2k+2-\nu)_{i_{k-1}}(2k+1)_{i_k}}\Bigg\}$$
$$\times \sum_{i_n=i_{n-1}}^{\infty} \frac{(\omega+2n+1-\nu)_{i_n}(2n+2-\nu)_{i_{n-1}}(2n+1)_{i_{n-1}}}{(\omega+2n+1-\nu)_{i_{n-1}}(2n+2-\nu)_{i_{n-1}}(2n+1)_{i_n}}\eta^{i_n}\Bigg\}\rho^n\Bigg\}. \tag{2.12}$$

It is required that $\nu \neq 0, -1, -2, \dots$ for the first kind of independent solutions of the GCH equation for an infinite series and a polynomial. But if it is not the case, its solutions will be divergent. And it is required that $\nu \neq 2, 3, 4, \dots$ for the second kind of independent solutions of the GCH equation for all cases.

Infinite series in this paper are equivalent to those in [11,12]. In this paper, B_n is the leading term in the sequence c_n of analytic function $y(x)$. In [11,12], A_n is the leading term in the sequence c_n of analytic function $y(x)$.*

2.2 Integral representation

2.2.1 Polynomial of type 2

Now I consider the combined definite and contour integral representation of the GCH equation by using R3TRF. There is a generalized hypergeometric function such as

$$\begin{aligned} I_l &= \sum_{i_l=i_{l-1}}^{\omega_l} \frac{(-\omega_l)_{i_l}(2l+1+\lambda)_{i_{l-1}}(2l+\nu+\lambda)_{i_{l-1}}}{(-\omega_l)_{i_{l-1}}(2l+1+\lambda)_{i_l}(2l+\nu+\lambda)_{i_l}}\eta^{i_l} \\ &= \sum_{j=0}^{\infty} \frac{B_{1,j}B_{2,j}(i_{l-1}-\omega_l)_j\eta^{i_{l-1}}}{(i_{l-1}+2l+\lambda)^{-1}(i_{l-1}+2l-1+\nu+\lambda)^{-1}(1)_j\, j!}\eta^j . \end{aligned} \tag{2.13}$$

By using integral form of the beta function,

$$B_{1,j} = B(i_{l-1}+2l+\lambda, j+1) = \int_0^1 dt_l\, t_l^{i_{l-1}+2l-1+\lambda}(1-t_l)^j, \tag{2.14a}$$

$$B_{2,j} = B(i_{l-1}+2l-1+\nu+\lambda, j+1) = \int_0^1 du_l\, u_l^{i_{l-1}+2l-2+\nu+\lambda}(1-u_l)^j. \tag{2.14b}$$

Substitute (2.14a) and (2.14b) into (2.13) and the result divide by $(i_{l-1}+2l+\lambda)(i_{l-1}+2l-1+\nu+\lambda)$. We get

$$\begin{aligned} &\frac{(i_{l-1}+2l+\lambda)^{-1}}{(i_{l-1}+2l-1+\nu+\lambda)} \sum_{i_l=i_{l-1}}^{\omega_l} \frac{(-\omega_l)_{i_l}(2l+1+\lambda)_{i_{l-1}}(2l+\nu+\lambda)_{i_{l-1}}}{(-\omega_l)_{i_{l-1}}(2l+1+\lambda)_{i_l}(2l+\nu+\lambda)_{i_l}}\eta^{i_l} \\ &\quad = \int_0^1 dt_l\, t_l^{2l-1+\lambda} \int_0^1 du_l\, u_l^{2l-2+\nu+\lambda}(\eta t_l u_l)^{i_{l-1}} \sum_{j=0}^{\infty} \frac{(i_{l-1}-\omega_l)_j}{(1)_j\, j!}\big(\eta(1-t_l)(1-u_l)\big)^j. \end{aligned} \tag{2.15}$$

The integral form of the confluent hypergeometric function of the first kind is given by

$$\sum_{j=0}^{\infty} \frac{(-\alpha_0)_j}{(\gamma)_j j!} z^j = \frac{\Gamma(\alpha_0+1)\Gamma(\gamma)}{2\pi i\,\Gamma(\alpha_0+\gamma)} \oint dv_l\, \frac{\exp(-\frac{z v_l}{(1-v_l)})}{v_l^{\alpha_0+1}(1-v_l)^{\gamma}}. \tag{2.16}$$

*As $\Gamma(1/2+\nu/2-\Omega/(2\mu))/\Gamma(1/2+\nu/2)$ is multiplied by (2.11), the new (2.11) is equivalent to the first kind solution of the GCH equation for an infinite series using 3TRF [11]. Again, as $(-\mu/2)^{1/2(1-\nu)}\,\Gamma(1-\Omega/(2\mu))/\Gamma(3/2-\nu/2)$ is multiplied by (2.12), the new (2.12) corresponds to the second kind solution of the GCH equation for an infinite series using 3TRF [11].

Replacing α_0, γ and z in (2.16), respectively, by $\omega_l - i_{l-1}$, 1 and $\eta(1-t_l)(1-u_l)$, we obtain

$$\sum_{j=0}^{\infty} \frac{(i_{l-1}-\omega_l)_j}{(1)_j\, j!} \left(\eta(1-t_l)(1-u_l)\right)^j = \frac{1}{2\pi i} \oint dv_l\, \frac{\exp\left(-\frac{v_l}{(1-v_l)}\,\eta(1-t_l)(1-u_l)\right)}{v_l^{\omega_l+1-i_{l-1}}(1-v_l)}\,. \tag{2.17}$$

Substitute (2.17) into (2.15):

$$\begin{aligned} K_l &= \frac{(i_{l-1}+2l+\lambda)^{-1}}{(i_{l-1}+2l-1+\nu+\lambda)} \sum_{i_l=i_{l-1}}^{\omega_l} \frac{(-\omega_l)_{i_l}(2l+1+\lambda)_{i_{l-1}}(2l+\nu+\lambda)_{i_{l-1}}}{(-\omega_l)_{i_{l-1}}(2l+1+\lambda)_{i_l}(2l+\nu+\lambda)_{i_l}}\, \eta^{i_l} \\ &= \int_0^1 dt_l\; t_l^{2l-1+\lambda} \int_0^1 du_l\; u_l^{2l-2+\nu+\lambda}\, \frac{1}{2\pi i} \oint dv_l\, \frac{\exp\left(-\frac{v_l}{(1-v_l)}\,\eta(1-t_l)(1-u_l)\right)}{v_l^{\omega_l+1}(1-v_l)}\, (\eta t_l u_l v_l)^{i_{l-1}}. \end{aligned} \tag{2.18}$$

Substitute (2.18) into (2.6), where $l = 1, 2, 3, \ldots$: apply K_1 into the second summation of the sub-power series $y_1(x)$; apply K_2 into the third summation and K_1 into the second summation of the sub-power series $y_2(x)$; apply K_3 into the forth summation, K_2 into the third summation and K_1 into the second summation of the sub-power series $y_3(x)$, etc.*

Theorem 2.5. *The general representation in the form of an integral of the GCH polynomial of type* 2 *is given by*

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} y_n(x) = y_0(x) + y_1(x) + y_2(x) + y_3(x) + \cdots \\ &= c_0 x^{\lambda} \left\{ \sum_{i_0=0}^{\omega_0} \frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}}\, \eta^{i_0} + \sum_{n=1}^{\infty} \left\{ \prod_{k=0}^{n-1} \left\{ \int_0^1 dt_{n-k}\; t_{n-k}^{2(n-k)-1+\lambda} \int_0^1 du_{n-k}\; u_{n-k}^{2(n-k-1)+\nu+\lambda} \right.\right.\right. \\ &\quad \times \frac{1}{2\pi i} \oint dv_{n-k}\, \frac{\exp\left(-\frac{v_{n-k}}{(1-v_{n-k})}\, w_{n-k+1,n}(1-t_{n-k})(1-u_{n-k})\right)}{v_{n-k}^{\omega_{n-k}+1}(1-v_{n-k})} \\ &\quad \left. \times w_{n-k,n}^{-(\Omega/\mu+2(n-k-1)+\lambda)} (w_{n-k,n}\partial_{w_{n-k,n}}) w_{n-k,n}^{\Omega/\mu+2(n-k-1)+\lambda} \right\} \\ &\quad \left.\left. \times \sum_{i_0=0}^{\omega_0} \frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}}\, w_{1,n}^{i_0} \right\} \rho^n \right\}, \end{aligned} \tag{2.19}$$

where

$$w_{a,b} = \begin{cases} \eta \prod_{l=a}^{b} t_l u_l v_l, & \textit{where } a \le b, \\ \eta \;\; \textit{only if } a > b. \end{cases}$$

Here the first sub-integral form contains one term of B_n's, the second one contains two terms of B_n's, the third one contains three terms of B_n's, etc.

Proof. In (2.6), the power series expansions of sub-summation terms $y_0(x)$, $y_1(x)$, $y_2(x)$ and $y_3(x)$ of the GCH polynomial of type 2 are

$$y(x) = \sum_{n=0}^{\infty} y_n(x) = y_0(x) + y_1(x) + y_2(x) + y_3(x) + \cdots\,, \tag{2.20}$$

where

$$y_0(x) = c_0 x^{\lambda} \sum_{i_0=0}^{\omega_0} \frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}}\, \eta^{i_0}, \tag{2.21a}$$

*$y_1(x)$ means the sub-power series in (2.6), contains one term of B_n's; $y_2(x)$ means the sub-power series in (2.6), contains two terms of B_n's; $y_3(x)$ means the sub-power series in (2.6), contains three terms of B_n's, etc.

$$
\begin{aligned}
y_1(x) = c_0 x^{\lambda} \Bigg\{ \sum_{i_0=0}^{\omega_0} \frac{(i_0+\Omega/\mu+\lambda)}{(i_0+2+\lambda)(i_0+1+\nu+\lambda)} \frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}} \\
\times \sum_{i_1=i_0}^{\omega_1} \frac{(-\omega_1)_{i_1}(3+\lambda)_{i_0}(2+\nu+\lambda)_{i_0}}{(-\omega_1)_{i_0}(3+\lambda)_{i_1}(2+\nu+\lambda)_{i_1}} \eta^{i_1} \Bigg\} \rho,
\end{aligned} \tag{2.21b}
$$

$$
\begin{aligned}
y_2(x) = c_0 x^{\lambda} \Bigg\{ \sum_{i_0=0}^{\omega_0} \frac{(i_0+\Omega/\mu+\lambda)}{(i_0+2+\lambda)(i_0+1+\nu+\lambda)} \frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}} \\
\times \sum_{i_1=i_0}^{\omega_1} \frac{(i_1+2+\Omega/\mu+\lambda)}{(i_1+4+\lambda)(i_1+3+\nu+\lambda)} \frac{(-\omega_1)_{i_1}(3+\lambda)_{i_0}(2+\nu+\lambda)_{i_0}}{(-\omega_1)_{i_0}(3+\lambda)_{i_1}(2+\nu+\lambda)_{i_1}} \\
\times \sum_{i_2=i_1}^{\omega_2} \frac{(-\omega_2)_{i_2}(5+\lambda)_{i_1}(4+\nu+\lambda)_{i_1}}{(-\omega_2)_{i_1}(5+\lambda)_{i_2}(4+\nu+\lambda)_{i_2}} \eta^{i_2} \Bigg\} \rho^2,
\end{aligned} \tag{2.21c}
$$

$$
\begin{aligned}
y_3(x) = c_0 x^{\lambda} \Bigg\{ \sum_{i_0=0}^{\omega_0} \frac{(i_0+\Omega/\mu+\lambda)}{(i_0+2+\lambda)(i_0+1+\nu+\lambda)} \frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}} \\
\times \sum_{i_1=i_0}^{\omega_1} \frac{(i_1+2+\Omega/\mu+\lambda)}{(i_1+4+\lambda)(i_1+3+\nu+\lambda)} \frac{(-\omega_1)_{i_1}(3+\lambda)_{i_0}(2+\nu+\lambda)_{i_0}}{(-\omega_1)_{i_0}(3+\lambda)_{i_1}(2+\nu+\lambda)_{i_1}} \\
\times \sum_{i_2=i_1}^{\omega_2} \frac{(i_2+4+\Omega/\mu+\lambda)}{(i_2+6+\lambda)(i_2+5+\nu+\lambda)} \frac{(-\omega_2)_{i_2}(5+\lambda)_{i_1}(4+\nu+\lambda)_{i_1}}{(-\omega_2)_{i_1}(5+\lambda)_{i_2}(4+\nu+\lambda)_{i_2}} \\
\times \sum_{i_3=i_2}^{\omega_3} \frac{(-\omega_3)_{i_3}(7+\lambda)_{i_2}(6+\nu+\lambda)_{i_2}}{(-\omega_3)_{i_2}(7+\lambda)_{i_3}(6+\nu+\lambda)_{i_3}} \eta^{i_3} \Bigg\} \rho^3.
\end{aligned} \tag{2.21d}
$$

Put $l = 1$ in (2.18) and insert it into (2.21b):

$$
\begin{aligned}
y_1(x) &= c_0 x^{\lambda} \rho \int_0^1 dt_1\, t_1^{1+\lambda} \int_0^1 du_1\, \frac{u_1^{\nu+\lambda}}{2\pi i} \oint dv_1\, \frac{\exp\left(-\frac{v_1}{(1-v_1)}\eta(1-t_1)(1-u_1)\right)}{v_1^{\omega_1+1}(1-v_1)} \\
&\quad \times \Bigg\{ \sum_{i_0=0}^{\omega_0} (i_0+\Omega/\mu+\lambda) \frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}} (\eta t_1 u_1 v_1)^{i_0} \Bigg\} \rho \\
&= c_0 x^{\lambda} \int_0^1 dt_1\, t_1^{1+\lambda} \int_0^1 du_1\, \frac{u_1^{\nu+\lambda}}{2\pi i} \oint dv_1 \frac{\exp\left(-\frac{v_1}{(1-v_1)}\eta(1-t_1)(1-u_1)\right)}{v_1^{\omega_1+1}(1-v_1)} \\
&\quad \times w_{1,1}^{-(\Omega/\mu+\lambda)} (w_{1,1}\partial_{w_{1,1}}) w_{1,1}^{\Omega/\mu+\lambda} \sum_{i_0=0}^{\omega_0} \frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}} w_{1,1}^{i_0},
\end{aligned} \tag{2.22}
$$

where

$$
w_{1,1} = \eta \prod_{l=1}^{1} t_l u_l v_l .
$$

Put $l = 2$ in (2.18) and insert it into (2.21c):

$$
\begin{aligned}
y_2(x) &= c_0 x^{\lambda} \rho^2 \int_0^1 dt_2\, t_2^{3+\lambda} \int_0^1 du_2\, \frac{u_2^{2+\nu+\lambda}}{2\pi i} \oint dv_2\, \frac{\exp\left(-\frac{v_2}{(1-v_2)}\eta(1-t_2)(1-u_2)\right)}{v_2^{\omega_2+1}(1-v_2)} \\
&\quad \times w_{2,2}^{-(\Omega/\mu+2+\lambda)} (w_{2,2}\partial_{w_{2,2}}) w_{2,2}^{\Omega/\mu+2+\lambda} \sum_{i_0=0}^{\omega_0} \frac{(i_0+\Omega/\mu+\lambda)}{(i_0+2+\lambda)(i_0+1+\nu+\lambda)} \frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}} \\
&\quad \times \sum_{i_1=i_0}^{\omega_1} \frac{(-\omega_1)_{i_1}(3+\lambda)_{i_0}(2+\nu+\lambda)_{i_0}}{(-\omega_1)_{i_0}(3+\lambda)_{i_1}(2+\nu+\lambda)_{i_1}} w_{2,2}^{i_1},
\end{aligned} \tag{2.23}
$$

where

$$w_{2,2} = \eta \prod_{l=2}^{2} t_l u_l v_l.$$

Put $l = 1$ and $\eta = w_{2,2}$ in (2.18) and insert it into (2.23). We get

$$\begin{aligned}
y_2(x) = c_0 x^{\lambda} \rho^2 \int_0^1 dt_2\; t_2^{3+\lambda} \int_0^1 du_2\; \frac{u_2^{2+\nu+\lambda}}{2\pi i} \\
\times \oint dv_2\; \frac{\exp\left(-\frac{v_2}{(1-v_2)}\,\eta(1-t_2)(1-u_2)\right)}{v_2^{\omega_2+1}(1-v_2)}\, w_{2,2}^{-(\Omega/\mu+2+\lambda)}(w_{2,2}\partial_{w_{2,2}}) w_{2,2}^{\Omega/\mu+2+\lambda} \\
\times \int_0^1 dt_1\; t_1^{1+\lambda} \int_0^1 du_1\; \frac{u_1^{\nu+\lambda}}{2\pi i} \oint dv_1\; \frac{\exp\left(-\frac{v_1}{(1-v_1)}\, w_{2,2}(1-t_1)(1-u_1)\right)}{v_1^{\omega_1+1}(1-v_1)} \\
\times w_{1,2}^{-(\Omega/\mu+\lambda)}(w_{1,2}\partial_{w_{1,2}}) w_{2,2}^{\Omega/\mu+\lambda} \sum_{i_0=0}^{\omega_0} \frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}}\, w_{1,2}^{i_0},
\end{aligned} \tag{2.24}$$

where

$$w_{1,2} = \eta \prod_{l=1}^{2} t_l u_l v_l.$$

By using similar process as in the previous cases for integral forms of $y_1(x)$ and $y_2(x)$, we obtain the following integral form of the sub-power series expansion $y_3(x)$:

$$\begin{aligned}
y_3(x) = c_0 x^{\lambda} \rho^3 \int_0^1 dt_3\; t_3^{5+\lambda} \int_0^1 du_3\; \frac{u_3^{4+\nu+\lambda}}{2\pi i} \\
\times \oint dv_3\; \frac{\exp\left(-\frac{v_3}{(1-v_3)}\,\eta(1-t_3)(1-u_3)\right)}{v_3^{\omega_3+1}(1-v_3)}\, w_{3,3}^{-(\Omega/\mu+4+\lambda)}(w_{3,3}\partial_{w_{3,3}}) w_{3,3}^{\Omega/\mu+4+\lambda} \\
\times \int_0^1 dt_2\; t_2^{3+\lambda} \int_0^1 du_2\; \frac{u_2^{2+\nu+\lambda}}{2\pi i} \\
\times \oint dv_2\; \frac{\exp\left(-\frac{v_2}{(1-v_2)}\, w_{3,3}(1-t_2)(1-u_2)\right)}{v_2^{\omega_2+1}(1-v_2)}\, w_{2,3}^{-(\Omega/\mu+2+\lambda)}(w_{2,3}\partial_{w_{2,3}}) w_{2,3}^{\Omega/\mu+2+\lambda} \\
\times \int_0^1 dt_1\; t_1^{1+\lambda} \int_0^1 du_1\; \frac{u_1^{\nu+\lambda}}{2\pi i} \\
\times \oint dv_1\; \frac{\exp\left(-\frac{v_1}{(1-v_1)}\, w_{2,3}(1-t_1)(1-u_1)\right)}{v_1^{\omega_1+1}(1-v_1)}\, w_{1,3}^{-(\Omega/\mu+\lambda)}(w_{1,3}\partial_{w_{1,3}})\, w_{1,3}^{\Omega/\mu+\lambda} \\
\times \sum_{i_0=0}^{\omega_0} \frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}}\, w_{1,3}^{i_0},
\end{aligned} \tag{2.25}$$

where

$$\begin{cases}
w_{3,3} = \eta \prod_{l=3}^{3} t_l u_l v_l, \\
w_{2,3} = \eta \prod_{l=2}^{3} t_l u_l v_l, \\
w_{1,3} = \eta \prod_{l=1}^{3} t_l u_l v_l\,.
\end{cases}$$

By repeating the above process, we obtain integral forms of all higher sub-summation terms $y_m(x)$, where $m \geq 4$. Substituting (2.21a), (2.22), (2.24), (2.25) and including integral forms of $y_m(x)$, $m \geq 4$, into (2.20), we obtain (2.19). □

Put $c_0 = 1$ as $\lambda = 0$ for the first kind of independent solution of the GCH equation and as $\lambda = 1-\nu$ for the second kind one in (2.19).

Remark 2.6. The integral representation of the first kind GCH equation for a polynomial of type 2 about $x = 0$ as $\omega = -(\omega_j + 2j)$, where $j, \omega_j = 0, 1, 2, \ldots$, is

$$\begin{aligned} y(x) &= QW^R_{\omega_j}\Big(\mu, \varepsilon, \nu, \Omega, \omega = -(\omega_j + 2j); \rho = -\mu x^2, \eta = -\varepsilon x\Big) \\ &= {}_1F_1(-\omega_0; \nu; \eta) + \sum_{n=1}^{\infty} \left\{ \prod_{k=0}^{n-1} \left\{ \int_0^1 dt_{n-k}\; t_{n-k}^{2(n-k)-1} \int_0^1 du_{n-k}\; u_{n-k}^{2(n-k-1)+\nu} \right.\right. \\ &\quad \times \frac{1}{2\pi i} \oint dv_{n-k}\; \frac{\exp\left(-\frac{v_{n-k}}{(1-v_{n-k})}\, w_{n-k+1,n}(1-t_{n-k})(1-u_{n-k})\right)}{v_{n-k}^{\omega_{n-k}+1}(1-v_{n-k})} \\ &\quad \left.\left. \times\, w_{n-k,n}^{-(\Omega/\mu+2(n-k-1))}(w_{n-k,n}\partial_{w_{n-k,n}})w_{n-k,n}^{\Omega/\mu+2(n-k-1)} \right\} {}_1F_1(-\omega_0; \nu; w_{1,n}) \right\} \rho^n. \end{aligned} \tag{2.26}$$

Remark 2.7. The integral representation of the second kind GCH equation for a polynomial of type 2 about $x = 0$ as $\omega = -(\omega_j + 2j + 1 - \nu)$, where $j, \omega_j = 0, 1, 2, \ldots$, is

$$\begin{aligned} y(x) &= RW^R_{\omega_j}\Big(\mu, \varepsilon, \nu, \Omega, \omega = -(\omega_j + 2j + 1 - \nu); \rho = -\mu x^2, \eta = -\varepsilon x\Big) \\ &= x^{1-\nu}\left\{ {}_1F_1(-\omega_0; 2-\nu; \eta) + \sum_{n=1}^{\infty} \left\{ \prod_{k=0}^{n-1} \left\{ \int_0^1 dt_{n-k}\; t_{n-k}^{2(n-k)-\nu} \int_0^1 du_{n-k}\; u_{n-k}^{2(n-k)-1} \right.\right.\right. \\ &\quad \times \frac{1}{2\pi i} \oint dv_{n-k}\; \frac{\exp\left(-\frac{v_{n-k}}{(1-v_{n-k})}\, w_{n-k+1,n}(1-t_{n-k})(1-u_{n-k})\right)}{v_{n-k}^{\omega_{n-k}+1}(1-v_{n-k})} \\ &\quad \left.\left.\left. \times w_{n-k,n}^{-(\Omega/\mu+2(n-k)-1-\nu)}(w_{n-k,n}\partial_{w_{n-k,n}})w_{n-k,n}^{\Omega/\mu+2(n-k)-1-\nu} \right\} {}_1F_1(-\omega_0; 2-\nu; w_{1,n}) \right\} \rho^n \right\}. \end{aligned} \tag{2.27}$$

In the above equalities, ${}_1F_1\,(a; b; z)$ is a Kummer function of the first kind defined as

$$\begin{aligned} {}_1F_1(a; b; z) = M(a, b, z) &= \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n n!}\;, z^n = e^z M(b-a, b, -z) \\ &= -\frac{1}{2\pi i}\;, \frac{\Gamma(1-a)\Gamma(b)}{\Gamma(b-a)} \oint dv_j\; e^{zv_j}(-v_j)^{a-1}(1-v_j)^{b-a-1} \\ &= \frac{\Gamma(a)}{2\pi i} \oint dv_j\; e^{v_j} v_j^{-b}(1-\frac{z}{v_j})^{-a} \\ &= \frac{1}{2\pi i}\;, \frac{\Gamma(1-a)\Gamma(b)}{\Gamma(b-a)} \oint dv_j\; e^{-\frac{z\, v_j}{1-v_j}}\; v_j^{a-1}(1-v_j)^{-b}. \end{aligned} \tag{2.28}$$

2.2.2 Infinite Series

Let us consider the integral representation of the GCH equation about $x = 0$ for an infinite series by applying R3TRF. There is a generalized hypergeometric function which is given by

$$\begin{aligned} M_l &= \sum_{i_l=i_{l-1}}^{\infty} \frac{(\omega+2l+\lambda)_{i_l}(2l+1+\lambda)_{i_{l-1}}(2l+\nu+\lambda)_{i_{l-1}}}{(\omega+2l+\lambda)_{i_{l-1}}(2l+1+\lambda)_{i_l}(2l+\nu+\lambda)_{i_l}}\, \eta^{i_l} \\ &= \sum_{j=0}^{\infty} \frac{B_{i_{l-1},j}(\omega+2l+\lambda+i_{l-1})_j \eta^{i_{l-1}}}{(i_{l-1}+2l+\lambda)^{-1}(i_{l-1}+2l-1+\nu+\lambda)^{-1}(1)_j\, j!}\, \eta^j, \end{aligned} \tag{2.29}$$

where

$$B_{i_{l-1},j} = B(i_{l-1}+2l+\lambda, j+1)B(i_{l-1}+2l-1+\nu+\lambda, j+1).$$

Substituting (2.14a) and (2.14b) into (2.29) and dividing the obtained equality by $(i_{l-1}+2l+\lambda)(i_{l-1}+2l-1+\nu+\lambda)$, we get

$$\sum_{i_l=i_{l-1}}^{\infty} \frac{A_{i_{l-1}}(\omega+2l+\lambda)_{i_l}(2l+1+\lambda)_{i_{l-1}}(2l+\nu+\lambda)_{i_{l-1}}}{(\omega+2l+\lambda)_{i_{l-1}}(2l+1+\lambda)_{i_l}(2l+\nu+\lambda)_{i_l}} \eta^{i_l}$$
$$= \int_0^1 dt_l\; t_l^{2l-1+\lambda} \int_0^1 du_l\; u_l^{2l-2+\nu+\lambda} (\eta t_l u_l)^{i_{l-1}} \sum_{j=0}^{\infty} \frac{(\omega+2l+\lambda+i_{l-1})_j}{(1)_j\, j!} \left(\eta(1-t_l)(1-u_l)\right)^j, \quad (2.30)$$

where

$$A_{i_{l-1}} = \frac{1}{(i_{l-1}+2l+\lambda)(i_{l-1}+2l-1+\nu+\lambda)}\,.$$

In (2.28), replacing a, b and z, respectively, by $\omega+2l+\lambda+i_{l-1}$, 1 and $\eta(1-t_j)(1-u_j)$, and inserting the resulting equality into (2.30), we obtain

$$V_l = \sum_{i_l=i_{l-1}}^{\infty} \frac{A_{i_{l-1}}(\omega+2l+\lambda)_{i_l}(2l+1+\lambda)_{i_{l-1}}(2l+\nu+\lambda)_{i_{l-1}}}{(\omega+2l+\lambda)_{i_{l-1}}(2l+1+\lambda)_{i_l}(2l+\nu+\lambda)_{i_l}} \eta^{i_l}$$
$$= \int_0^1 dt_l\; t_l^{2l-1+\lambda} \int_0^1 du_l\; u_l^{2l-2+\nu+\lambda} \frac{1}{2\pi i} \oint dv_l\; \frac{\exp\left(-\frac{v_l}{(1-v_l)}\,\eta(1-t_l)(1-u_l)\right)}{v_l^{-(\omega+2l-1+\lambda)}(1-v_l)} (\eta t_l u_l v_l)^{i_{l-1}}\,. \quad (2.31)$$

We substitute (2.31) into (2.10), where $l = 1, 2, 3, \ldots$: apply V_1 into the second summation of the sub-power series $y_1(x)$; apply V_2 into the third summation and V_1 into the second summation of the sub-power series $y_2(x)$; apply V_3 into the forth summation, V_2 into the third summation and V_1 into the second summation of the sub-power series $y_3(x)$, etc.*

Theorem 2.8. *The general representation in the form of an integral of the GCH equation for an infinite series about $x=0$ using R3TRF is given by*

$$y(x) = \sum_{n=0}^{\infty} y_n(x) = y_0(x)+y_1(x)+y_2(x)+y_3(x)+\cdots$$
$$= c_0 x^{\lambda}\left\{ \sum_{i_0=0}^{\infty} \frac{(\omega+\lambda)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}} \eta^{i_0} \right.$$
$$+ \sum_{n=1}^{\infty} \left\{ \prod_{k=0}^{n-1} \left\{ \int_0^1 dt_{n-k}\; t_{n-k}^{2(n-k)-1+\lambda} \int_0^1 du_{n-k}\; u_{n-k}^{2(n-k-1)+\nu+\lambda} \right.\right.$$
$$\times \frac{1}{2\pi i} \oint dv_{n-k}\; \frac{\exp\left(-\frac{v_{n-k}}{(1-v_{n-k})}\, w_{n-k+1,n}(1-t_{n-k})(1-u_{n-k})\right)}{v_{n-k}^{-(\omega+2(n-k)-1+\lambda)}(1-v_{n-k})}$$
$$\left. \times\, w_{n-k,n}^{-(\Omega/\mu+2(n-k-1)+\lambda)} (w_{n-k,n}\partial_{w_{n-k,n}}) w_{n-k,n}^{\Omega/\mu+2(n-k-1)+\lambda} \right\}$$
$$\left.\left. \times \sum_{i_0=0}^{\infty} \frac{(\omega+\lambda)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}} w_{1,n}^{i_0} \right\} \rho^n \right\}. \quad (2.32)$$

*$y_1(x)$ means the sub-power series in (2.10), contains one term of B_n's; $y_2(x)$ means the sub-power series in (2.10), contains two terms of B_n's; $y_3(x)$ means the sub-power series in (2.10), contains three terms of B_n's, etc. Or we replace the finite summation with an interval $[0,\omega_0]$ by an infinite summation with an interval $[0,\infty]$ in (2.19). We also replace ω_0 and ω_{n-j} by $-(\omega+\lambda)$ and substitute $-(\omega+2(n-k)+\lambda)$ into the new (2.19). Its solution is likewise equivalent to (2.32).

*Here the first sub-integral form contains one term of B_n's, the second one contains two terms of B_n's, the third one contains three terms of B_n's, etc.**

Put c_0= 1 as $\lambda = 0$ for the first kind of independent solution of the GCH equation and as $\lambda = 1-\nu$, for the second kind one in (2.32).

Remark 2.9. The integral representation of the first kind GCH equation for an infinite series about $x = 0$ applying R3TRF is

$$\begin{aligned} y(x) &= QW^R\big(\mu, \varepsilon, \nu, \Omega, \omega; \rho = -\mu x^2, \eta = -\varepsilon x\big) \\ &= {}_1F_1(\omega; \nu; \eta) + \sum_{n=1}^{\infty} \left\{ \prod_{k=0}^{n-1} \left\{ \int_0^1 dt_{n-k}\, t_{n-k}^{2(n-k)-1} \int_0^1 du_{n-k}\, u_{n-k}^{2(n-k-1)+\nu} \right.\right. \\ &\quad \times \frac{1}{2\pi i} \oint dv_{n-k}\, \frac{\exp\big(-\frac{v_{n-k}}{(1-v_{n-k})}\, w_{n-k+1,n}(1-t_{n-k})(1-u_{n-k})\big)}{v_{n-k}^{-(\omega+2(n-k)-1)}(1-v_{n-k})} \\ &\quad \left.\left. \times\, w_{n-k,n}^{-(\Omega/\mu+2(n-k-1))}(w_{n-k,n}\partial_{w_{n-k,n}})w_{n-k,n}^{\Omega/\mu+2(n-k-1)} \right\} {}_1F_1(\omega; \nu; w_{1,n}) \right\}\rho^n. \end{aligned} \tag{2.33}$$

Remark 2.10. The integral representation of the second kind GCH equation for an infinite series about $x = 0$ applying R3TRF is

$$\begin{aligned} y(x) &= RW^R\big(\mu, \varepsilon, \nu, \Omega, \omega; \rho = -\mu x^2, \eta = -\varepsilon x\big) \\ &= x^{1-\nu}\left\{ {}_1F_1(\omega+1-\nu; 2-\nu; \eta) + \sum_{n=1}^{\infty} \left\{ \prod_{k=0}^{n-1} \left\{ \int_0^1 dt_{n-k}\, t_{n-k}^{2(n-k)-\nu} \int_0^1 du_{n-k}\, u_{n-k}^{2(n-k)-1} \right.\right.\right. \\ &\quad \times \frac{1}{2\pi i} \oint dv_{n-k}\, \frac{\exp\big(-\frac{v_{n-k}}{(1-v_{n-k})}\, w_{n-k+1,n}(1-t_{n-k})(1-u_{n-k})\big)}{v_{n-k}^{-(\omega+2(n-k)-\nu)}(1-v_{n-k})} \\ &\left.\left.\left. \times w_{n-k,n}^{-(\Omega/\mu+2(n-k)-1-\nu)}(w_{n-k,n}\partial_{w_{n-k,n}})w_{n-k,n}^{\Omega/\mu+2(n-k)-1-\nu} \right\} {}_1F_1(\omega+1-\nu; 2-\nu; w_{1,n}) \right\}\rho^n \right\}. \end{aligned} \tag{2.34}$$

(2.33) multiplied by $\frac{\Gamma(1/2+\nu/2-\Omega/(2\mu))}{\Gamma(1/2+\nu/2)}$ is equivalent to the integral form of the first kind solution of the GCH equation for an infinite series applying 3TRF [11]. Also, (2.34) multiplied by $(-\mu/2)^{1/2(1-\nu)}\frac{\Gamma(1-\Omega/(2\mu))}{\Gamma(3/2-\nu/2)}$ corresponds to the integral representation of the second kind solution of the GCH equation for an infinite series applying 3TRF [11].

2.3 Generating function for the GCH polynomial of type 2

Now let us investigate generating functions for the type 2 GCH polynomials of the first and second kind around $x = 0$.

Definition 2.11. Define

$$\begin{cases} s_{a,b} = \begin{cases} s_a \cdot s_{a+1} \cdot s_{a+2} \cdots s_{b-2} \cdot s_{b-1} \cdot s_b, & \text{where } \ a < b, \\ s_a \ \text{ only if } \ a = b, \end{cases} \\ \widetilde{w}_{i,j} = \eta s_{i,\infty} \displaystyle\prod_{l=i}^{j} t_l u_l, \end{cases} \tag{2.35}$$

where $a, b, i, j \in \mathbb{N}_0$, $0 \le a \le b \le \infty$ and $1 \le i \le j \le \infty$.

*The method how to prove an integral for an infinite series is similar as an integral for a fixed value of ω at Subsection 2.2.1. Explicit proof for this integral is available on pages 250–253 in Chapter 6 [13].

We have

$$\sum_{\omega_i=\omega_j}^{\infty} s_i^{\omega_i} = \frac{s_i^{\omega_j}}{(1-s_i)} \quad \text{at } |s_i|<1. \tag{2.36}$$

Theorem 2.12. *The general expression of the generating function for the GCH polynomial of type 2 about $x=0$ is given by*

$$\begin{aligned}
&\sum_{\omega_0=0}^{\infty} \frac{s_0^{\omega_0}}{\omega_0!} \frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')} \prod_{n=1}^{\infty} \Big\{ \sum_{\omega_n=\omega_{n-1}}^{\infty} s_n^{\omega_n} \Big\} y(x) = \prod_{k=1}^{\infty} \frac{1}{(1-s_{k,\infty})} \boldsymbol{\Upsilon}(\lambda; s_{0,\infty}; \eta) \\
&\quad + \Bigg\{ \prod_{k=1}^{\infty} \frac{1}{(1-s_{k,\infty})} \int_0^1 dt_1\, t_1^{1+\lambda} \int_0^1 du_1\, u_1^{\nu+\lambda} \exp\Big(-\frac{s_{1,\infty}}{(1-s_{1,\infty})} \eta(1-t_1)(1-u_1) \Big) \\
&\qquad \times \widetilde{w}_{1,1}^{-(\Omega/\mu+\lambda)} (\widetilde{w}_{1,1} \partial_{\widetilde{w}_{1,1}}) \widetilde{w}_{1,1}^{\Omega/\mu+\lambda} \boldsymbol{\Upsilon}(\lambda; s_0; \widetilde{w}_{1,1}) \Bigg\} \rho \\
&+ \sum_{n=2}^{\infty} \Bigg\{ \prod_{k=n}^{\infty} \frac{1}{(1-s_{k,\infty})} \int_0^1 dt_n\, t_n^{2n-1+\lambda} \int_0^1 du_n\, u_n^{2(n-1)+\nu+\lambda} \exp\Big(-\frac{s_{n,\infty}}{(1-s_{n,\infty})} \eta(1-t_n)(1-u_n) \Big) \\
&\qquad \times \widetilde{w}_{n,n}^{-(\Omega/\mu+2(n-1)+\lambda)} (\widetilde{w}_{n,n} \partial_{\widetilde{w}_{n,n}}) \widetilde{w}_{n,n}^{\Omega/\mu+2(n-1)+\lambda} \\
&\times \prod_{j=1}^{n-1} \Bigg\{ \int_0^1 dt_{n-j}\, t_{n-j}^{2(n-j)-1+\lambda} \int_0^1 du_{n-j}\, u_{n-j}^{2(n-j-1)+\nu+\lambda} \frac{\exp\big(-\frac{s_{n-j}}{(1-s_{n-j})} \widetilde{w}_{n-j+1,n} (1-t_{n-j})(1-u_{n-j}) \big)}{(1-s_{n-j})} \\
&\qquad \times \widetilde{w}_{n-j,n}^{-(\Omega/\mu+2(n-j-1)+\lambda)} (\widetilde{w}_{n-j,n} \partial_{\widetilde{w}_{n-j,n}}) \widetilde{w}_{n-j,n}^{\Omega/\mu+2(n-j-1)+\lambda} \Bigg\} \boldsymbol{\Upsilon}(\lambda; s_0; \widetilde{w}_{1,n}) \Bigg\} \rho^n,
\end{aligned} \tag{2.37}$$

where

$$\begin{cases}
\boldsymbol{\Upsilon}(\lambda; s_{0,\infty}; \eta) = \displaystyle\sum_{\omega_0=0}^{\infty} \frac{s_{0,\infty}^{\omega_0}}{\omega_0!} \frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')} \Big\{ c_0 x^{\lambda} \sum_{i_0=0}^{\omega_0} \frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}} \eta^{i_0} \Big\}, \\
\boldsymbol{\Upsilon}(\lambda; s_0; \widetilde{w}_{1,1}) = \displaystyle\sum_{\omega_0=0}^{\infty} \frac{s_0^{\omega_0}}{\omega_0!} \frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')} \Big\{ c_0 x^{\lambda} \sum_{i_0=0}^{\omega_0} \frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}} \widetilde{w}_{1,1}^{i_0} \Big\}, \\
\boldsymbol{\Upsilon}(\lambda; s_0; \widetilde{w}_{1,n}) = \displaystyle\sum_{\omega_0=0}^{\infty} \frac{s_0^{\omega_0}}{\omega_0!} \frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')} \Big\{ c_0 x^{\lambda} \sum_{i_0=0}^{\omega_0} \frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}} \widetilde{w}_{1,n}^{i_0} \Big\}.
\end{cases}$$

Proof. Applying the summation operator $\sum_{\omega_0=0}^{\infty} \frac{s_0^{\omega_0}}{\omega_0!} \frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')} \prod_{n=1}^{\infty} \Big\{ \sum_{\omega_n=\omega_{n-1}}^{\infty} s_n^{\omega_n} \Big\}$ to the form of a general integral of type 2 GCH polynomial $y(x)$, we get

$$\begin{aligned}
&\sum_{\omega_0=0}^{\infty} \frac{s_0^{\omega_0}}{\omega_0!} \frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')} \prod_{n=1}^{\infty} \Big\{ \sum_{\omega_n=\omega_{n-1}}^{\infty} s_n^{\omega_n} \Big\} y(x) \\
&\qquad = \sum_{\omega_0=0}^{\infty} \frac{s_0^{\omega_0}}{\omega_0!} \frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')} \prod_{n=1}^{\infty} \Big\{ \sum_{\omega_n=\omega_{n-1}}^{\infty} s_n^{\omega_n} \Big\} (y_0(x)+y_1(x)+y_2(x)+\cdots).
\end{aligned} \tag{2.38}$$

Applying the summation operator $\sum_{\omega_0=0}^{\infty} \frac{s_0^{\omega_0}}{\omega_0!} \frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')} \prod_{n=1}^{\infty} \Big\{ \sum_{\omega_n=\omega_{n-1}}^{\infty} s_n^{\omega_n} \Big\}$ to (2.21a) and using (2.35) and (2.36), we obtain

$$\sum_{\omega_0=0}^{\infty}\frac{s_0^{\omega_0}}{\omega_0!}\frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')}\prod_{n=1}^{\infty}\Big\{\sum_{\omega_n=\omega_{n-1}}^{\infty}s_n^{\omega_n}\Big\}y_0(x)$$
$$=\prod_{k=1}^{\infty}\frac{1}{(1-s_{k,\infty})}\sum_{\omega_0=0}^{\infty}\frac{s_{0,\infty}^{\omega_0}}{\omega_0!}\frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')}c_0x^{\lambda}\sum_{i_0=0}^{\omega_0}\frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}}\eta^{i_0}. \quad (2.39)$$

Applying the summation operator $\sum_{\omega_0=0}^{\infty}\frac{s_0^{\omega_0}}{\omega_0!}\frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')}\prod_{n=1}^{\infty}\Big\{\sum_{\omega_n=\omega_{n-1}}^{\infty}s_n^{\omega_n}\Big\}$ to (2.22) and using (2.35) and (2.36), we get

$$\sum_{\omega_0=0}^{\infty}\frac{s_0^{\omega_0}}{\omega_0!}\frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')}\prod_{n=1}^{\infty}\Big\{\sum_{\omega_n=\omega_{n-1}}^{\infty}s_n^{\omega_n}\Big\}y_1(x)=\prod_{k=2}^{\infty}\frac{1}{(1-s_{k,\infty})}\int_0^1 dt_1\,t_1^{1+\lambda}\int_0^1 du_1\,u_1^{\nu+\lambda}$$
$$\times\frac{1}{2\pi i}\oint dv_1\,\frac{\exp\left(-\frac{v_1}{(1-v_1)}\eta(1-t_1)(1-u_1)\right)}{v_1(1-v_1)}\sum_{\omega_1=\omega_0}^{\infty}\left(\frac{s_{1,\infty}}{v_1}\right)^{\omega_1}w_{1,1}^{-(\Omega/\mu+\lambda)}(w_{1,1}\partial_{w_{1,1}})w_{1,1}^{\Omega/\mu+\lambda}$$
$$\times\sum_{\omega_0=0}^{\infty}\frac{s_0^{\omega_0}}{\omega_0!}\frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')}\left\{c_0x^{\lambda}\sum_{i_0=0}^{\omega_0}\frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}}w_{1,1}^{i_0}\right\}\rho. \quad (2.40)$$

Replacing ω_i, ω_j and s_i, respectively, by ω_1, ω_0 and $\frac{s_{1,\infty}}{v_1}$ in (2.36) and inserting it into (2.40), we have

$$\sum_{\omega_0=0}^{\infty}\frac{s_0^{\omega_0}}{\omega_0!}\frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')}\prod_{n=1}^{\infty}\Big\{\sum_{\omega_n=\omega_{n-1}}^{\infty}s_n^{\omega_n}\Big\}y_1(x)=\prod_{k=2}^{\infty}\frac{1}{(1-s_{k,\infty})}\int_0^1 dt_1\,t_1^{1+\lambda}\int_0^1 du_1\,u_1^{\nu+\lambda}$$
$$\times\frac{1}{2\pi i}\oint dv_1\,\frac{\exp\left(-\frac{v_1}{(1-v_1)}\eta(1-t_1)(1-u_1)\right)}{(1-v_1)(v_1-s_{1,\infty})}w_{1,1}^{-(\Omega/\mu+\lambda)}(w_{1,1}\partial_{w_{1,1}})w_{1,1}^{\Omega/\mu+\lambda}$$
$$\times\sum_{\omega_0=0}^{\infty}\frac{1}{\omega_0!}\left(\frac{s_{0,\infty}}{v_1}\right)^{\omega_0}\frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')}\left\{c_0x^{\lambda}\sum_{i_0=0}^{\omega_0}\frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}}w_{1,1}^{i_0}\right\}\rho. \quad (2.41)$$

By using Cauchy's integral formula, the contour integrand has poles at $v_1=1$ or $s_{1,\infty}$, where $s_{1,\infty}$ is only inside the unit circle. As we compute the residue in (2.41), we obtain

$$\sum_{\omega_0=0}^{\infty}\frac{s_0^{\omega_0}}{\omega_0!}\frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')}\prod_{n=1}^{\infty}\Big\{\sum_{\omega_n=\omega_{n-1}}^{\infty}s_n^{\omega_n}\Big\}y_1(x)=\prod_{k=1}^{\infty}\frac{1}{(1-s_{k,\infty})}\int_0^1 dt_1\,t_1^{1+\lambda}\int_0^1 du_1\,u_1^{\nu+\lambda}$$
$$\times\exp\left(-\frac{s_{1,\infty}}{(1-s_{1,\infty})}\eta(1-t_1)(1-u_1)\right)\widetilde{w}_{1,1}^{-(\Omega/\mu+\lambda)}(\widetilde{w}_{1,1}\partial_{\widetilde{w}_{1,1}})\widetilde{w}_{1,1}^{\Omega/\mu+\lambda}$$
$$\times\sum_{\omega_0=0}^{\infty}\frac{s_0^{\omega_0}}{\omega_0!}\frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')}\left\{c_0x^{\lambda}\sum_{i_0=0}^{\omega_0}\frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}}\widetilde{w}_{1,1}^{i_0}\right\}\rho, \quad (2.42)$$

where

$$\widetilde{w}_{1,1}=\eta s_{1,\infty}\prod_{l=1}^{1}t_l u_l.$$

Applying the summation operator $\sum_{\omega_0=0}^{\infty}\frac{s_0^{\omega_0}}{\omega_0!}\frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')}\prod_{n=1}^{\infty}\Big\{\sum_{\omega_n=\omega_{n-1}}^{\infty}s_n^{\omega_n}\Big\}$ to (2.24) and using (2.35) and (2.36), we obtain

$$
\begin{aligned}
&\sum_{\omega_0=0}^{\infty} \frac{s_0^{\omega_0}}{\omega_0!} \frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')} \prod_{n=1}^{\infty} \Big\{ \sum_{\omega_n=\omega_{n-1}}^{\infty} s_n^{\omega_n} \Big\} y_2(x) \\
&\quad = \prod_{k=3}^{\infty} \frac{1}{(1-s_{k,\infty})} \int_0^1 dt_2\; t_2^{3+\lambda} \int_0^1 du_2\; u_2^{2+\nu+\lambda} \frac{1}{2\pi i} \oint dv_2\; \frac{\exp\left(-\frac{v_2}{(1-v_2)}\,\eta(1-t_2)(1-u_2)\right)}{v_2(1-v_2)} \\
&\qquad \times \sum_{\omega_2=\omega_1}^{\infty} \left(\frac{s_{2,\infty}}{v_2}\right)^{\omega_2} w_{2,2}^{-(\Omega/\mu+2+\lambda)} (w_{2,2}\partial_{w_{2,2}}) w_{2,2}^{\Omega/\mu+2+\lambda} \int_0^1 dt_1\; t_1^{1+\lambda} \int_0^1 du_1\; u_1^{\nu+\lambda} \\
&\quad \times \frac{1}{2\pi i} \oint dv_1\; \frac{\exp\left(-\frac{v_1}{(1-v_1)}\, w_{2,2}(1-t_1)(1-u_1)\right)}{v_1(1-v_1)} \sum_{\omega_1=\omega_0}^{\infty} \left(\frac{s_1}{v_1}\right)^{\omega_1} w_{1,2}^{-(\Omega/\mu+\lambda)} (w_{1,2}\partial_{w_{1,2}}) w_{1,2}^{\Omega/\mu+\lambda} \\
&\qquad \times \sum_{\omega_0=0}^{\infty} \frac{s_0^{\omega_0}}{\omega_0!} \frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')} \left\{ c_0 x^{\lambda} \sum_{i_0=0}^{\omega_0} \frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}}\, w_{1,2}^{i_0} \right\} \rho^2. \qquad (2.43)
\end{aligned}
$$

Replacing in (2.36) ω_i, ω_j and s_i, respectively, by ω_2, ω_1 and $\frac{s_{2,\infty}}{v_2}$ and inserting the obtained formula into (2.43), we get

$$
\begin{aligned}
&\sum_{\omega_0=0}^{\infty} \frac{s_0^{\omega_0}}{\omega_0!} \frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')} \prod_{n=1}^{\infty} \Big\{ \sum_{\omega_n=\omega_{n-1}}^{\infty} s_n^{\omega_n} \Big\} y_2(x) = \prod_{k=3}^{\infty} \frac{1}{(1-s_{k,\infty})} \int_0^1 dt_2\; t_2^{3+\lambda} \int_0^1 du_2\; u_2^{2+\nu+\lambda} \\
&\quad \times \frac{1}{2\pi i} \oint dv_2\; \frac{\exp\left(-\frac{v_2}{(1-v_2)}\,\eta(1-t_2)(1-u_2)\right)}{(1-v_2)(v_2-s_{2,\infty})} w_{2,2}^{-(\Omega/\mu+2+\lambda)} (w_{2,2}\partial_{w_{2,2}}) w_{2,2}^{\Omega/\mu+2+\lambda} \\
&\quad \times \int_0^1 dt_1\; t_1^{1+\lambda} \int_0^1 du_1\; u_1^{\nu+\lambda} \frac{1}{2\pi i} \oint dv_1\; \frac{\exp\left(-\frac{v_1}{(1-v_1)}\, w_{2,2}(1-t_1)(1-u_1)\right)}{v_1(1-v_1)} \\
&\qquad \times \sum_{\omega_1=\omega_0}^{\infty} \left(\frac{s_{1,\infty}}{v_1 v_2}\right)^{\omega_1} w_{1,2}^{-(\Omega/\mu+\lambda)} (w_{1,2}\partial_{w_{1,2}}) w_{1,2}^{\Omega/\mu+\lambda} \\
&\qquad \times \sum_{\omega_0=0}^{\infty} \frac{s_0^{\omega_0}}{\omega_0!} \frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')} \left\{ c_0 x^{\lambda} \sum_{i_0=0}^{\omega_0} \frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}} w_{1,2}^{i_0} \right\} \rho^2. \qquad (2.44)
\end{aligned}
$$

By using Cauchy's integral formula, the contour integrand has poles at $v_2 = 1$ or $s_{2,\infty}$, where $s_{2,\infty}$ is only inside the unit circle. As we compute the residue in (2.44), we obtain

$$
\begin{aligned}
&\sum_{\omega_0=0}^{\infty} \frac{s_0^{\omega_0}}{\omega_0!} \frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')} \prod_{n=1}^{\infty} \Big\{ \sum_{\omega_n=\omega_{n-1}}^{\infty} s_n^{\omega_n} \Big\} y_2(x) = \prod_{k=2}^{\infty} \frac{1}{(1-s_{k,\infty})} \int_0^1 dt_2\; t_2^{3+\lambda} \int_0^1 du_2\; u_2^{2+\nu+\lambda} \\
&\quad \times \exp\left(-\frac{s_{2,\infty}}{(1-s_{2,\infty})}\,\eta(1-t_2)(1-u_2)\right) \widetilde{w}_{2,2}^{-(\Omega/\mu+2+\lambda)} (\widetilde{w}_{2,2}\partial_{\widetilde{w}_{2,2}}) \widetilde{w}_{2,2}^{\Omega/\mu+2+\lambda} \\
&\quad \times \int_0^1 dt_1\; t_1^{1+\lambda} \int_0^1 du_1\; u_1^{\nu+\lambda} \frac{1}{2\pi i} \oint dv_1\; \frac{\exp\left(-\frac{v_1}{(1-v_1)}\, \widetilde{w}_{2,2}(1-t_1)(1-u_1)\right)}{v_1(1-v_1)} \\
&\qquad \times \sum_{\omega_1=\omega_0}^{\infty} \left(\frac{s_1}{v_1}\right)^{\omega_1} \ddot{w}_{1,2}^{-(\Omega/\mu+\lambda)} (\ddot{w}_{1,2}\partial_{\ddot{w}_{1,2}}) \ddot{w}_{1,2}^{\Omega/\mu+\lambda} \\
&\qquad \times \sum_{\omega_0=0}^{\infty} \frac{s_0^{\omega_0}}{\omega_0!} \frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')} \left\{ c_0 x^{\lambda} \sum_{i_0=0}^{\omega_0} \frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}}\, \ddot{w}_{1,2}^{i_0} \right\} \rho^2, \qquad (2.45)
\end{aligned}
$$

where

$$
\widetilde{w}_{2,2} = \eta s_{2,\infty} \prod_{l=2}^{2} t_l u_l, \quad \ddot{w}_{1,2} = \eta s_{2,\infty} v_1 \prod_{l=1}^{2} t_l u_l.
$$

Replace in (2.36) ω_i, ω_j and s_i, respectively, by ω_1, ω_0 and $\frac{s_1}{v_1}$ and insert the result into (2.45). We have

$$\begin{aligned}
&\sum_{\omega_0=0}^{\infty}\frac{s_0^{\omega_0}}{\omega_0!}\frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')}\prod_{n=1}^{\infty}\Big\{\sum_{\omega_n=\omega_{n-1}}^{\infty}s_n^{\omega_n}\Big\}y_2(x)=\prod_{k=2}^{\infty}\frac{1}{(1-s_{k,\infty})}\int_0^1 dt_2\ t_2^{3+\lambda}\int_0^1 du_2\ u_2^{2+\nu+\lambda}\\
&\times\exp\Big(-\frac{s_{2,\infty}}{(1-s_{2,\infty})}\eta(1-t_2)(1-u_2)\Big)\widetilde{w}_{2,2}^{-(\Omega/\mu+2+\lambda)}(\widetilde{w}_{2,2}\partial_{\widetilde{w}_{2,2}})\widetilde{w}_{2,2}^{\Omega/\mu+2+\lambda}\int_0^1 dt_1\ t_1^{1+\lambda}\int_0^1 du_1\ u_1^{\nu+\lambda}\\
&\times\frac{1}{2\pi i}\oint dv_1\ \frac{\exp\left(-\frac{v_1}{(1-v_1)}\widetilde{w}_{2,2}(1-t_1)(1-u_1)\right)}{(1-v_1)(v_1-s_1)}\ddot{w}_{1,2}^{-(\Omega/\mu+\lambda)}(\ddot{w}_{1,2}\partial_{\ddot{w}_{1,2}})\ddot{w}_{1,2}^{\Omega/\mu+\lambda}\\
&\times\sum_{\omega_0=0}^{\infty}\frac{1}{\omega_0!}\Big(\frac{s_{0,1}}{v_1}\Big)^{\omega_0}\frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')}\left\{c_0x^{\lambda}\sum_{i_0=0}^{\omega_0}\frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}}\ddot{w}_{1,2}^{i_0}\right\}\rho^2. \qquad (2.46)
\end{aligned}$$

By using Cauchy's integral formula, the contour integrand has poles at $v_1 = 1$ or s_1, where s_1 is only inside the unit circle. As we compute the residue in (2.46), we obtain

$$\begin{aligned}
&\sum_{\omega_0=0}^{\infty}\frac{s_0^{\omega_0}}{\omega_0!}\frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')}\prod_{n=1}^{\infty}\Big\{\sum_{\omega_n=\omega_{n-1}}^{\infty}s_n^{\omega_n}\Big\}y_2(x)=\prod_{k=2}^{\infty}\frac{1}{(1-s_{k,\infty})}\int_0^1 dt_2\ t_2^{3+\lambda}\int_0^1 du_2\ u_2^{2+\nu+\lambda}\\
&\times\exp\Big(-\frac{s_{2,\infty}}{(1-s_{2,\infty})}\eta(1-t_2)(1-u_2)\Big)\widetilde{w}_{2,2}^{-(\Omega/\mu+2+\lambda)}(\widetilde{w}_{2,2}\partial_{\widetilde{w}_{2,2}})\widetilde{w}_{2,2}^{\Omega/\mu+2+\lambda}\\
&\times\int_0^1 dt_1\ t_1^{1+\lambda}\int_0^1 du_1\ u_1^{\nu+\lambda}\frac{\exp\left(-\frac{s_1}{(1-s_1)}\widetilde{w}_{2,2}(1-t_1)(1-u_1)\right)}{(1-s_1)}\widetilde{w}_{1,2}^{-(\Omega/\mu+\lambda)}(\widetilde{w}_{1,2}\partial_{\widetilde{w}_{1,2}})\widetilde{w}_{1,2}^{\Omega/\mu+\lambda}\\
&\times\sum_{\omega_0=0}^{\infty}\frac{s_0^{\omega_0}}{\omega_0!}\frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')}\left\{c_0x^{\lambda}\sum_{i_0=0}^{\omega_0}\frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}}\widetilde{w}_{1,2}^{i_0}\right\}\rho^2, \qquad (2.47)
\end{aligned}$$

where

$$\widetilde{w}_{1,2}=\eta s_{1,\infty}\prod_{l=1}^{2}t_l u_l.$$

Applying the summation operator $\sum_{\omega_0=0}^{\infty}\frac{s_0^{\omega_0}}{\omega_0!}\frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')}\prod_{n=1}^{\infty}\Big\{\sum_{\omega_n=\omega_{n-1}}^{\infty}s_n^{\omega_n}\Big\}$ to (2.47) and using (2.35) and (2.36), we get

$$\begin{aligned}
&\sum_{\omega_0=0}^{\infty}\frac{s_0^{\omega_0}}{\omega_0!}\frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')}\prod_{n=1}^{\infty}\Big\{\sum_{\omega_n=\omega_{n-1}}^{\infty}s_n^{\omega_n}\Big\}y_3(x)=\prod_{k=3}^{\infty}\frac{1}{(1-s_{k,\infty})}\int_0^1 dt_3\ t_3^{5+\lambda}\int_0^1 du_3\ u_3^{4+\nu+\lambda}\\
&\times\exp\Big(-\frac{s_{3,\infty}}{(1-s_{3,\infty})}\eta(1-t_3)(1-u_3)\Big)\widetilde{w}_{3,3}^{-(\Omega/\mu+4+\lambda)}(\widetilde{w}_{3,3}\partial_{\widetilde{w}_{3,3}})\widetilde{w}_{3,3}^{\Omega/\mu+4+\lambda}\\
&\times\int_0^1 dt_2\ t_2^{3+\lambda}\int_0^1 du_2\ u_2^{2+\nu+\lambda}\frac{\exp\left(-\frac{s_2}{(1-s_2)}\widetilde{w}_{3,3}(1-t_2)(1-u_2)\right)}{(1-s_2)}\widetilde{w}_{2,3}^{-(\Omega/\mu+2+\lambda)}(\widetilde{w}_{2,3}\partial_{\widetilde{w}_{2,3}})\widetilde{w}_{2,3}^{\Omega/\mu+2+\lambda}\\
&\times\int_0^1 dt_1\ t_1^{1+\lambda}\int_0^1 du_1\ u_1^{\nu+\lambda}\frac{\exp\left(-\frac{s_1}{(1-s_1)}\widetilde{w}_{2,3}(1-t_1)(1-u_1)\right)}{(1-s_1)}\widetilde{w}_{1,3}^{-(\Omega/\mu+\lambda)}(\widetilde{w}_{1,3}\partial_{\widetilde{w}_{1,3}})\widetilde{w}_{1,3}^{\Omega/\mu+\lambda}\\
&\times\sum_{\omega_0=0}^{\infty}\frac{s_0^{\omega_0}}{\omega_0!}\frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')}\left\{c_0x^{\lambda}\sum_{i_0=0}^{\omega_0}\frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}}\widetilde{w}_{1,3}^{i_0}\right\}\rho^3, \qquad (2.48)
\end{aligned}$$

where

$$\widetilde{w}_{3,3} = \eta s_{3,\infty} \prod_{l=3}^{3} t_l u_l, \quad \widetilde{w}_{2,3} = \eta s_{2,\infty} \prod_{l=2}^{3} t_l u_l, \quad \widetilde{w}_{1,3} = \eta s_{1,\infty} \prod_{l=1}^{3} t_l u_l.$$

By repeating the above process for all integral forms of higher sub-summation terms $y_m(x)$, $m > 3$, we obtain every term $\sum_{\omega_0=0}^{\infty} \frac{s_0^{\omega_0}}{\omega_0!} \frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')} \prod_{n=1}^{\infty} \left\{ \sum_{\omega_n=\omega_{n-1}}^{\infty} s_n^{\omega_n} \right\} y_m(x)$. If into (2.38) along with (2.39), (2.42), (2.47), (2.48) we substitute all such terms, we obtain (2.37). □

Remark 2.13. The generating function for the first kind GCH polynomial of type 2 about $x = 0$ as $\omega = -(\omega_j + 2j)$, where $j, \omega_j = 0, 1, 2, \dots$, is

$$\begin{aligned}
&\sum_{\omega_0=0}^{\infty} \frac{s_0^{\omega_0}}{\omega_0!} \frac{\Gamma(\omega_0+\nu)}{\Gamma(\nu)} \prod_{n=1}^{\infty} \left\{ \sum_{\omega_n=\omega_{n-1}}^{\infty} s_n^{\omega_n} \right\} QW^R_{\omega_j}(\mu,\varepsilon,\nu,\Omega,\omega;\rho,\eta) = \prod_{k=1}^{\infty} \frac{1}{(1-s_{k,\infty})} \mathbf{A}(s_{0,\infty};\eta) \\
&+ \left\{ \prod_{k=1}^{\infty} \frac{1}{(1-s_{k,\infty})} \int_0^1 dt_1\; t_1 \int_0^1 du_1\; u_1^{\nu} \overleftrightarrow{\mathbf{\Gamma}}_1(s_{1,\infty};t_1,u_1,\eta) \widetilde{w}_{1,1}^{-\Omega/\mu} (\widetilde{w}_{1,1}\partial_{\widetilde{w}_{1,1}}) \widetilde{w}_{1,1}^{\Omega/\mu} \mathbf{A}(s_0;\widetilde{w}_{1,1}) \right\} \rho \\
&+ \sum_{n=2}^{\infty} \left\{ \prod_{k=n}^{\infty} \frac{1}{(1-s_{k,\infty})} \int_0^1 dt_n\; t_n^{2n-1} \int_0^1 du_n\; u_n^{2(n-1)+\nu} \overleftrightarrow{\mathbf{\Gamma}}_n(s_{n,\infty};t_n,u_n,\eta) \right. \\
&\qquad \times \widetilde{w}_{n,n}^{-(\Omega/\mu+2(n-1))} (\widetilde{w}_{n,n}\partial_{\widetilde{w}_{n,n}}) \widetilde{w}_{n,n}^{\Omega/\mu+2(n-1)} \\
&\times \prod_{j=1}^{n-1} \left\{ \int_0^1 dt_{n-j}\; t_{n-j}^{2(n-j)-1} \int_0^1 du_{n-j}\; u_{n-j}^{2(n-j-1)+\nu} \overleftrightarrow{\mathbf{\Gamma}}_{n-j}(s_{n-j};t_{n-j},u_{n-j},\widetilde{w}_{n-j+1,n}) \right. \\
&\qquad \left.\left. \times \widetilde{w}_{n-j,n}^{-(\Omega/\mu+2(n-j-1))} (\widetilde{w}_{n-j,n}\partial_{\widetilde{w}_{n-j,n}}) \widetilde{w}_{n-j,n}^{\Omega/\mu+2(n-j-1)} \right\} \mathbf{A}(s_0;\widetilde{w}_{1,n}) \right\} \rho^n,
\end{aligned} \tag{2.49}$$

where

$$\begin{cases}
\omega = -(\omega_j + 2j), \quad \rho = -\mu x^2, \quad \eta = -\varepsilon x; \\
\overleftrightarrow{\mathbf{\Gamma}}_1(s_{1,\infty};t_1,u_1,\eta) = \exp\Big(-\frac{s_{1,\infty}}{(1-s_{1,\infty})}\, \eta(1-t_1)(1-u_1) \Big); \\
\overleftrightarrow{\mathbf{\Gamma}}_n(s_{n,\infty};t_n,u_n,\eta) = \exp\Big(-\frac{s_{n,\infty}}{(1-s_{n,\infty})}\, \eta(1-t_n)(1-u_n) \Big); \\
\overleftrightarrow{\mathbf{\Gamma}}_{n-j}(s_{n-j};t_{n-j},u_{n-j},\widetilde{w}_{n-j+1,n}) = \dfrac{\exp\Big(-\frac{s_{n-j}}{(1-s_{n-j})}\, \widetilde{w}_{n-j+1,n}(1-t_{n-j})(1-u_{n-j}) \Big)}{(1-s_{n-j})}
\end{cases}$$

and

$$\begin{cases}
\mathbf{A}(s_{0,\infty};\eta) = (1-s_{0,\infty})^{-\nu} \exp\Big(-\frac{\eta s_{0,\infty}}{(1-s_{0,\infty})} \Big), \\
\mathbf{A}(s_0;\widetilde{w}_{1,1}) = (1-s_0)^{-\nu} \exp\Big(-\frac{\widetilde{w}_{1,1} s_0}{(1-s_0)} \Big), \\
\mathbf{A}(s_0;\widetilde{w}_{1,n}) = (1-s_0)^{-\nu} \exp\Big(-\frac{\widetilde{w}_{1,n} s_0}{(1-s_0)} \Big).
\end{cases}$$

Proof. The generating function for a confluent first kind Hypergeometric polynomial is given by

$$\sum_{\omega_0=0}^{\infty} \frac{t^{\omega_0}}{\omega_0!} \frac{\Gamma(\omega_0+\gamma)}{\Gamma(\gamma)}\, {}_1F_1(-\omega_0;\gamma;z) = (1-t)^{-\gamma} \exp\Big(-\frac{zt}{(1-t)} \Big). \tag{2.50}$$

Replacing t, γ and z, respectively, by $s_{0,\infty}$, ν and η in (2.50), we get

$$\sum_{\omega_0=0}^{\infty} \frac{s_{0,\infty}^{\omega_0}}{\omega_0!} \frac{\Gamma(\omega_0+\nu)}{\Gamma(\nu)} \, {}_1F_1(-\omega_0;\nu;\eta) = (1-s_{0,\infty})^{-\nu} \exp\Big(-\frac{\eta s_{0,\infty}}{(1-s_{0,\infty})}\Big). \tag{2.51}$$

Replacing t, γ and z, respectively, by s_0, ν and $\widetilde{w}_{1,1}$ in (2.50), we get

$$\sum_{\omega_0=0}^{\infty} \frac{s_0^{\omega_0}}{\omega_0!} \frac{\Gamma(\omega_0+\nu)}{\Gamma(\nu)} \, {}_1F_1(-\omega_0;\nu;\widetilde{w}_{1,1}) = (1-s_0)^{-\nu} \exp\Big(-\frac{\widetilde{w}_{1,1} s_0}{(1-s_0)}\Big). \tag{2.52}$$

Replacing t, γ and z, respectively, by s_0, ν and $\widetilde{w}_{1,n}$ in (2.50), we get

$$\sum_{\omega_0=0}^{\infty} \frac{s_0^{\omega_0}}{\omega_0!} \frac{\Gamma(\omega_0+\nu)}{\Gamma(\nu)} \, {}_1F_1(-\omega_0;\nu;\widetilde{w}_{1,n}) = (1-s_0)^{-\nu} \exp\Big(-\frac{\widetilde{w}_{1,n} s_0}{(1-s_0)}\Big). \tag{2.53}$$

Taking $c_0 = 1$, λ=0 and $\gamma' = \nu$ in (2.37) and substituting (2.51), (2.52) and (2.53) into the obtained equality, we get the desired result. □

Remark 2.14. The generating function for the second kind GCH polynomial of type 2 about $x = 0$ as $\omega = -(\omega_j + 2j + 1 - \nu)$, where $j, \omega_j = 0, 1, 2, \dots$, is

$$\begin{aligned}
&\sum_{\omega_0=0}^{\infty} \frac{s_0^{\omega_0}}{\omega_0!} \frac{\Gamma(\omega_0+2-\nu)}{\Gamma(2-\nu)} \prod_{n=1}^{\infty} \Big\{ \sum_{\omega_n=\omega_{n-1}}^{\infty} s_n^{\omega_n} \Big\} RW^R_{\omega_j}(\mu,\varepsilon,\nu,\Omega,\omega;\rho,\eta) \\
&= x^{1-\nu} \Bigg\{ \prod_{k=1}^{\infty} \frac{1}{(1-s_{k,\infty})} \mathbf{B}(s_{0,\infty};\eta) + \Bigg\{ \prod_{k=1}^{\infty} \frac{1}{(1-s_{k,\infty})} \int_0^1 dt_1\, t_1^{2-\nu} \int_0^1 du_1\, u_1 \overleftrightarrow{\mathbf{\Gamma}}_1(s_{1,\infty};t_1,u_1,\eta) \\
&\qquad \times \widetilde{w}_{1,1}^{-(\Omega/\mu+1-\nu)} (\widetilde{w}_{1,1}\partial_{\widetilde{w}_{1,1}}) \widetilde{w}_{1,1}^{\Omega/\mu+1-\nu} \mathbf{B}(s_0;\widetilde{w}_{1,1}) \Bigg\} \rho \\
&\quad + \sum_{n=2}^{\infty} \Bigg\{ \prod_{k=n}^{\infty} \frac{1}{(1-s_{k,\infty})} \int_0^1 dt_n\, t_n^{2n-\nu} \int_0^1 du_n\, u_n^{2n-1} \overleftrightarrow{\mathbf{\Gamma}}_n(s_{n,\infty};t_n,u_n,\eta) \\
&\qquad \times \widetilde{w}_{n,n}^{-(\Omega/\mu+2n-1-\nu)} (\widetilde{w}_{n,n}\partial_{\widetilde{w}_{n,n}}) \widetilde{w}_{n,n}^{\Omega/\mu+2n-1-\nu} \\
&\quad \times \prod_{j=1}^{n-1} \Bigg\{ \int_0^1 dt_{n-j}\, t_{n-j}^{2(n-j)-\nu} \int_0^1 du_{n-j}\, u_{n-j}^{2(n-j)-1} \overleftrightarrow{\mathbf{\Gamma}}_{n-j}(s_{n-j};t_{n-j},u_{n-j},\widetilde{w}_{n-j+1,n}) \\
&\qquad \times \widetilde{w}_{n-j,n}^{-(\Omega/\mu+2(n-j)-1-\nu)} (\widetilde{w}_{n-j,n}\partial_{\widetilde{w}_{n-j,n}}) \widetilde{w}_{n-j,n}^{\Omega/\mu+2(n-j)-1-\nu} \Bigg\} \mathbf{B}(s_0;\widetilde{w}_{1,n}) \Bigg\} \rho^n \Bigg\},
\end{aligned} \tag{2.54}$$

where

$$\begin{cases}
\omega = -(\omega_j+2j+1-\nu); \quad \rho = -\mu x^2, \quad \eta = -\varepsilon x; \\
\overleftrightarrow{\mathbf{\Gamma}}_1(s_{1,\infty};t_1,u_1,\eta) = \exp\Big(-\frac{s_{1,\infty}}{(1-s_{1,\infty})} \eta(1-t_1)(1-u_1)\Big); \\
\overleftrightarrow{\mathbf{\Gamma}}_n(s_{n,\infty};t_n,u_n,\eta) = \exp\Big(-\frac{s_{n,\infty}}{(1-s_{n,\infty})} \eta(1-t_n)(1-u_n)\Big); \\
\overleftrightarrow{\mathbf{\Gamma}}_{n-j}(s_{n-j};t_{n-j},u_{n-j},\widetilde{w}_{n-j+1,n}) = \dfrac{\exp\big(-\frac{s_{n-j}}{(1-s_{n-j})} \widetilde{w}_{n-j+1,n}(1-t_{n-j})(1-u_{n-j})\big)}{(1-s_{n-j})}
\end{cases}$$

and

$$\begin{cases} \mathbf{B}(s_{0,\infty};\eta) = (1-s_{0,\infty})^{\nu-2}\exp\Big(-\dfrac{\eta s_{0,\infty}}{(1-s_{0,\infty})}\Big), \\ \mathbf{B}(s_0;\widetilde{w}_{1,1}) = (1-s_0)^{\nu-2}\exp\Big(-\dfrac{\widetilde{w}_{1,1}s_0}{(1-s_0)}\Big), \\ \mathbf{B}(s_0;\widetilde{w}_{1,n}) = (1-s_0)^{\nu-2}\exp\Big(-\dfrac{\widetilde{w}_{1,n}s_0}{(1-s_0)}\Big). \end{cases}$$

Proof. Replacing t, γ and z, respectively, by $s_{0,\infty}$, $2-\nu$ and η in (2.50), we get

$$\sum_{\omega_0=0}^{\infty}\frac{s_{0,\infty}^{\omega_0}}{\omega_0!}\frac{\Gamma(\omega_0+2-\nu)}{\Gamma(2-\nu)}\,{}_1F_1(-\omega_0;2-\nu;\eta) = (1-s_{0,\infty})^{\nu-2}\exp\Big(-\frac{\eta s_{0,\infty}}{(1-s_{0,\infty})}\Big). \tag{2.55}$$

Replacing t, γ and z, respectively, by s_0, $2-\nu$ and $\widetilde{w}_{1,1}$ in (2.50), we get

$$\sum_{\omega_0=0}^{\infty}\frac{s_0^{\omega_0}}{\omega_0!}\frac{\Gamma(\omega_0+2-\nu)}{\Gamma(2-\nu)}\,{}_1F_1(-\omega_0;2-\nu;\widetilde{w}_{1,1}) = (1-s_0)^{\nu-2}\exp\Big(-\frac{\widetilde{w}_{1,1}s_0}{(1-s_0)}\Big). \tag{2.56}$$

Replacing t, γ and z, respectively, by s_0, $2-\nu$ and $\widetilde{w}_{1,n}$ in (2.50), we get

$$\sum_{\omega_0=0}^{\infty}\frac{s_0^{\omega_0}}{\omega_0!}\frac{\Gamma(\omega_0+2-\nu)}{\Gamma(2-\nu)}\,{}_1F_1(-\omega_0;2-\nu;\widetilde{w}_{1,n}) = (1-s_0)^{\nu-2}\exp\Big(-\frac{\widetilde{w}_{1,n}s_0}{(1-s_0)}\Big). \tag{2.57}$$

Taking $c_0=1$, $\lambda=1-\nu$ and $\gamma'=2-\nu$ in (2.37) and substituting (2.55), (2.56) and (2.57) into the obtained equality, we get the desired result. □

3 GCH equation about an irregular singular point at infinity

Let $z=\frac{1}{x}$ in (1.1) in order to get an analytic solution of the GCH equation about $x=\infty$:

$$z^4\frac{d^2y}{dz^2} + ((2-\nu)z^3 - \varepsilon z^2 - \mu z)\frac{dy}{dz} + (\Omega + \varepsilon\omega z)y = 0. \tag{3.1}$$

Assume that its solution is

$$y(z) = \sum_{n=0}^{\infty} c_n z^{n+\lambda}, \tag{3.2}$$

where λ is indicial root. Substitute (3.2) into (3.1). For the coefficients c_n, we get the following three-term recurrence relation:

$$c_{n+1} = A_n c_n + B_n c_{n-1}, \quad n \geq 1, \tag{3.3}$$

where

$$A_n = -\frac{\varepsilon}{\mu}\,\frac{(n-\omega+\lambda)}{(n+1-\Omega/\mu+\lambda)}\,, \tag{3.4a}$$

$$B_n = \frac{1}{\mu}\,\frac{(n-1+\lambda)(n-\nu+\lambda)}{(n+1-\Omega/\mu+\lambda)}\,, \tag{3.4b}$$

$$c_1 = A_0 c_0. \tag{3.4c}$$

We have an indicial root $\lambda = \Omega/\mu$.

Now, let us test for the convergence of the analytic function $y(z)$. As $n\to\infty$, from (3.4a) and (3.4b), we get

$$\lim_{n\to\infty} A_n = -\frac{\varepsilon}{\mu}\,, \tag{3.5a}$$

$$\lim_{n\to\infty} B_n = \frac{n}{\mu} \to \infty. \tag{3.5b}$$

There are no analytic solutions for a polynomial of type 2 and infinite series. Since, by (3.5b), $y(z)$ is divergent as $n \to \infty$, there are only two types of analytic solutions of the GCH equation about $x = \infty$ such as polynomials of type 1 and of type 3. In Chapter 10 [14], the polynomial of type 3 about $x = \infty$ is derived: μ, ε, Ω are treated as free variables and ν, ω as fixed values. In this section, we have constructed the power series expansion, an integral form and the generating function for the GCH polynomial of type 1 about $x = \infty$: μ, ε, ω and Ω are treated as free variables and ν as a fixed value.

3.1 Power series for a polynomial of type 1

In [10], the general expression of a power series of $y(x)$ for a polynomial of type 1 is given by

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} y_n(x) = y_0(x) + y_1(x) + y_2(x) + y_3(x) + \cdots \\ &= c_0 x^{\lambda} \Bigg\{ \sum_{i_0=0}^{\beta_0} \Big(\prod_{i_1=0}^{i_0-1} B_{2i_1+1} \Big) x^{2i_0} + \sum_{i_0=0}^{\beta_0} \Big\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \sum_{i_2=i_0}^{\beta_1} \Big(\prod_{i_3=i_0}^{i_2-1} B_{2i_3+2} \Big) \Big\} x^{2i_2+1} \\ &\quad + \sum_{N=2}^{\infty} \Big\{ \sum_{i_0=0}^{\beta_0} \Big\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \prod_{k=1}^{N-1} \Big(\sum_{i_{2k}=i_{2(k-1)}}^{\beta_k} A_{2i_{2k}+k} \prod_{i_{2k+1}=i_{2(k-1)}}^{i_{2k}-1} B_{2i_{2k+1}+(k+1)} \Big) \\ &\quad \times \sum_{i_{2N}=i_{2(N-1)}}^{\beta_N} \Big(\prod_{i_{2N+1}=i_{2(N-1)}}^{i_{2N}-1} B_{2i_{2N+1}+(N+1)} \Big) \Big\} \Big\} x^{2i_{2N}+N} \Bigg\}. \end{aligned} \tag{3.6}$$

Here $\beta_i \le \beta_j$ only if $i \le j$, where $i, j, \beta_i, \beta_j \in \mathbb{N}_0$.

For a polynomial we need the following condition:

$$B_{2\beta_i+(i+1)} = 0, \quad \text{where } i = 0, 1, 2, \ldots, \quad \beta_i = 0, 1, 2, \ldots. \tag{3.7}$$

Here β_i is an eigenvalue that makes B_n term terminated at a certain value of the index n. (3.7) turns each $y_i(x)$, where $i = 0, 1, 2, \ldots$, into the polynomial in (3.6). Replace β_i by ν_i in (3.7) and put $n = 2\nu_i + (i+1)$ in (3.4b) with the condition $B_{2\nu_i+(i+1)} = 0$. Then we obtain eigenvalues ν of the form

$$\nu = 2\nu_i + i + 1 + \lambda.$$

In (3.4b), we replace ν by $2\nu_i + i + 1 + \lambda$, and insert the obtained result and (3.4a) into (3.6), where a variable x and an index β_i are, respectively, replaced by z and ν_i. Hence the general expression of a power series of the GCH equation for a polynomial of type 1 about $x = \infty$ is given by

$$\begin{aligned} y(z) &= \sum_{n=0}^{\infty} y_n(z) = y_0(z) + y_1(z) + y_2(z) + y_3(z) + \cdots \\ &= c_0 z^{\lambda} \Bigg\{ \sum_{i_0=0}^{\nu_0} \frac{(-\nu_0)_{i_0} (\frac{\lambda}{2})_{i_0}}{(1 - \frac{\Omega}{2\mu} + \frac{\lambda}{2})_{i_0}} \eta^{i_0} \\ &\quad + \Big\{ \sum_{i_0=0}^{\nu_0} \frac{(i_0 - \frac{\omega}{2} + \frac{\lambda}{2})}{(i_0 + \frac{1}{2} - \frac{\Omega}{2\mu} + \frac{\lambda}{2})} \frac{(-\nu_0)_{i_0} (\frac{\lambda}{2})_{i_0}}{(1 - \frac{\Omega}{2\mu} + \frac{\lambda}{2})_{i_0}} \sum_{i_1=i_0}^{\nu_1} \frac{(-\nu_1)_{i_1} (\frac{1}{2} + \frac{\lambda}{2})_{i_1} (\frac{3}{2} - \frac{\Omega}{2\mu} + \frac{\lambda}{2})_{i_0}}{(-\nu_1)_{i_0} (\frac{1}{2} + \frac{\lambda}{2})_{i_0} (\frac{3}{2} - \frac{\Omega}{2\mu} + \frac{\lambda}{2})_{i_1}} \eta^{i_1} \Big\} \xi \\ &\quad + \sum_{n=2}^{\infty} \Big\{ \sum_{i_0=0}^{\nu_0} \frac{(i_0 - \frac{\omega}{2} + \frac{\lambda}{2})}{(i_0 + \frac{1}{2} - \frac{\Omega}{2\mu} + \frac{\lambda}{2})} \frac{(-\nu_0)_{i_0} (\frac{\lambda}{2})_{i_0}}{(1 - \frac{\Omega}{2\mu} + \frac{\lambda}{2})_{i_0}} \\ &\quad \times \prod_{k=1}^{n-1} \Big\{ \sum_{i_k=i_{k-1}}^{\nu_k} \frac{(i_k + \frac{k}{2} - \frac{\omega}{2} + \frac{\lambda}{2})}{(i_k + \frac{k}{2} + \frac{1}{2} - \frac{\Omega}{2\mu} + \frac{\lambda}{2})} \frac{(-\nu_k)_{i_k} (\frac{k}{2} + \frac{\lambda}{2})_{i_k} (\frac{k}{2} + 1 - \frac{\Omega}{2\mu} + \frac{\lambda}{2})_{i_{k-1}}}{(-\nu_k)_{i_{k-1}} (\frac{k}{2} + \frac{\lambda}{2})_{i_{k-1}} (\frac{k}{2} + 1 - \frac{\Omega}{2\mu} + \frac{\lambda}{2})_{i_k}} \Big\} \\ &\quad \times \sum_{i_n=i_{n-1}}^{\nu_n} \frac{(-\nu_n)_{i_n} (\frac{n}{2} + \frac{\lambda}{2})_{i_n} (\frac{n}{2} + 1 - \frac{\Omega}{2\mu} + \frac{\lambda}{2})_{i_{n-1}}}{(-\nu_n)_{i_{n-1}} (\frac{n}{2} + \frac{\lambda}{2})_{i_{n-1}} (\frac{n}{2} + 1 - \frac{\Omega}{2\mu} + \frac{\lambda}{2})_{i_n}} \eta^{i_n} \Big\} \xi^n \Bigg\}, \end{aligned} \tag{3.8}$$

where

$$\begin{cases} \eta = \dfrac{2}{\mu}\, z^2, \\ \xi = -\dfrac{\varepsilon}{\mu}\, z, \\ \nu = 2\nu_j + j + 1 + \lambda, \\ z = \dfrac{1}{x}\,, \\ \nu_i \le \nu_j \;\text{ only if }\; i \le j, \;\text{ where }\; i, j, \nu_i, \nu_j \in \mathbb{N}_0 \ldots . \end{cases}$$

Put c_0= 1, as $\lambda = \Omega/\mu$ in (3.8).

Remark 3.1. The power series expansion of the first kind GCH equation for a polynomial of type 1 about $x = \infty$, as $\nu = 2\nu_j + j + 1 + \Omega/\mu$, where $j, \nu_j \in \mathbb{N}_0$, is

$$\begin{aligned} y(z) &= Q^{(i)} W_{\nu_j}\Big(\mu, \varepsilon, \Omega, \omega, \nu = 2\nu_j + j + 1 + \frac{\Omega}{\mu}; z = \frac{1}{x}, \xi = -\frac{\varepsilon}{\mu}\, z, \eta = \frac{2}{\mu}\, z^2\Big) \\ &= z^{\frac{\Omega}{\mu}} \Bigg\{ \sum_{i_0=0}^{\nu_0} \frac{(-\nu_0)_{i_0} (\frac{\Omega}{2\mu})_{i_0}}{(1)_{i_0}}\, \eta^{i_0} \\ &\quad + \Bigg\{ \sum_{i_0=0}^{\nu_0} \frac{(i_0 - \frac{\omega}{2} + \frac{\Omega}{2\mu})}{(i_0 + \frac{1}{2})} \frac{(-\nu_0)_{i_0} (\frac{\Omega}{2\mu})_{i_0}}{(1)_{i_0}} \sum_{i_1=i_0}^{\nu_1} \frac{(-\nu_1)_{i_1} (\frac{1}{2} + \frac{\Omega}{2\mu})_{i_1} (\frac{3}{2})_{i_0}}{(-\nu_1)_{i_0} (\frac{1}{2} + \frac{\Omega}{2\mu})_{i_0} (\frac{3}{2})_{i_1}}\, \eta^{i_1} \Bigg\} \xi \\ &\quad + \sum_{n=2}^{\infty} \Bigg\{ \sum_{i_0=0}^{\nu_0} \frac{(i_0 - \frac{\omega}{2} + \frac{\Omega}{2\mu})}{(i_0 + \frac{1}{2})} \frac{(-\nu_0)_{i_0} (\frac{\Omega}{2\mu})_{i_0}}{(1)_{i_0}} \\ &\qquad \times \prod_{k=1}^{n-1} \Bigg\{ \sum_{i_k=i_{k-1}}^{\nu_k} \frac{(i_k + \frac{k}{2} - \frac{\omega}{2} + \frac{\Omega}{2\mu})}{(i_k + \frac{k}{2} + \frac{1}{2})} \frac{(-\nu_k)_{i_k} (\frac{k}{2} + \frac{\Omega}{2\mu})_{i_k} (\frac{k}{2} + 1)_{i_{k-1}}}{(-\nu_k)_{i_{k-1}} (\frac{k}{2} + \frac{\Omega}{2\mu})_{i_{k-1}} (\frac{k}{2} + 1)_{i_k}} \Bigg\} \\ &\qquad \times \sum_{i_n=i_{n-1}}^{\nu_n} \frac{(-\nu_n)_{i_n} (\frac{n}{2} + \frac{\Omega}{2\mu})_{i_n} (\frac{n}{2} + 1)_{i_{n-1}}}{(-\nu_n)_{i_{n-1}} (\frac{n}{2} + \frac{\Omega}{2\mu})_{i_{n-1}} (\frac{n}{2} + 1)_{i_n}}\, \eta^{i_n} \Bigg\} \xi^n \Bigg\}. \end{aligned} \tag{3.9}$$

For the minimum value of the first kind GCH equation for a polynomial of type 1 about $x = \infty$, in (3.9) we set $\nu_0 = \nu_1 = \nu_2 = \cdots = 0$ and get

$$\begin{aligned} y(z) &= Q^{(i)} W_0\Big(\mu, \varepsilon, \Omega, \omega, \nu = j + 1 + \frac{\Omega}{\mu}; z = \frac{1}{x}, \xi = -\frac{\varepsilon}{\mu}\, z, \eta = \frac{2}{\mu}\, z^2\Big) \\ &= z^{\frac{\Omega}{\mu}} \sum_{n=0}^{\infty} \frac{(\Omega/\mu - \omega)_n}{n!}\, \xi^n = z^{\frac{\Omega}{\mu}} \Big(1 + \frac{\varepsilon}{\mu}\, z\Big)^{-(\frac{\Omega}{\mu} - \omega)}. \end{aligned}$$

From the above it follows that a polynomial of type 1 requires $|\frac{\varepsilon}{\mu}\, z| < 1$ for the convergence of the radius.

3.2 Integral representation for a polynomial of type 1

There is a generalized hypergeometric function such that

$$\begin{aligned} L_l &= \sum_{i_l=i_{l-1}}^{\nu_l} \frac{(-\nu_l)_{i_l} (\frac{l}{2} + \frac{\lambda}{2})_{i_l} (\frac{l}{2} + 1 - \frac{\Omega}{2\mu} + \frac{\lambda}{2})_{i_{l-1}}}{(-\nu_l)_{i_{l-1}} (\frac{l}{2} + \frac{\lambda}{2})_{i_{l-1}} (\frac{l}{2} + 1 - \frac{\Omega}{2\mu} + \frac{\lambda}{2})_{i_l}}\, \eta^{i_l} \\ &= \eta^{i_{l-1}} \sum_{j=0}^{\infty} \frac{A_j (i_{l-1} - \nu_l)_j (i_{l-1} + \frac{l}{2} + \frac{\lambda}{2})_j}{(i_{l-1} + \frac{l}{2} - \frac{\Omega}{2\mu} + \frac{\lambda}{2})^{-1} (1)_j}\, \eta^j, \end{aligned} \tag{3.10}$$

where

$$A_j = B\Big(i_{l-1} + \frac{l}{2} - \frac{\Omega}{2\mu} + \frac{\lambda}{2}, j + 1\Big).$$

By using integral form of the beta function, we have

$$B\Big(i_{l-1}+\frac{l}{2}-\frac{\Omega}{2\mu}+\frac{\lambda}{2},j+1\Big)=\int_0^1 dt_l\; t_l^{i_{l-1}+\frac{l}{2}-1-\frac{\Omega}{2\mu}+\frac{\lambda}{2}}(1-t_l)^j. \tag{3.11}$$

Substitute (3.11) into (3.10), and divide L_l by $(i_{l-1}+\frac{l}{2}-\frac{\Omega}{2\mu}+\frac{\lambda}{2})$. We obtain

$$\begin{aligned} G_l &= \frac{1}{(i_{l-1}+\frac{l}{2}-\frac{\Omega}{2\mu}+\frac{\lambda}{2})}\sum_{i_l=i_{l-1}}^{\nu_l}\frac{(-\nu_l)_{i_l}(\frac{l}{2}+\frac{\lambda}{2})_{i_l}(\frac{l}{2}+1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_{l-1}}}{(-\nu_l)_{i_{l-1}}(\frac{l}{2}+\frac{\lambda}{2})_{i_{l-1}}(\frac{l}{2}+1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_l}}\,\eta^{i_l} \\ &= \int_0^1 dt_l\; t_l^{\frac{l}{2}-1-\frac{\Omega}{2\mu}+\frac{\lambda}{2}}(\eta t_l)^{i_{l-1}}\sum_{j=0}^{\infty}\frac{(i_{l-1}-\nu_l)_j(i_{l-1}+\frac{l}{2}+\frac{\lambda}{2})_j}{(1)_j}\,(\eta(1-t_l))^j. \end{aligned} \tag{3.12}$$

Tricomi's function is defined by

$$U(a,b,z)=\frac{\Gamma(1-b)}{\Gamma(a-b+1)}\,M(a,b,z)+\frac{\Gamma(b-1)}{\Gamma(a)}\,z^{1-b}M(a-b+1,2-b,z). \tag{3.13}$$

The contour integral form of (3.13) is given by (see [33])

$$U(a,b,z)=e^{-a\pi i}\,\frac{\Gamma(1-a)}{2\pi i}\int_{\infty}^{(0+)} dp_l\; e^{-zp_l}p_l^{a-1}(1+p_l)^{b-a-1}, \tag{3.14}$$

where

$$a\neq 1,2,3,\dots,\quad |\mathrm{ph}\, z|<\frac{1}{2}\,\pi.$$

Also (3.13) is written as (see [33])

$$U(a,b,z)=z^{-a}\sum_{j=0}^{\infty}\frac{(a)_j(a-b+1)_j}{(1)_j}\,(-z^{-1})^j=z^{-a}\;{}_2F_0(a,a-b+1;-;-z^{-1}). \tag{3.15}$$

Replace a, b and z in (3.15), respectively, by $i_{l-1}-\nu_l$, $-\nu_l+1-\frac{l}{2}-\frac{\lambda}{2}$ and $\frac{-1}{\eta(1-t_l)}$. We get

$$\begin{aligned} &\sum_{j=0}^{\infty}\frac{(i_{l-1}-\nu_l)_j(i_{l-1}+\frac{l}{2}+\frac{\lambda}{2})_j}{(1)_j}\,(\eta(1-t_l))^j \\ &\qquad =\Big(\frac{-1}{\eta(1-t_l)}\Big)^{i_{l-1}-\nu_l}U\Big(i_{l-1}-\nu_l,-\nu_l+1-\frac{l}{2}-\frac{\lambda}{2},\varpi\Big), \end{aligned} \tag{3.16}$$

where

$$\varpi=\frac{-1}{\eta(1-t_l)}\,.$$

Replace a, b and z in (3.14), respectively, by $i_{l-1}-\nu_l$, $-\nu_l+1-\frac{l}{2}-\frac{\lambda}{2}$ and $\frac{-1}{\eta(1-t_l)}$ and insert the result into (3.16). We obtain

$$\begin{aligned} &\sum_{j=0}^{\infty}\frac{(i_{l-1}-\nu_l)_j(i_{l-1}+\frac{l}{2}+\frac{\lambda}{2})_j}{(1)_j}\,(\eta(1-t_l))^j=\frac{\Gamma(\nu_l-i_{l-1}+1)}{2\pi i} \\ &\quad\times\int_{\infty}^{(0+)} dp_l\;\exp\Big(\frac{p_l}{\eta(1-t_l)}\Big)p_l^{-1}(1+p_l)^{-\frac{1}{2}(l+\lambda)}\Big(\frac{\eta(1-t_l)}{p_l}\Big)^{\nu_l}\Big(\frac{p_l}{\eta(1-t_l)(1+p_l)}\Big)^{i_{l-1}}. \end{aligned} \tag{3.17}$$

The Gamma function $\Gamma(z)$ is defined as follows:

$$\Gamma(z) = \int_0^\infty du_l\; e^{-u_l} u_l^{z-1}, \quad \text{where } \operatorname{Re}(z) > 0. \tag{3.18}$$

Put $z = \nu_l - i_{l-1} + 1$ in (3.18). We have

$$\Gamma(\nu_l - i_{l-1} + 1) = \int_0^\infty du_l\; e^{-u_l} u_l^{\nu_l - i_{l-1}}. \tag{3.19}$$

Substitute (3.19) in (3.17) and insert the result into (3.12). We get

$$\begin{aligned} G_l &= \frac{1}{(i_{l-1} + \frac{l}{2} - \frac{\Omega}{2\mu} + \frac{\lambda}{2})} \sum_{i_l = i_{l-1}}^{\nu_l} \frac{(-\nu_l)_{i_l} (\frac{l}{2} + \frac{\lambda}{2})_{i_l} (\frac{l}{2} + 1 - \frac{\Omega}{2\mu} + \frac{\lambda}{2})_{i_{l-1}}}{(-\nu_l)_{i_{l-1}} (\frac{l}{2} + \frac{\lambda}{2})_{i_{l-1}} (\frac{l}{2} + 1 - \frac{\Omega}{2\mu} + \frac{\lambda}{2})_{i_l}} \eta^{i_l} \\ &= \int_0^1 dt_l\; t_l^{\frac{1}{2}(l - 2 - \frac{\Omega}{\mu} + \lambda)} \int_0^\infty du_l\; \frac{1}{2\pi i} \\ &\quad \times \int_\infty^{(0+)} dp_l\; \exp\left(\frac{p_l}{\eta(1 - t_l)} \right) p_l^{-1} (1 + p_l)^{-\frac{1}{2}(l + \lambda)} \left(\frac{\eta u_l (1 - t_l)}{p_l} \right)^{\nu_l} \left(\frac{t_l p_l}{u_l (1 - t_l)(1 + p_l)} \right)^{i_{l-1}}. \end{aligned} \tag{3.20}$$

Substitute (3.20) into (3.8), where $l = 1, 2, 3, \ldots$: apply G_1 into the second summation of the sub-power series $y_1(z)$; apply G_2 into the third summation and G_1 into the second summation of the sub-power series $y_2(z)$; apply G_3 into the forth summation, G_2 into the third summation and G_1 into the second summation of the sub-power series $y_3(z)$, etc.*

Theorem 3.2. *The general representation in the form of an integral of the GCH polynomial of type* 1 *about $x = \infty$ is given by*

$$\begin{aligned} y(z) &= \sum_{n=0}^\infty y_n(z) = y_0(z) + y_1(z) + y_2(z) + y_3(z) + \cdots \\ &= c_0 z^\lambda \Bigg\{ \sum_{i_0=0}^{\nu_0} \frac{(-\nu_0)_{i_0} (\frac{\lambda}{2})_{i_0}}{(1 - \frac{\Omega}{2\mu} + \frac{\lambda}{2})_{i_0}} \eta^{i_0} + \sum_{n=1}^\infty \Bigg\{ \prod_{k=0}^{n-1} \Bigg\{ \int_0^1 dt_{n-k}\; t_{n-k}^{\frac{1}{2}(n-k-2-\frac{\Omega}{\mu}+\lambda)} \int_0^\infty du_{n-k}\; e^{-u_{n-k}} \\ &\quad \times \frac{1}{2\pi i} \int_\infty^{(0+)} dp_{n-k}\; p_{n-k}^{-1} (1 + p_{n-k})^{-\frac{1}{2}(n-k+\lambda)} \\ &\quad \times \exp\left(\frac{p_{n-k}}{w_{n-k+1,n}(1 - t_{n-k})} \right) \left(\frac{w_{n-k+1,n} u_{n-k} (1 - t_{n-k})}{p_{n-k}} \right)^{\nu_{n-k}} \\ &\quad \times w_{n-k,n}^{-\frac{1}{2}(n-k-1-\omega+\lambda)} (w_{n-k,n} \partial_{w_{n-k,n}}) w_{n-k,n}^{\frac{1}{2}(n-k-1-\omega+\lambda)} \Bigg\} \sum_{i_0=0}^{\nu_0} \frac{(-\nu_0)_{i_0} (\frac{\lambda}{2})_{i_0}}{(1 - \frac{\Omega}{2\mu} + \frac{\lambda}{2})_{i_0}} w_{1,n}^{i_0} \Bigg\} \xi^n \Bigg\}, \end{aligned} \tag{3.21}$$

where

$$w_{i,j} = \begin{cases} \dfrac{t_i p_i}{u_i (1 - t_i)(1 + p_i)}, & \text{where } i \le j, \\ \eta & \text{only if } i > j. \end{cases}$$

Here the first sub-integral form contains one term of A_n's, the second one contains two terms of A_n's, the third one contains three terms of A_n's, etc.

*$y_1(z)$ means the sub-power series in (3.8), contains one term of A_n's; $y_2(z)$ means the sub-power series in (3.8), contains two terms of A_n's; $y_3(z)$ means the sub-power series in (3.8), contains three terms of A_n's, etc.

Proof. In (3.8), the power series expansions of the sub-summation terms $y_0(z)$, $y_1(z)$, $y_2(z)$ and $y_3(z)$ of the GCH polynomial of type 1 about $x=\infty$ are

$$y(z)=\sum_{n=0}^{\infty} y_n(z) = y_0(z)+y_1(z)+y_2(z)+y_3(z)+\cdots\,, \tag{3.22}$$

where

$$y_0(z)=\sum_{i_0=0}^{\nu_0}\frac{(-\nu_0)_{i_0}(\frac{\lambda}{2})_{i_0}}{(1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}}\,\eta^{i_0}, \tag{3.23a}$$

$$\begin{aligned} y_1(z)=c_0z^{\lambda}\Bigg\{&\sum_{i_0=0}^{\nu_0}\frac{(i_0-\frac{\omega}{2}+\frac{\lambda}{2})}{(i_0+\frac{1}{2}-\frac{\Omega}{2\mu}+\frac{\lambda}{2})}\,\frac{(-\nu_0)_{i_0}(\frac{\lambda}{2})_{i_0}}{(1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}}\\ &\times\sum_{i_1=i_0}^{\nu_1}\frac{(-\nu_1)_{i_1}(\frac{1}{2}+\frac{\lambda}{2})_{i_1}(\frac{3}{2}-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}}{(-\nu_1)_{i_0}(\frac{1}{2}+\frac{\lambda}{2})_{i_0}(\frac{3}{2}-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_1}}\,\eta^{i_1}\Bigg\}\xi, \end{aligned} \tag{3.23b}$$

$$\begin{aligned} y_2(z)=c_0z^{\lambda}\Bigg\{&\sum_{i_0=0}^{\nu_0}\frac{(i_0-\frac{\omega}{2}+\frac{\lambda}{2})}{(i_0+\frac{1}{2}-\frac{\Omega}{2\mu}+\frac{\lambda}{2})}\,\frac{(-\nu_0)_{i_0}(\frac{\lambda}{2})_{i_0}}{(1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}}\\ &\times\sum_{i_1=i_0}^{\nu_1}\frac{(i_1+\frac{1}{2}-\frac{\omega}{2}+\frac{\lambda}{2})}{(i_1+1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})}\,\frac{(-\nu_1)_{i_1}(\frac{1}{2}+\frac{\lambda}{2})_{i_1}(\frac{3}{2}-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}}{(-\nu_1)_{i_0}(\frac{1}{2}+\frac{\lambda}{2})_{i_0}(\frac{3}{2}-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_1}}\\ &\times\sum_{i_2=i_1}^{\nu_2}\frac{(-\nu_2)_{i_2}(1+\frac{\lambda}{2})_{i_2}(2-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_1}}{(-\nu_2)_{i_1}(1+\frac{\lambda}{2})_{i_1}(2-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_2}}\,\eta^{i_2}\Bigg\}\xi^2, \end{aligned} \tag{3.23c}$$

$$\begin{aligned} y_3(z)=c_0z^{\lambda}\Bigg\{&\sum_{i_0=0}^{\nu_0}\frac{(i_0-\frac{\omega}{2}+\frac{\lambda}{2})}{(i_0+\frac{1}{2}-\frac{\Omega}{2\mu}+\frac{\lambda}{2})}\,\frac{(-\nu_0)_{i_0}(\frac{\lambda}{2})_{i_0}}{(1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}}\\ &\times\sum_{i_1=i_0}^{\nu_1}\frac{(i_1+\frac{1}{2}-\frac{\omega}{2}+\frac{\lambda}{2})}{(i_1+1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})}\,\frac{(-\nu_1)_{i_1}(\frac{1}{2}+\frac{\lambda}{2})_{i_1}(\frac{3}{2}-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}}{(-\nu_1)_{i_0}(\frac{1}{2}+\frac{\lambda}{2})_{i_0}(\frac{3}{2}-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_1}}\\ &\times\sum_{i_2=i_1}^{\nu_2}\frac{(i_2+1-\frac{\omega}{2}+\frac{\lambda}{2})}{(i_2+\frac{3}{2}-\frac{\Omega}{2\mu}+\frac{\lambda}{2})}\,\frac{(-\nu_2)_{i_2}(1+\frac{\lambda}{2})_{i_2}(2-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_1}}{(-\nu_2)_{i_1}(1+\frac{\lambda}{2})_{i_1}(2-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_2}}\\ &\times\sum_{i_3=i_2}^{\nu_3}\frac{(-\nu_3)_{i_3}(\frac{3}{2}+\frac{\lambda}{2})_{i_3}(\frac{5}{2}-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_2}}{(-\nu_3)_{i_2}(\frac{3}{2}+\frac{\lambda}{2})_{i_2}(\frac{5}{2}-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_3}}\,\eta^{i_3}\Bigg\}\xi^3. \end{aligned} \tag{3.23d}$$

Put $l=1$ in (3.20) and insert the result into (3.23b). We get

$$\begin{aligned} y_1(z) &= c_0z^{\lambda}\xi\int_0^1 dt_1\,t_1^{\frac{1}{2}(-1-\frac{\Omega}{\mu}+\lambda)}\int_0^{\infty}du_1\,e^{-u_1}\,\frac{1}{2\pi i}\int_{\infty}^{(0+)}dp_1\\ &\quad\times\exp\Big(\frac{p_1}{\eta(1-t_1)}\Big)p_1^{-1}(1+p_1)^{-\frac{1}{2}(1+\lambda)}\Big(\frac{\eta u_1(1-t_1)}{p_1}\Big)^{\nu_1}\\ &\quad\times\sum_{i_0=0}^{\nu_0}\Big(i_0-\frac{\omega}{2}+\frac{\lambda}{2}\Big)\frac{(-\nu_0)_{i_0}(\frac{\lambda}{2})_{i_0}}{(1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}}\Big(\frac{t_1p_1}{u_1(1-t_1)(1+p_1)}\Big)^{i_0}\\ &= c_0z^{\lambda}\xi\int_0^1 dt_1\,t_1^{\frac{1}{2}(-1-\frac{\Omega}{\mu}+\lambda)}\int_0^{\infty}du_1\,e^{-u_1}\,\frac{1}{2\pi i}\int_{\infty}^{(0+)}dp_1\\ &\quad\times\exp\Big(\frac{p_1}{\eta(1-t_1)}\Big)p_1^{-1}(1+p_1)^{-\frac{1}{2}(1+\lambda)}\Big(\frac{\eta u_1(1-t_1)}{p_1}\Big)^{\nu_1}\\ &\quad\times w_{1,1}^{-\frac{1}{2}(-\omega+\lambda)}(w_{1,1}\partial_{w_{1,1}})w_{1,1}^{\frac{1}{2}(-\omega+\lambda)}\sum_{i_0=0}^{\nu_0}\frac{(-\nu_0)_{i_0}(\frac{\lambda}{2})_{i_0}}{(1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}}w_{1,1}^{i_0}, \end{aligned} \tag{3.24}$$

where $w_{1,1} = \frac{t_1 p_1}{u_1(1-t_1)(1+p_1)}$. Put $l = 2$ in (3.20) and insert the result into (3.23c). We get

$$\begin{aligned} y_2(z) &= c_0 z^\lambda \xi^2 \int_0^1 dt_2\; t_2^{\frac{1}{2}\left(-\frac{\Omega}{\mu}+\lambda\right)} \int_0^\infty du_2\; e^{-u_2} \frac{1}{2\pi i} \int_\infty^{(0+)} dp_2 \\ &\times \exp\left(\frac{p_2}{\eta(1-t_2)}\right) p_2^{-1}(1+p_2)^{-\frac{1}{2}(2+\lambda)} \left(\frac{\eta u_2(1-t_2)}{p_2}\right)^{\nu_2} w_{2,2}^{-\frac{1}{2}(1-\omega+\lambda)} (w_{2,2}\partial_{w_{2,2}}) w_{2,2}^{\frac{1}{2}(1-\omega+\lambda)} \\ &\times \sum_{i_0=0}^{\nu_0} \frac{(i_0 - \frac{\omega}{2} + \frac{\lambda}{2})}{(i_0 + \frac{1}{2} - \frac{\Omega}{2\mu} + \frac{\lambda}{2})} \frac{(-\nu_0)_{i_0}(\frac{\lambda}{2})_{i_0}}{(1 - \frac{\Omega}{2\mu} + \frac{\lambda}{2})_{i_0}} \sum_{i_1=i_0}^{\nu_1} \frac{(-\nu_1)_{i_1}(\frac{1}{2}+\frac{\lambda}{2})_{i_1}(\frac{3}{2}-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}}{(-\nu_1)_{i_0}(\frac{1}{2}+\frac{\lambda}{2})_{i_0}(\frac{3}{2}-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_1}} w_{2,2}^{i_1}. \end{aligned} \tag{3.25}$$

where $w_{2,2} = \frac{t_2 p_2}{u_2(1-t_2)(1+p_2)}$. Put $l = 1$ and $\eta = w_{2,2}$ in (3.20) and insert the result into (3.25). We get

$$\begin{aligned} y_2(z) &= c_0 z^\lambda \xi^2 \int_0^1 dt_2\; t_2^{\frac{1}{2}\left(-\frac{\Omega}{\mu}+\lambda\right)} \int_0^\infty du_2\; e^{-u_2} \frac{1}{2\pi i} \int_\infty^{(0+)} dp_2 \\ &\times \exp\left(\frac{p_2}{\eta(1-t_2)}\right) p_2^{-1}(1+p_2)^{-\frac{1}{2}(2+\lambda)} \left(\frac{\eta u_2(1-t_2)}{p_2}\right)^{\nu_2} w_{2,2}^{-\frac{1}{2}(1-\omega+\lambda)} (w_{2,2}\partial_{w_{2,2}}) w_{2,2}^{\frac{1}{2}(1-\omega+\lambda)} \\ &\times \int_0^1 dt_1\; t_1^{\frac{1}{2}\left(-1-\frac{\Omega}{\mu}+\lambda\right)} \int_0^\infty du_1\; e^{-u_1} \frac{1}{2\pi i} \int_\infty^{(0+)} dp_1 \exp\left(\frac{p_1}{w_{2,2}(1-t_1)}\right) p_1^{-1}(1+p_1)^{-\frac{1}{2}(1+\lambda)} \\ &\times \left(\frac{w_{2,2} u_1(1-t_1)}{p_1}\right)^{\nu_1} w_{1,2}^{-\frac{1}{2}(-\omega+\lambda)} (w_{1,2}\partial_{w_{1,2}}) w_{1,2}^{\frac{1}{2}(-\omega+\lambda)} \sum_{i_0=0}^{\nu_0} \frac{(-\nu_0)_{i_0}(\frac{\lambda}{2})_{i_0}}{(1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}} w_{1,2}^{i_0}, \end{aligned} \tag{3.26}$$

where $w_{1,2} = \frac{t_1 p_1}{u_1(1-t_1)(1+p_1)}$.

By using similar process as for the previous cases of integral forms of $y_1(z)$ and $y_2(z)$, the integral form of the sub-power series expansion $y_3(z)$ takes the form

$$\begin{aligned} y_3(z) &= c_0 z^\lambda \xi^3 \int_0^1 dt_3\; t_3^{\frac{1}{2}\left(1-\frac{\Omega}{\mu}+\lambda\right)} \int_0^\infty du_3\; e^{-u_3} \frac{1}{2\pi i} \int_\infty^{(0+)} dp_3 \\ &\times \exp\left(\frac{p_3}{\eta(1-t_3)}\right) p_3^{-1}(1+p_3)^{-\frac{1}{2}(3+\lambda)} \left(\frac{\eta u_3(1-t_3)}{p_3}\right)^{\nu_3} w_{3,3}^{-\frac{1}{2}(2-\omega+\lambda)} (w_{3,3}\partial_{w_{3,3}}) w_{3,3}^{\frac{1}{2}(2-\omega+\lambda)} \\ &\times \int_0^1 dt_2\; t_2^{\frac{1}{2}\left(-\frac{\Omega}{\mu}+\lambda\right)} \int_0^\infty du_2\; e^{-u_2} \frac{1}{2\pi i} \int_\infty^{(0+)} dp_2 \\ &\times \exp\left(\frac{p_2}{w_{3,3}(1-t_2)}\right) p_2^{-1}(1+p_2)^{-\frac{1}{2}(2+\lambda)} \left(\frac{w_{3,3} u_2(1-t_2)}{p_2}\right)^{\nu_2} w_{2,3}^{-\frac{1}{2}(1-\omega+\lambda)} (w_{2,3}\partial_{w_{2,3}}) w_{2,3}^{\frac{1}{2}(1-\omega+\lambda)} \\ &\times \int_0^1 dt_1\; t_1^{\frac{1}{2}\left(-1-\frac{\Omega}{\mu}+\lambda\right)} \int_0^\infty du_1\; e^{-u_1} \frac{1}{2\pi i} \int_\infty^{(0+)} dp_1 \exp\left(\frac{p_1}{w_{2,3}(1-t_1)}\right) p_1^{-1}(1+p_1)^{-\frac{1}{2}(1+\lambda)} \\ &\times \left(\frac{w_{2,3} u_1(1-t_1)}{p_1}\right)^{\nu_1} w_{1,3}^{-\frac{1}{2}(-\omega+\lambda)} (w_{1,3}\partial_{w_{1,3}}) w_{1,3}^{\frac{1}{2}(-\omega+\lambda)} \sum_{i_0=0}^{\nu_0} \frac{(-\nu_0)_{i_0}(\frac{\lambda}{2})_{i_0}}{(1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}} w_{1,3}^{i_0}, \end{aligned} \tag{3.27}$$

where

$$\begin{cases} w_{3,3} = \dfrac{t_3 p_3}{u_3(1-t_3)(1+p_3)}\,, \\ w_{2,3} = \dfrac{t_2 p_2}{u_2(1-t_2)(1+p_2)}\,, \\ w_{1,3} = \dfrac{t_1 p_1}{u_1(1-t_1)(1+p_1)}\,. \end{cases}$$

By repeating this process for all higher terms of integral forms of sub-summation terms $y_m(z)$, $m \geq 4$, we obtain their integral forms. If we substitute (3.23a), (3.24), (3.26), (3.27) and the integral forms of $y_m(z)$, $m \geq 4$, into (3.22), we obtain (3.21). □

Remark 3.3. The integral representation of the first kind GCH equation for a polynomial of type 1 about $x = \infty$ as $\nu = 2\nu_j + j + 1 + \Omega/\mu$ where $j, \nu_j \in \mathbb{N}_0$ is

$$\begin{aligned} y(z) &= Q^{(i)} W_{\nu_j}\Big(\mu, \varepsilon, \Omega, \omega, \nu = 2\nu_j + j + 1 + \frac{\Omega}{\mu}; z, \xi, \eta\Big) \\ &= z^{\frac{\Omega}{\mu}} \Bigg\{ (-\eta)^{\nu_0} U\Big(-\nu_0, -\nu_0 + 1 - \frac{\Omega}{\mu}, -\eta^{-1}\Big) + \sum_{n=1}^{\infty} \Bigg\{ \prod_{k=0}^{n-1} \Bigg\{ \int_0^1 dt_{n-k}\, t_{n-k}^{\frac{1}{2}(n-k-2)} \\ &\quad \times \int_0^\infty du_{n-k}\, e^{-u_{n-k}} \frac{1}{2\pi i} \oint_\infty^{(0+)} dp_{n-k}\, p_{n-k}^{-1} (1 + p_{n-k})^{-\frac{1}{2}(n-k+\frac{\Omega}{\mu})} \\ &\quad \times \exp\Big(\frac{p_{n-k}}{w_{n-k+1,n}(1-t_{n-k})} \Big) \Big(\frac{w_{n-k+1,n} u_{n-k} (1-t_{n-k})}{p_{n-k}} \Big)^{\nu_{n-k}} w_{n-k,n}^{-\frac{1}{2}(n-k-1-\omega+\frac{\Omega}{\mu})} \\ &\quad \times (w_{n-k,n} \partial_{w_{n-k,n}}) w_{n-k,n}^{\frac{1}{2}(n-k-1-\omega+\frac{\Omega}{\mu})} \Bigg\} (-w_{1,n})^{\nu_0} U\Big(-\nu_0, -\nu_0 + 1 - \frac{\Omega}{\mu}, -w_{1,n}^{-1}\Big) \Bigg\} \xi^n \Bigg\}, \end{aligned} \tag{3.28}$$

where

$$\begin{cases} z = \dfrac{1}{x}, \\ \xi = -\dfrac{\varepsilon}{\mu} z, \\ \eta = \dfrac{2}{\mu} z^2. \end{cases}$$

Proof. Replace a, b and z, respectively, by $-\nu_0$, $-\nu_0 + 1 - \frac{\Omega}{2\mu}$ and $-\eta^{-1}$ into (3.15):

$$\sum_{j=0}^{\nu_0} \frac{(-\nu_0)_{i_0} (\frac{\lambda}{2})_{i_0}}{(1 - \frac{\Omega}{2\mu} + \frac{\lambda}{2})_{i_0}} \eta^j = (-\eta)^{\nu_0} U\Big(-\nu_0, -\nu_0 + 1 - \frac{\Omega}{\mu}, -\eta^{-1}\Big). \tag{3.29}$$

Replace a, b and z, respectively, by $-\nu_0$, $-\nu_0 + 1 - \frac{\Omega}{2\mu}$ and $-w_{1,n}^{-1}$ into (3.15):

$$\sum_{j=0}^{\nu_0} \frac{(-\nu_0)_{i_0} (\frac{\lambda}{2})_{i_0}}{(1 - \frac{\Omega}{2\mu} + \frac{\lambda}{2})_{i_0}} w_{1,n}^j = (-w_{1,n})^{\nu_0} U\Big(-\nu_0, -\nu_0 + 1 - \frac{\Omega}{\mu}, -w_{1,n}^{-1}\Big). \tag{3.30}$$

Putting $c_0 = 1$ and $\lambda = \Omega/\mu$ in (3.21) and substituting (3.29) and (3.30) into obtained equality we get the result. □

3.3 Generating function of the GCH polynomial of type 1

Let us investigate the generating function for the first kind GCH polynomial of type 1 about $x = \infty$.

Definition 3.4. Define

$$\begin{cases} s_{a,b} = \begin{cases} s_a \cdot s_{a+1} \cdot s_{a+2} \cdots s_{b-2} \cdot s_{b-1} \cdot s_b, & \text{where } a < b, \\ s_a \text{ only if } a = b, \end{cases} \\ \widetilde{w}_{i,j} = \begin{cases} \dfrac{s_i t_i \widetilde{w}_{i+1,j}}{1 + s_i u_i (1-t_i) \widetilde{w}_{i+1,j}}, & \text{where } i < j, \\ \dfrac{s_{i,\infty} t_i \eta}{1 + s_{i,\infty} u_i (1-t_i) \eta} & \text{only if } i = j, \end{cases} \end{cases} \tag{3.31}$$

where $a, b, i, j \in \mathbb{N}_0$, $0 \leq a \leq b \leq \infty$ and $1 \leq i \leq j \leq \infty$.

We have

$$\sum_{\nu_i=\nu_j}^{\infty} s_i^{\nu_i} = \frac{s_i^{\nu_j}}{(1-s_i)} \quad \text{at } |s_i|<1. \tag{3.32}$$

Theorem 3.5. *The general expression of the generating function for the GCH polynomial of type* 1 *about* $x=\infty$ *is given by*

$$\begin{aligned}
&\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\prod_{n=1}^{\infty}\Big\{\sum_{\nu_n=\nu_{n-1}}^{\infty}s_n^{\nu_n}\Big\}y(z)=\prod_{k=1}^{\infty}\frac{1}{(1-s_{k,\infty})}\mathbf{\Upsilon}(\lambda;s_{0,\infty};\eta)\\
&+\Bigg\{\prod_{k=2}^{\infty}\frac{1}{(1-s_{k,\infty})}\int_0^1 dt_1\; t_1^{\frac{1}{2}(-1-\frac{\Omega}{\mu}+\lambda)}\int_0^{\infty}du_1\;\exp(-(1-s_{1,\infty})u_1)(1+s_{1,\infty}u_1(1-t_1)\eta)^{-\frac{1}{2}(1+\lambda)}\\
&\qquad\times\widetilde{w}_{1,1}^{-\frac{1}{2}(-\omega+\lambda)}(\widetilde{w}_{1,1}\partial_{\widetilde{w}_{1,1}})\widetilde{w}_{1,1}^{\frac{1}{2}(-\omega+\lambda)}\mathbf{\Upsilon}(\lambda;s_0;\widetilde{w}_{1,1})\Bigg\}\xi\\
&+\sum_{n=2}^{\infty}\Bigg\{\prod_{k=n+1}^{\infty}\frac{1}{(1-s_{k,\infty})}\int_0^1 dt_n\; t_n^{\frac{1}{2}(n-2-\frac{\Omega}{\mu}+\lambda)}\int_0^{\infty}du_n\;\exp(-(1-s_{n,\infty})u_n)(1+s_{n,\infty}u_n(1-t_n)\eta)^{-\frac{1}{2}(n+\lambda)}\\
&\qquad\times\widetilde{w}_{n,n}^{-\frac{1}{2}(n-1-\omega+\lambda)}(\widetilde{w}_{n,n}\partial_{\widetilde{w}_{n,n}})\widetilde{w}_{n,n}^{\frac{1}{2}(n-1-\omega+\lambda)}\prod_{j=1}^{n-1}\Bigg\{\int_0^1 dt_{n-j}\;t_{n-j}^{\frac{1}{2}(n-j-2-\frac{\Omega}{\mu}+\lambda)}\int_0^{\infty}du_{n-j}\\
&\qquad\times\exp(-(1-s_{n-j})u_{n-j})(1+s_{n-j}u_{n-j}(1-t_{n-j})\widetilde{w}_{n-j+1,n})^{-\frac{1}{2}(n-j+\lambda)}\\
&\qquad\times\widetilde{w}_{n-j,n}^{-\frac{1}{2}(n-j-1-\omega+\lambda)}(\widetilde{w}_{n-j,n}\partial_{\widetilde{w}_{n-j,n}})\widetilde{w}_{n-j,n}^{\frac{1}{2}(n-j-1-\omega+\lambda)}\Bigg\}\mathbf{\Upsilon}(\lambda;s_0;\widetilde{w}_{1,n})\Bigg\}\xi^n,
\end{aligned}\tag{3.33}$$

where

$$\begin{cases}
\mathbf{\Upsilon}(\lambda;s_{0,\infty};\eta)=\sum_{\nu_0=0}^{\infty}\frac{s_{0,\infty}^{\nu_0}}{\nu_0!}\Big\{c_0z^{\lambda}\sum_{i_0=0}^{\nu_0}\frac{(-\nu_0)_{i_0}(\frac{\lambda}{2})_{i_0}}{(1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}}\eta^{i_0}\Big\},\\
\mathbf{\Upsilon}(\lambda;s_0;\widetilde{w}_{1,1})=\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\Big\{c_0z^{\lambda}\sum_{i_0=0}^{\nu_0}\frac{(-\nu_0)_{i_0}(\frac{\lambda}{2})_{i_0}}{(1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}}\widetilde{w}_{1,1}^{i_0}\Big\},\\
\mathbf{\Upsilon}(\lambda;s_0;\widetilde{w}_{1,n})=\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\Big\{c_0z^{\lambda}\sum_{i_0=0}^{\nu_0}\frac{(-\nu_0)_{i_0}(\frac{\lambda}{2})_{i_0}}{(1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}}\widetilde{w}_{1,n}^{i_0}\Big\}.
\end{cases}$$

Proof. Applying the summation operator $\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\prod_{n=1}^{\infty}\Big\{\sum_{\nu_n=\nu_{n-1}}^{\infty}s_n^{\nu_n}\Big\}$ to the form of a general integral of type 1 GCH polynomial $y(z)$, we obtain

$$\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\prod_{n=1}^{\infty}\Big\{\sum_{\nu_n=\nu_{n-1}}^{\infty}s_n^{\nu_n}\Big\}y(z)=\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\prod_{n=1}^{\infty}\Big\{\sum_{\nu_n=\nu_{n-1}}^{\infty}s_n^{\nu_n}\Big\}\big(y_0(z)+y_1(z)+y_2(z)+\cdots\big). \tag{3.34}$$

Applying the summation operator $\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\prod_{n=1}^{\infty}\Big\{\sum_{\nu_n=\nu_{n-1}}^{\infty}s_n^{\nu_n}\Big\}$ to (3.23a) by using (3.31) and (3.32), we get

$$\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\prod_{n=1}^{\infty}\Big\{\sum_{\nu_n=\nu_{n-1}}^{\infty}s_n^{\nu_n}\Big\}y_0(z)=\prod_{k=1}^{\infty}\frac{1}{(1-s_{k,\infty})}\sum_{\nu_0=0}^{\infty}\frac{s_{0,\infty}^{\nu_0}}{\nu_0!}\Big\{c_0z^{\lambda}\sum_{i_0=0}^{\nu_0}\frac{(-\nu_0)_{i_0}(\frac{\lambda}{2})_{i_0}}{(1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}}\eta^{i_0}\Big\} \tag{3.35}$$

Applying the summation operator $\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\prod_{n=1}^{\infty}\Big\{\sum_{\nu_n=\nu_{n-1}}^{\infty}s_n^{\nu_n}\Big\}$ to (3.24), by using (3.31) and (3.32), we get

$$\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\prod_{n=1}^{\infty}\Big\{\sum_{\nu_n=\nu_{n-1}}^{\infty}s_n^{\nu_n}\Big\}y_1(z)=\prod_{k=2}^{\infty}\frac{1}{(1-s_{k,\infty})}\int_0^1 dt_1\, t_1^{\frac{1}{2}\left(-1-\frac{\Omega}{\mu}+\lambda\right)}\int_0^{\infty}du_1\, e^{-u_1}\frac{1}{2\pi i}\int_{\infty}^{(0+)}dp_1$$
$$\times\exp\Big(\frac{p_1}{\eta(1-t_1)}\Big)\frac{(1+p_1)^{-\frac{1}{2}(1+\lambda)}}{p_1}\sum_{\nu_1=\nu_0}^{\infty}\Big(\frac{s_{1,\infty}\eta u_1(1-t_1)}{p_1}\Big)^{\nu_1}$$
$$\times w_{1,1}^{-\frac{1}{2}(-\omega+\lambda)}(w_{1,1}\partial_{w_{1,1}})w_{1,1}^{\frac{1}{2}(-\omega+\lambda)}\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\left\{c_0z^{\lambda}\sum_{i_0=0}^{\nu_0}\frac{(-\nu_0)_{i_0}(\frac{\lambda}{2})_{i_0}}{(1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}}w_{1,1}^{i_0}\right\}\xi. \quad (3.36)$$

Replace ν_i, ν_j and s_i by ν_1, ν_0 and $\frac{s_{1,\infty}\eta u_1(1-t_1)}{p_1}$ in (3.32). Substitute the new (3.32) into (3.36),

$$\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\prod_{n=1}^{\infty}\Big\{\sum_{\nu_n=\nu_{n-1}}^{\infty}s_n^{\nu_n}\Big\}y_1(z)=\prod_{k=2}^{\infty}\frac{1}{(1-s_{k,\infty})}\int_0^1 dt_1\, t_1^{\frac{1}{2}\left(-1-\frac{\Omega}{\mu}+\lambda\right)}\int_0^{\infty}du_1\, e^{-u_1}$$
$$\times\frac{1}{2\pi i}\int_{\infty}^{(0+)}dp_1\exp\Big(\frac{p_1}{\eta(1-t_1)}\Big)\frac{(1+p_1)^{-\frac{1}{2}(1+\lambda)}}{p_1-s_{1,\infty}\eta u_1(1-t_1)}w_{1,1}^{-\frac{1}{2}(-\omega+\lambda)}(w_{1,1}\partial_{w_{1,1}})w_{1,1}^{\frac{1}{2}(-\omega+\lambda)}$$
$$\times\sum_{\nu_0=0}^{\infty}\Big(\frac{s_{0,\infty}\eta u_1(1-t_1)}{p_1}\Big)^{\nu_0}\frac{1}{\nu_0!}\left\{c_0z^{\lambda}\sum_{i_0=0}^{\nu_0}\frac{(-\nu_0)_{i_0}(\frac{\lambda}{2})_{i_0}}{(1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}}w_{1,1}^{i_0}\right\}\xi. \quad (3.37)$$

By using Cauchy's integral formula, the contour integrand has poles at $p_1=s_{1,\infty}\eta u_1(1-t_1)$, where $s_{1,\infty}\eta u_1(1-t_1)$ is inside the unit circle. Computing the residue in (3.37), we obtain

$$\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\prod_{n=1}^{\infty}\Big\{\sum_{\nu_n=\nu_{n-1}}^{\infty}s_n^{\nu_n}\Big\}y_1(z)=\prod_{k=2}^{\infty}\frac{1}{(1-s_{k,\infty})}\int_0^1 dt_1\, t_1^{\frac{1}{2}\left(-1-\frac{\Omega}{\mu}+\lambda\right)}$$
$$\times\int_0^{\infty}du_1\ \exp(-(1-s_{1,\infty})u_1)(1+s_{1,\infty}u_1(1-t_1)\eta)^{-\frac{1}{2}(1+\lambda)}$$
$$\times\widetilde{w}_{1,1}^{-\frac{1}{2}(-\omega+\lambda)}(\widetilde{w}_{1,1}\partial_{\widetilde{w}_{1,1}})\widetilde{w}_{1,1}^{\frac{1}{2}(-\omega+\lambda)}\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\left\{c_0z^{\lambda}\sum_{i_0=0}^{\nu_0}\frac{(-\nu_0)_{i_0}(\frac{\lambda}{2})_{i_0}}{(1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}}\widetilde{w}_{1,1}^{i_0}\right\}\xi, \quad (3.38)$$

where

$$\widetilde{w}_{1,1}=\frac{t_1p_1}{u_1(1-t_1)(1+p_1)}\Bigg|_{p_1=s_{1,\infty}\eta u_1(1-t_1)}=\frac{s_{1,\infty}t_1\eta}{1+s_{1,\infty}u_1(1-t_1)\eta}\,.$$

Applying the summation operator $\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\prod_{n=1}^{\infty}\Big\{\sum_{\nu_n=\nu_{n-1}}^{\infty}s_n^{\nu_n}\Big\}$ to (3.26), by using (3.31) and (3.32), we have

$$\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\prod_{n=1}^{\infty}\Big\{\sum_{\nu_n=\nu_{n-1}}^{\infty}s_n^{\nu_n}\Big\}y_2(z)=\prod_{k=3}^{\infty}\frac{1}{(1-s_{k,\infty})}\int_0^1 dt_2\, t_2^{\frac{1}{2}\left(-\frac{\Omega}{\mu}+\lambda\right)}\int_0^{\infty}du_2\, e^{-u_2}$$
$$\times\frac{1}{2\pi i}\int_{\infty}^{(0+)}dp_2\ \exp\Big(\frac{p_2}{\eta(1-t_2)}\Big)\frac{(1+p_2)^{-\frac{1}{2}(2+\lambda)}}{p_2}$$
$$\times\sum_{\nu_2=\nu_1}^{\infty}\Big(\frac{s_{2,\infty}\eta u_2(1-t_2)}{p_2}\Big)^{\nu_2}w_{2,2}^{-\frac{1}{2}(1-\omega+\lambda)}(w_{2,2}\partial_{w_{2,2}})w_{2,2}^{\frac{1}{2}(1-\omega+\lambda)}$$

$$\times \int_0^1 dt_1\; t_1^{\frac{1}{2}\left(-1-\frac{\Omega}{\mu}+\lambda\right)} \int_0^\infty du_1\; e^{-u_1} \frac{1}{2\pi i} \int_\infty^{(0+)} dp_1\; \exp\left(\frac{p_1}{w_{2,2}(1-t_1)}\right) \frac{(1+p_1)^{-\frac{1}{2}(1+\lambda)}}{p_1}$$
$$\times \sum_{\nu_1=\nu_0}^{\infty} \left(\frac{s_1 w_{2,2} u_1(1-t_1)}{p_1}\right)^{\nu_1} w_{1,2}^{-\frac{1}{2}(-\omega+\lambda)} (w_{1,2}\partial_{w_{1,2}}) w_{1,2}^{\frac{1}{2}(-\omega+\lambda)}$$
$$\times \sum_{\nu_0=0}^{\infty} \frac{s_0^{\nu_0}}{\nu_0!} \left\{ c_0 z^\lambda \sum_{i_0=0}^{\nu_0} \frac{(-\nu_0)_{i_0} (\frac{\lambda}{2})_{i_0}}{(1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}}\, w_{1,2}^{i_0} \right\} \xi^2. \tag{3.39}$$

Replace ν_i, ν_j and s_i, respectively, by ν_2, ν_1 and $\frac{s_{2,\infty}\,\eta u_2(1-t_2)}{p_2}$ in (3.32) and the insert the result into (3.39). We have

$$\sum_{\nu_0=0}^{\infty} \frac{s_0^{\nu_0}}{\nu_0!} \prod_{n=1}^{\infty} \Big\{ \sum_{\nu_n=\nu_{n-1}}^{\infty} s_n^{\nu_n} \Big\} y_2(z) = \prod_{k=3}^{\infty} \frac{1}{(1-s_{k,\infty})} \int_0^1 dt_2\; t_2^{\frac{1}{2}\left(-\frac{\Omega}{\mu}+\lambda\right)} \int_0^\infty du_2\; e^{-u_2}$$
$$\times \frac{1}{2\pi i} \int_\infty^{(0+)} dp_2\; \exp\left(\frac{p_2}{\eta(1-t_2)}\right) \frac{(1+p_2)^{-\frac{1}{2}(2+\lambda)}}{p_2 - s_{2,\infty}\eta u_2(1-t_2)}\, w_{2,2}^{-\frac{1}{2}(1-\omega+\lambda)} (w_{2,2}\partial_{w_{2,2}}) w_{2,2}^{\frac{1}{2}(1-\omega+\lambda)}$$
$$\times \int_0^1 dt_1\; t_1^{\frac{1}{2}\left(-1-\frac{\Omega}{\mu}+\lambda\right)} \int_0^\infty du_1\; e^{-u_1} \frac{1}{2\pi i} \int_\infty^{(0+)} dp_1\; \exp\left(\frac{p_1}{w_{2,2}(1-t_1)}\right) \frac{(1+p_1)^{-\frac{1}{2}(1+\lambda)}}{p_1}$$
$$\times \sum_{\nu_1=\nu_0}^{\infty} \left(\frac{s_{1,\infty}\eta u_2(1-t_2) w_{2,2} u_1(1-t_1)}{p_1 p_2}\right)^{\nu_1} w_{1,2}^{-\frac{1}{2}(-\omega+\lambda)} (w_{1,2}\partial_{w_{1,2}}) w_{1,2}^{\frac{1}{2}(-\omega+\lambda)}$$
$$\times \sum_{\nu_0=0}^{\infty} \frac{s_0^{\nu_0}}{\nu_0!} \left\{ c_0 z^\lambda \sum_{i_0=0}^{\nu_0} \frac{(-\nu_0)_{i_0} (\frac{\lambda}{2})_{i_0}}{(1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}}\, w_{1,2}^{i_0} \right\} \xi^2. \tag{3.40}$$

By using Cauchy's integral formula, the contour integrand has poles at $p_2 = s_{2,\infty}\eta u_2(1-t_2)$, where $s_{2,\infty}\eta u_2(1-t_2)$ is inside the unit circle. Computing the residue in (3.40), we obtain

$$\sum_{\nu_0=0}^{\infty} \frac{s_0^{\nu_0}}{\nu_0!} \prod_{n=1}^{\infty} \Big\{ \sum_{\nu_n=\nu_{n-1}}^{\infty} s_n^{\nu_n} \Big\} y_2(z) = \prod_{k=3}^{\infty} \frac{1}{(1-s_{k,\infty})} \int_0^1 dt_2\; t_2^{\frac{1}{2}\left(-\frac{\Omega}{\mu}+\lambda\right)}$$
$$\times \int_0^\infty du_2\; \exp\Big(-(1-s_{2,\infty})u_2\Big) \Big(1+s_{2,\infty}u_2(1-t_2)\eta\Big)^{-\frac{1}{2}(2+\lambda)} \widetilde{w}_{2,2}^{-\frac{1}{2}(1-\omega+\lambda)} (\widetilde{w}_{2,2}\partial_{\widetilde{w}_{2,2}}) \widetilde{w}_{2,2}^{\frac{1}{2}(1-\omega+\lambda)}$$
$$\times \int_0^1 dt_1\; t_1^{\frac{1}{2}\left(-1-\frac{\Omega}{\mu}+\lambda\right)} \int_0^\infty du_1\; e^{-u_1} \frac{1}{2\pi i} \int_\infty^{(0+)} dp_1\; \exp\left(\frac{p_1}{\widetilde{w}_{2,2}(1-t_1)}\right) \frac{(1+p_1)^{-\frac{1}{2}(1+\lambda)}}{p_1}$$
$$\times \sum_{\nu_1=\nu_0}^{\infty} \left(\frac{s_1 \widetilde{w}_{2,2} u_1(1-t_1)}{p_1}\right)^{\nu_1} w_{1,2}^{-\frac{1}{2}(-\omega+\lambda)} (w_{1,2}\partial_{w_{1,2}}) w_{1,2}^{\frac{1}{2}(-\omega+\lambda)}$$
$$\times \sum_{\nu_0=0}^{\infty} \frac{s_0^{\nu_0}}{\nu_0!} \left\{ c_0 z^\lambda \sum_{i_0=0}^{\nu_0} \frac{(-\nu_0)_{i_0} (\frac{\lambda}{2})_{i_0}}{(1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}}\, w_{1,2}^{i_0} \right\} \xi^2, \tag{3.41}$$

where

$$\widetilde{w}_{2,2} = \left.\frac{t_2 p_2}{u_2(1-t_2)(1+p_2)}\right|_{p_2=s_{2,\infty}\eta u_2(1-t_2)} = \frac{s_{2,\infty} t_2 \eta}{1+s_{2,\infty}u_2(1-t_2)\eta}\,.$$

Replace ν_i, ν_j and s_i, respectively, by ν_1, ν_0 and $\frac{s_1 \widetilde{w}_{2,2} u_1(1-t_1)}{p_1}$ in (3.32) and insert the result in (3.41):

$$\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\prod_{n=1}^{\infty}\Big\{\sum_{\nu_n=\nu_{n-1}}^{\infty}s_n^{\nu_n}\Big\}y_2(z)=\prod_{k=3}^{\infty}\frac{1}{(1-s_{k,\infty})}\int_0^1 dt_2\; t_2^{\frac{1}{2}\left(-\frac{\Omega}{\mu}+\lambda\right)}$$
$$\times\int_0^{\infty}du_2\;\exp(-(1-s_{2,\infty})u_2)\big(1+s_{2,\infty}u_2(1-t_2)\eta\big)^{-\frac{1}{2}(2+\lambda)}\widetilde{w}_{2,2}^{-\frac{1}{2}(1-\omega+\lambda)}(\widetilde{w}_{2,2}\partial_{\widetilde{w}_{2,2}})\widetilde{w}_{2,2}^{\frac{1}{2}(1-\omega+\lambda)}$$
$$\times\int_0^1 dt_1\; t_1^{\frac{1}{2}\left(-1-\frac{\Omega}{\mu}+\lambda\right)}\int_0^{\infty}du_1\; e^{-u_1}\frac{1}{2\pi i}\int_{\infty}^{(0+)}dp_1\;\exp\Big(\frac{p_1}{\widetilde{w}_{2,2}(1-t_1)}\Big)$$
$$\times\frac{(1+p_1)^{-\frac{1}{2}(1+\lambda)}}{p_1-s_1\widetilde{w}_{2,2}u_1(1-t_1)}\,w_{1,2}^{-\frac{1}{2}(-\omega+\lambda)}(w_{1,2}\partial_{w_{1,2}})w_{1,2}^{\frac{1}{2}(-\omega+\lambda)}$$
$$\times\sum_{\nu_0=0}^{\infty}\Big(\frac{s_{0,1}\widetilde{w}_{2,2}u_1(1-t_1)}{p_1}\Big)^{\nu_0}\frac{1}{\nu_0!}\left\{c_0z^{\lambda}\sum_{i_0=0}^{\nu_0}\frac{(-\nu_0)_{i_0}(\frac{\lambda}{2})_{i_0}}{(1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}}\,w_{1,2}^{i_0}\right\}\xi^2. \tag{3.42}$$

By using Cauchy's integral formula, the contour integrand has poles at $p_1=s_1\widetilde{w}_{2,2}u_1(1-t_1)$, where $s_1\widetilde{w}_{2,2}u_1(1-t_1)$ is inside the unit circle. Computing the residue in (3.42), we obtain

$$\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\prod_{n=1}^{\infty}\Big\{\sum_{\nu_n=\nu_{n-1}}^{\infty}s_n^{\nu_n}\Big\}y_2(z)=\prod_{k=3}^{\infty}\frac{1}{(1-s_{k,\infty})}\int_0^1 dt_2\; t_2^{\frac{1}{2}\left(-\frac{\Omega}{\mu}+\lambda\right)}$$
$$\times\int_0^{\infty}du_2\;\exp(-(1-s_{2,\infty})u_2)\big(1+s_{2,\infty}u_2(1-t_2)\eta\big)^{-\frac{1}{2}(2+\lambda)}\widetilde{w}_{2,2}^{-\frac{1}{2}(1-\omega+\lambda)}(\widetilde{w}_{2,2}\partial_{\widetilde{w}_{2,2}})\widetilde{w}_{2,2}^{\frac{1}{2}(1-\omega+\lambda)}$$
$$\times\int_0^1 dt_1\; t_1^{\frac{1}{2}\left(-1-\frac{\Omega}{\mu}+\lambda\right)}\int_0^{\infty}du_1\;\exp(-(1-s_1)u_1)\big(1+s_1u_1(1-t_1)\widetilde{w}_{2,2}\big)^{-\frac{1}{2}(1+\lambda)}$$
$$\times\widetilde{w}_{1,2}^{-\frac{1}{2}(-\omega+\lambda)}(\widetilde{w}_{1,2}\partial_{\widetilde{w}_{1,2}})\widetilde{w}_{1,2}^{\frac{1}{2}(-\omega+\lambda)}\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\left\{c_0z^{\lambda}\sum_{i_0=0}^{\nu_0}\frac{(-\nu_0)_{i_0}(\frac{\lambda}{2})_{i_0}}{(1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}}\,\widetilde{w}_{1,2}^{i_0}\right\}\xi^2, \tag{3.43}$$

where

$$\widetilde{w}_{1,2}=\frac{t_1p_1}{u_1(1-t_1)(1+p_1)}\bigg|_{p_1=s_1\widetilde{w}_{2,2}u_1(1-t_1)}=\frac{s_1t_1\widetilde{w}_{2,2}}{1+s_1u_1(1-t_1)\widetilde{w}_{2,2}}\,.$$

Applying the summation operator $\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\prod_{n=1}^{\infty}\Big\{\sum_{\nu_n=\nu_{n-1}}^{\infty}s_n^{\nu_n}\Big\}$ to (3.27), by using (3.31) and (3.32), we have

$$\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\prod_{n=1}^{\infty}\Big\{\sum_{\nu_n=\nu_{n-1}}^{\infty}s_n^{\nu_n}\Big\}y_3(z)=\prod_{k=4}^{\infty}\frac{1}{(1-s_{k,\infty})}\int_0^1 dt_3\; t_3^{\frac{1}{2}\left(1-\frac{\Omega}{\mu}+\lambda\right)}$$
$$\times\int_0^{\infty}du_3\;\exp(-(1-s_{3,\infty})u_3)\big(1+s_{3,\infty}u_3(1-t_3)\eta\big)^{-\frac{1}{2}(3+\lambda)}\widetilde{w}_{3,3}^{-\frac{1}{2}(2-\omega+\lambda)}(\widetilde{w}_{3,3}\partial_{\widetilde{w}_{3,3}})\widetilde{w}_{3,3}^{\frac{1}{2}(2-\omega+\lambda)}$$
$$\times\int_0^1 dt_2\; t_2^{\frac{1}{2}\left(-\frac{\Omega}{\mu}+\lambda\right)}\int_0^{\infty}du_2\;\exp(-(1-s_2)u_2)(1+s_2u_2(1-t_2)\widetilde{w}_{3,3})^{-\frac{1}{2}(2+\lambda)}\widetilde{w}_{2,3}^{-\frac{1}{2}(1-\omega+\lambda)}$$
$$\times(\widetilde{w}_{2,3}\partial_{\widetilde{w}_{2,3}})\widetilde{w}_{2,3}^{\frac{1}{2}(1-\omega+\lambda)}\int_0^1 dt_1\; t_1^{\frac{1}{2}\left(-1-\frac{\Omega}{\mu}+\lambda\right)}\int_0^{\infty}du_1\;\exp(-(1-s_1)u_1)\big(1+s_1u_1(1-t_1)\widetilde{w}_{2,3}\big)^{-\frac{1}{2}(1+\lambda)}$$
$$\times\widetilde{w}_{1,3}^{-\frac{1}{2}(-\omega+\lambda)}(\widetilde{w}_{1,3}\partial_{\widetilde{w}_{1,3}})\widetilde{w}_{1,3}^{\frac{1}{2}(-\omega+\lambda)}\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\left\{c_0z^{\lambda}\sum_{i_0=0}^{\nu_0}\frac{(-\nu_0)_{i_0}(\frac{\lambda}{2})_{i_0}}{(1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}}\,\widetilde{w}_{1,3}^{i_0}\right\}\xi^3, \tag{3.44}$$

where

$$\begin{cases} \widetilde{w}_{3,3} = \dfrac{t_3 p_3}{u_3(1-t_3)(1+p_3)}\Big|_{p_3 = s_{3,\infty}\eta u_3(1-t_3)} = \dfrac{s_{3,\infty} t_3 \eta}{1 + s_{3,\infty} u_3 (1-t_3)\eta}\,, \\ \widetilde{w}_{2,3} = \dfrac{t_2 p_2}{u_2(1-t_2)(1+p_2)}\Big|_{p_2 = s_2 \widetilde{w}_{3,3} u_2(1-t_2)} = \dfrac{s_2 t_2 \widetilde{w}_{3,3}}{1 + s_2 u_2 (1-t_2)\widetilde{w}_{3,3}}\,, \\ \widetilde{w}_{1,3} = \dfrac{t_1 p_1}{u_1(1-t_1)(1+p_1)}\Big|_{p_1 = s_1 \widetilde{w}_{2,3} u_1(1-t_1)} = \dfrac{s_1 t_1 \widetilde{w}_{2,3}}{1 + s_1 u_1 (1-t_1)\widetilde{w}_{2,3}}\,. \end{cases}$$

By repeating this process for all higher terms of integral forms of the sub-summation $y_m(z)$ terms, where $m > 3$, we obtain every $\sum_{\nu_0=0}^{\infty} \frac{s_0^{\nu_0}}{\nu_0!} \prod_{n=1}^{\infty} \{ \sum_{\nu_n=\nu_{n-1}}^{\infty} s_n^{\nu_n} \} y_m(z)$ terms. Since we substitute (3.35), (3.38), (3.43), (3.44) and include all $\sum_{\nu_0=0}^{\infty} \frac{s_0^{\nu_0}}{\nu_0!} \prod_{n=1}^{\infty} \{ \sum_{\nu_n=\nu_{n-1}}^{\infty} s_n^{\nu_n} \} y_m(z)$ terms, where $m > 3$, into (3.34), we obtain (3.33). □

Remark 3.6. The generating function for the first kind GCH polynomial of type 1 about $x = \infty$ as $\nu = 2\nu_j + j + 1 + \Omega/\mu$, where $j, \nu_j \in \mathbb{N}_0$, is

$$\begin{aligned} &\sum_{\nu_0=0}^{\infty} \frac{s_0^{\nu_0}}{\nu_0!} \prod y_{n=1}^{\infty} \Big\{ \sum_{\nu y_n=\nu_{n-1}}^{\infty} s_n^{\nu_n} \Big\} Q^{(i)} W_{\nu_j}(\mu,\varepsilon,\Omega,\omega,\nu;z,\xi,\eta) \\ &= z^{\frac{\Omega}{\mu}} \Bigg\{ \prod_{k=1}^{\infty} \frac{1}{(1-s_{k,\infty})} \mathbf{A}(s_{0,\infty};\eta) + \Bigg\{ \prod_{k=2}^{\infty} \frac{1}{(1-s_{k,\infty})} \int_0^1 dt_1\, t_1^{-\frac{1}{2}} \\ &\times \int_0^{\infty} du_1\, \overleftrightarrow{\mathbf{\Gamma}}_1(s_{1,\infty};t_1,u_1,\eta) \widetilde{w}_{1,1}^{-\frac{1}{2}(-\omega+\frac{\Omega}{\mu})} (\widetilde{w}_{1,1}\partial_{\widetilde{w}_{1,1}}) \widetilde{w}_{1,1}^{\frac{1}{2}(-\omega+\frac{\Omega}{\mu})} \mathbf{A}(s_0;\widetilde{w}_{1,1}) \Bigg\} \xi \\ &+ \sum_{n=2}^{\infty} \Bigg\{ \prod_{k=n+1}^{\infty} \frac{1}{(1-s_{k,\infty})} \int_0^1 dt_n\, t_n^{\frac{1}{2}(n-2)} \\ &\times \int_0^{\infty} du_n\, \overleftrightarrow{\mathbf{\Gamma}}_n(s_{n,\infty};t_n,u_n,\eta) \widetilde{w}_{n,n}^{-\frac{1}{2}(n-1-\omega+\frac{\Omega}{\mu})} (\widetilde{w}_{n,n}\partial_{\widetilde{w}_{n,n}}) \widetilde{w}_{n,n}^{\frac{1}{2}(n-1-\omega+\frac{\Omega}{\mu})} \\ &\times \prod_{j=1}^{n-1} \Bigg\{ \int_0^1 dt_{n-j}\, t_{n-j}^{\frac{1}{2}(n-j-2)} \int_0^{\infty} du_{n-j}\, \overleftrightarrow{\mathbf{\Gamma}}_{n-j}(s_{n-j};t_{n-j},u_{n-j},\widetilde{w}_{n-j+1,n}) \\ &\times \widetilde{w}_{n-j,n}^{-\frac{1}{2}(n-j-1-\omega+\frac{\Omega}{\mu})} (\widetilde{w}_{n-j,n}\partial_{\widetilde{w}_{n-j,n}}) \widetilde{w}_{n-j,n}^{\frac{1}{2}(n-j-1-\omega+\frac{\Omega}{\mu})} \Bigg\} \mathbf{A}(s_0;\widetilde{w}_{1,n}) \Bigg\} \xi^n \Bigg\}, \end{aligned} \tag{3.45}$$

where

$$\begin{cases} \nu = 2\nu_j + j + 1 + \frac{\Omega}{\mu}; \quad z = \frac{1}{x}\,, \quad \xi = -\frac{\varepsilon}{\mu} z, \quad \eta = \frac{2}{\mu} z^2; \\ \overleftrightarrow{\mathbf{\Gamma}}_1(s_{1,\infty};t_1,u_1,\eta) = \exp(-(1-s_{1,\infty})u_1)(1+s_{1,\infty}u_1(1-t_1)\eta)^{-\frac{1}{2}(1+\frac{\Omega}{\mu})}; \\ \overleftrightarrow{\mathbf{\Gamma}}_n(s_{n,\infty};t_n,u_n,\eta) = \exp(-(1-s_{n,\infty})u_n)(1+s_{n,\infty}u_n(1-t_n)\eta)^{-\frac{1}{2}(n+\frac{\Omega}{\mu})}; \\ \overleftrightarrow{\mathbf{\Gamma}}_{n-j}(s_{n-j};t_{n-j},u_{n-j},\widetilde{w}_{n-j+1,n}) \\ \qquad = \exp(-(1-s_{n-j})u_{n-j})(1+s_{n-j}u_{n-j}(1-t_{n-j})\widetilde{w}_{n-j+1,n})^{-\frac{1}{2}(n-j+\frac{\Omega}{\mu})} \end{cases}$$

and

$$\begin{cases} \mathbf{A}(s_{0,\infty};\eta) = \exp(s_{0,\infty})(1+s_{0,\infty}\eta)^{-\frac{\Omega}{2\mu}}, \\ \mathbf{A}(s_0;\widetilde{w}_{1,1}) = \exp(s_0)(1+s_0\widetilde{w}_{1,1})^{-\frac{\Omega}{2\mu}}, \\ \mathbf{A}(s_0;\widetilde{w}_{1,n}) = \exp(s_0)(1+s_0\widetilde{w}_{1,n})^{-\frac{\Omega}{2\mu}}. \end{cases}$$

Proof. Replace a, b, j and z, respectively, by $-\nu_0$, $-\nu_0+1-a$, i_0 and $-z^{-1}$ in (3.15). Applying the summation operator $\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}$ to the resulting equality, we have

$$\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\sum_{i_0=0}^{\nu_0}\frac{(-\nu_0)_{i_0}(a)_{i_0}}{(1)_{i_0}}z^{i_0} = \sum_{\nu_0=0}^{\infty}\frac{(-s_0z)^{\nu_0}}{\nu_0!}U(-\nu_0,-\nu_0+1-a,-z^{-1}). \tag{3.46}$$

Replace a, b, p_l and z, respectively, by $-\nu_0$, $-\nu_0+1-a$, p and $-z^{-1}$ in (3.14):

$$U(-\nu_0,-\nu_0+1-a,-z^{-1}) = e^{\nu_0\pi i}\frac{\nu_0!}{2\pi i}\int_{\infty}^{(0+)}dp\, e^{\frac{p}{z}}p^{-\nu_0-1}(1+p)^{-a}. \tag{3.47}$$

Insert (3.47) into (3.46):

$$\begin{aligned}\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\sum_{i_0=0}^{\nu_0}\frac{(-\nu_0)_{i_0}(a)_{i_0}}{(1)_{i_0}}z^{i_0} &= \frac{1}{2\pi i}\int_{\infty}^{(0+)}dp\, e^{\frac{p}{z}}p^{-1}(1+p)^{-a}\sum_{\nu_0=0}^{\infty}\left(\frac{s_0z}{p}\right)^{\nu_0} \\ &= \frac{1}{2\pi i}\int_{\infty}^{(0+)}dp\, e^{\frac{p}{z}}\frac{(1+p)^{-a}}{(p-s_0z)} = \exp(s_0)(1+s_0z)^{-a}.\end{aligned} \tag{3.48}$$

Replace s_0, a and z, respectively, by $s_{0,\infty}$, $\frac{\Omega}{2\mu}$ and η in (3.48):

$$\sum_{\nu_0=0}^{\infty}\frac{s_{0,\infty}^{\nu_0}}{\nu_0!}\sum_{i_0=0}^{\nu_0}\frac{(-\nu_0)_{i_0}(\frac{\Omega}{2\mu})_{i_0}}{(1)_{i_0}}\eta^{i_0} = \exp(s_{0,\infty})(1+s_{0,\infty}\eta)^{-\frac{\Omega}{2\mu}}. \tag{3.49}$$

Replace a and z, respectively, by $\frac{\Omega}{2\mu}$ and $\widetilde{w}_{1,1}$ in (3.48):

$$\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\sum_{i_0=0}^{\nu_0}\frac{(-\nu_0)_{i_0}(\frac{\Omega}{2\mu})_{i_0}}{(1)_{i_0}}\widetilde{w}_{1,1}^{i_0} = \exp(s_0)(1+s_0\widetilde{w}_{1,1})^{-\frac{\Omega}{2\mu}}. \tag{3.50}$$

Replace a and z, respectively, by $\frac{\Omega}{2\mu}$ and $\widetilde{w}_{1,n}$ in (3.48):

$$\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\sum_{i_0=0}^{\nu_0}\frac{(-\nu_0)_{i_0}(\frac{\Omega}{2\mu})_{i_0}}{(1)_{i_0}}\widetilde{w}_{1,n}^{i_0} = \exp(s_0)(1+s_0\widetilde{w}_{1,n})^{-\frac{\Omega}{2\mu}}. \tag{3.51}$$

Putting c_0= 1 and $\lambda = \frac{\Omega}{\mu}$ in (3.33) and substitute (3.49), (3.50) and (3.51) into obtained equality we get the result. □

4 Summary

The canonical form of the biconfluent Heun equation is defined by [26, 36]

$$x\frac{d^2y}{dx^2} + (1+\alpha-\beta x-2x^2)\frac{dy}{dx} + \Big((\gamma-\alpha-2)x - \frac{1}{2}[\delta+(1+\alpha)\beta]\Big)y = 0$$

in which $(\alpha, \beta, \gamma, \delta) \in \mathbb{C}^4$. This equation has two singular points: of a regular singularity at $x = 0$ and of an irregular singularity at ∞. This equation is derived from the GCH equation by replacing all coefficients μ, ε, ν, Ω and ω, respectively, by -2, $-\beta$, $1+\alpha$, $\gamma - \alpha - 2$ and $1/2(\delta/\beta + 1 + \alpha)$ in (1.1) [9].

In previous two papers of the author [11, 12], it was shown the way of deriving power series expansions in closed forms of the GCH equation about $x = 0$ by applying 3TRF for an infinite series of a polynomial of type 1 including their integral forms (each sub-integral $y_m(x)$ of a general integral $y(x) = \sum_{m=0}^{\infty} y_m(x)$ is composed of $2m$ terms of the definite integrals and m terms of the contour integrals), and generating functions for the GCH polynomials of type 1 were analyzed.

In the present paper, it is shown how one can construct power series expansions in closed forms and their integral forms of the GCH equation about $x = 0$ for an infinite series and a polynomial of type 2 by applying R3TRF. This is performed by letting B_n in the sequence c_n be the leading term in the analytic function $y(x)$. For a polynomial of type 2, we treat ω as a fixed value and μ, ε, ν, Ω as free variables.

The power series expansions and integral representations of the GCH equation about $x = 0$ for an infinite series in the present paper are equivalent to an infinite series of the GCH equation in [11, 12]. In this paper, B_n is the leading term in the sequence c_n in the analytic function $y(x)$. In [11, 12], A_n is the leading term in the sequence c_n in the analytic function $y(x)$.

As we can see in [11, 12], the power series expansions of the GCH equation for an infinite series and a polynomial of type 1, the denominators and numerators in all B_n terms of each sub-power series expansion $y_m(x)$, where $m = 0, 1, 2, \ldots$, arise with the Pochhammer symbol. In this paper, the denominators and numerators in all A_n terms of each sub-power series expansion $y_m(x)$ arise likewise with the Pochhammer symbol. Since we construct the power series expansions with Pochhammer symbols in numerators and denominators, we are able to describe integral representations of the GCH equation analytically. As we consider representations in closed form integrals of the GCH equation about $x = 0$ by applying either 3TRF or R3TRF, a ${}_1F_1$ function (the Kummer function of the first kind) recurs in each of its sub-integral forms. It means that we are able to transform the GCH (or BCH) functions about $x = 0$ into any well-known special functions having two term recursive relation between successive coefficients in the power series of their ODEs, because a ${}_1F_1$ function arises in each of sub-integral forms on the GCH equation. Having replaced ${}_1F_1$ functions in their integral forms by other special functions, we can rebuild the Frobenius solutions of the GCH equation about $x = 0$ in a backward.

In [12] and in this paper, it is shown how to derive generating functions for type 1 and type 2 GCH polynomials from their analytic integral representations. We are able to derive orthogonal relations, recursion relations and expectation values of physical quantities from these two generating functions; the processes for obtaining orthogonal and recursion relations of the GCH polynomials are similar to the case of a normalized wave function for the hydrogen-like atoms.*

In Section 3, we construct the Frobenius solution of the GCH equation about $x = \infty$ for the type 1 polynomial by applying 3TRF analytically [10]. Its integral representation and the generating function for the GCH polynomial are likewise derived analytically. There are no such solutions for an infinite series and for the type 2 polynomial, since the B_n term is divergent in (3.5b) and the index $n \to \infty$. Therefore, there are only two types of the analytic solution of the GCH equation about $x = \infty$ such as the type 1 and type 3 polynomials. In comparison with integral forms of the GCH polynomials of the type 1 and 2 around $x = 0$, a Tricomi's function (Kummer's function of the second kind) recurs in each of sub-integral forms of the GCH polynomial of type 1 about $x = \infty$.

*For instance, in the quantum mechanical aspects, if the eigenenergy is contained in B_n term in a 3-term recursive relation between successive coefficients of the power series expansion, we have to apply the type 1 GCH polynomial. If the eigenenergy is included in A_n term in a 3-term recursive relation, we should apply the type 2 GCH polynomial. If the first eigenenergy (mathematically, it is denoted by a spectral parameter) is included in A_n term and the second one is involved in B_n terms, we must apply the type 3 GCH polynomial. In Chapters 9 and 10 of [14] we discuss about the type 3 GCH polynomials.

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