Memoirs on Differential Equations and Mathematical Physics

Volume 72, 2017, 103–118

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SOME REMARKS ON FUNCTIONAL DIFFERENTIAL EQUATIONS IN ABSTRACT SPACES

Abstract. The aim of this paper is to present some remarks concerning the functional differential equation

$$v'(t) = G(v)(t)$$

in a Banach space $\mathbb{X}$, where $G : C([a,b];\mathbb{X}) \to B([a,b];\mathbb{X})$ is a continuous operator and $C([a,b];\mathbb{X})$, resp. $B([a,b];\mathbb{X})$, denotes the Banach space of continuous, resp. Bochner integrable, abstract functions.

It is proved, in particular, that both initial value problems (Darboux and Cauchy) for the hyperbolic functional differential equation

$$\frac{\partial^2 u(t,x)}{\partial t\,\partial x} = F(u)(t,x)$$

with a Carathéodory right-hand side on the rectangle $[a,b] \times [c,d]$ can be rewritten as initial value problems for abstract functional differential equation with a suitable operator G and $\mathbb{X} = C([c,d];\mathbb{R})$.*

2010 Mathematics Subject Classification. 34Gxx, 34Kxx, 35Lxx.

Key words and phrases. Functional differential equation in a Banach space, hyperbolic functional differential equation, initial value problem.

რეზიუმე. ნაშრომში მოყვანილია რამდენიმე შენიშვნა, რომელიც ეხება ფუნქციონალურ-დიფერენციალურ განტოლებას

$$v'(t) = G(v)(t)$$

ბანახის $\mathbb{X}$ სივრცეში, სადაც $G : C([a,b];\mathbb{X}) \to B([a,b];\mathbb{X})$ უწყვეტი ოპერატორია, ხოლო $C([a,b];\mathbb{X})$ და $B([a,b];\mathbb{X})$, შესაბამისად, უწყვეტ და ბოხნერის აზრით ინტეგრებად აბსტრაქტულ ფუნქციათა სივრცეებია.

კერძოდ, დადგენილია, რომ დარბუსა და კოშის ამოცანები ჰიპერბოლური ტიპის ფუნქციონალურ-დიფერენციალური განტოლებისთვის

$$\frac{\partial^2 u(t,x)}{\partial t\,\partial x} = F(u)(t,x)$$

კარათეოდორის მარჯვენა მხარით $[a,b] \times [c,d]$ მართკუთხედზე შეიძლება გადაიწეროს როგორც საწყისი ამოცანა აბსტრაქტული ფუნქიონალურ-დიფერენციალური განტოლებისთვის შესაფერისი G ოპერატორითა და $\mathbb{X} = C([c,d];\mathbb{R})$ სივრცით.

*Reported on Conference "Differential Equation and Applications", September 4-7, 2017, Brno

1 Statement of problem

On the interval $[a,b]$, we consider the functional differential equation

$$v'(t) = G(v)(t) \tag{1.1}$$

in a Banach space $\langle \mathbb{X}, \|\cdot\|_{\mathbb{X}} \rangle$, where $G : C([a,b];\mathbb{X}) \to B([a,b];\mathbb{X})$ is a continuous operator[1] satisfying the local Carathéodory condition (see Definition 2.9).

Definition 1.1. By a *solution* of equation (1.1) we understand an abstract function $v : [a,b] \to \mathbb{X}$ which is strongly absolutely continuous on $[a,b]$ (see Definition 2.1), differentiable a.e. on $[a,b]$ (see Definition 2.2), and satisfies equality (1.1) almost everywhere on $[a,b]$.

Remark 1.2. In Definition 1.1:

(a) Differentiability a.e. on $[a,b]$ has to be assumed, because it does not follow from the strong absolute continuity (in general). Indeed, let $\mathbb{X} = L([0,1];\mathbb{R})$ and

$$v(t)(x) = \begin{cases} 1 & \text{if } 0 \le x \le t \le 1, \\ 0 & \text{if } 0 \le t < x \le 1. \end{cases}$$

Then v is strongly absolutely continuous on $[0,1]$, but not differentiable a.e. on $[0,1]$ (see [3, Example 7.3.9]).

(b) Solutions of equation (1.1) are understood as global and strong ones, the notions like local existence and extendability of solutions have no sense in our concept.

Remark 1.3. In the existing literature, several kinds of abstract differential equations can be found and for each of them, a solution is defined in a different way. For instance, equation (1.1) differs from frequently studied abstract differential equations of the type

$$v' = A(t)v + f(t, v_t),$$

where $A(t)$ are usually densely closed linear operators with values in $\mathbb{X}$ that generate a semigroup etc. In those cases the so-called mild solutions are considered, i.e., the solutions of the corresponding integral equation

$$v(t) = \widehat{V}(t,0)v(0) + \int_0^t \widehat{V}(t,s) f(s, v_s)\,\mathrm{d}s,$$

where $\widehat{V}(t,s)$ denotes an evolution operator for $A(t)$.

We mention here two natural and straightforward particular cases of equation (1.1):

(A) $\mathbb{X} = \mathbb{R}$ – scalar first-order functional differential equations, for example,

- differential equation with an argument deviation

  $$v'(t) = f(t, v(t), v(\tau(t))),$$

  where $f : [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ is a Carathéodory function and $\tau : [a,b] \to [a,b]$ is a measurable function,

- integro-differential equation

  $$v'(t) = \int_a^b K(t,s) v(\tau(s))\,\mathrm{d}s,$$

  where $K : [a,b] \times [a,b] \to \mathbb{R}$ and $\tau : [a,b] \to [a,b]$ are suitable functions,

[1] For definition of the spaces $C([a,b];\mathbb{X})$ and $B([a,b];\mathbb{X})$, see Section 2.

- differential equation with a maximum

$$v'(t) = p(t) \max\big\{v(s) : \ \tau_1(t) \le s \le \tau_2(s)\big\} + q(s),$$

where $p, q \in L([a,b];\mathbb{R})$ and $\tau_1, \tau_2 : [a,b] \to [a,b]$ are measurable functions.

(B) $\mathbb{X} = \mathbb{R}^n$ – systems of first-order functional differential equations and scalar higher-order functional differential equations.

For both cases $\mathbb{R}$ and $\mathbb{R}^n$, there are plenty of results concerning solvability as well as unique solvability of various boundary value problems, theorems on differential inequalities (maximum principles in other terminology), oscillations, etc. In order to extend our results from those topics (as well as our methodology) for functional differential equations in abstract spaces, some additional operations and structures are needed in $\mathbb{X}$ (like ordering, positivity, monotonicity, unit element, ...). Therefore, we are interested in other particular cases of (1.1) besides (A) and (B) that can help one to find out what operations and structures a Banach space $\mathbb{X}$ should be endowed with. We will show in Section 4 that the hyperbolic functional differential equation

$$\frac{\partial^2 u(t,x)}{\partial t\, \partial x} = F(u)(t,x)$$

with a Carathéodory right-hand side on the rectangle $[a,b] \times [c,d]$ can be regarded as a particular case of the abstract equation (1.1) with $\mathbb{X} = C([c,d];\mathbb{R})$.

2 Notation and definitions

The following notation is used throughout the paper:

(1) $\langle \mathbb{X}, \|\cdot\|_{\mathbb{X}}\rangle$ is a Banach space.

(2) $C([a,b];\mathbb{X})$ is the Banach space of continuous abstract functions $v : [a,b] \to \mathbb{X}$ endowed with the norm $\|v\|_{C([a,b];\mathbb{X})} = \max\{\|v(t)\|_{\mathbb{X}} : \ t \in [a,b]\}$.

(3) $AC([a,b];\mathbb{X})$ is the set of strongly absolutely continuous abstract functions $v : [a,b] \to \mathbb{X}$ (see Definition 2.1 below).

(4) $B([a,b];\mathbb{X})$ is the Banach space of Bochner integrable abstract functions $g : [a,b] \to \mathbb{X}$ endowed with the norm $\|g\|_{B([a,b];\mathbb{X})} = \int_a^b \|g(t)\|_{\mathbb{X}}\, \mathrm{d}t$.

(5) $L([a,b];\mathbb{R}) = B([a,b];\mathbb{R})$, see Lemma 2.7 below.

(6) $\mathcal{D} = [a,b] \times [c,d]$.

(7) $C(\mathcal{D};\mathbb{R})$ is the Banach space of continuous functions $u : \mathcal{D} \to \mathbb{R}$ endowed with the norm $\|u\|_{C(\mathcal{D};\mathbb{R})} = \max\{|u(t,x)| : \ (t,x) \in \mathcal{D}\}$.

(8) The first- and the second-order partial derivatives of the function $v : \mathcal{D} \to \mathbb{R}$ at the point $(t,x) \in \mathcal{D}$ are denoted by $v'_{[1]}(t,x)$ (or $v'_t(t,x)$, $\frac{\partial v(t,x)}{\partial t}$), $v'_{[2]}(t,x)$ (or $v'_x(t,x)$, $\frac{\partial v(t,x)}{\partial x}$), $v''_{[1,2]}(t,x)$ (or $v''_{tx}(t,x)$, $\frac{\partial v(t,x)}{\partial t\, \partial x}$), and $v''_{[2,1]}(t,x)$ (or $v''_{xt}(t,x)$, $\frac{\partial v(t,x)}{\partial x\, \partial t}$).

(9) $AC(\mathcal{D};\mathbb{R})$ is the set of functions $u : \mathcal{D} \to \mathbb{R}$ absolutely continuous in the sense of Carathéodory (see Definition 2.4 and Proposition 2.5 below).

(10) $L(\mathcal{D};\mathbb{R})$ is the Banach space of Lebesgue integrable functions $p : \mathcal{D} \to \mathbb{R}$ endowed with the norm $\|p\|_{L(\mathcal{D};\mathbb{R})} = \iint_{\mathcal{D}} |p(t,x)|\, \mathrm{d}t\, \mathrm{d}x$.

(11) $\operatorname{meas} E$ denotes the Lebesgue measure of a (measurable) set $E \subset \mathbb{R}$.

Definition 2.1 ([3, Definition 7.1.7])**.** A function $v : [a, b] \to \mathbb{X}$ is said to be *strongly absolutely continuous*, if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\sum_i \|v(b_i) - v(a_i)\|_{\mathbb{X}} < \varepsilon$ whenever $\{[a_i, b_i]\}$ is a finite system of mutually non-overlapping subintervals of $[a, b]$ that satisfies $\sum_i (b_i - a_i) < \delta$.

Definition 2.2 ([3, Definition 7.3.2])**.** A function $v : [a, b] \to \mathbb{X}$ is said to be *differentiable at the point* $t \in [a, b]$, if there is $\chi \in \mathbb{X}$ such that

$$\lim_{\delta \to 0} \left\| \frac{v(t+\delta) - v(t)}{\delta} - \chi \right\|_{\mathbb{X}} = 0.$$

We denote $\chi = v'(t)$ the derivative of v at t. If v is differentiable at every point $t \in E \subseteq [a, b]$ with $\operatorname{meas} E = b - a$, then v is called *differentialbe almost everywhere (a.e.) on* $[a, b]$.

Definition 2.3 ([1, § 7.3])**.** Let $\mathcal{S}(\mathcal{D})$ denote the system of rectangles $[t_1, t_2] \times [x_1, x_2]$ contained in $\mathcal{D}$. A mapping $\Phi : \mathcal{S}(\mathcal{D}) \to \mathbb{R}$ is said to be *absolutely continuous function of rectangles*, if it is additive and for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any finite system $\{[a_i, b_i] \times [c_i, d_i]\}$ of mutually non-overlapping rectangles contained in $\mathcal{D}$, the implication

$$\sum_i (b_i - a_i)(d_i - c_i) < \delta \implies \sum_i \big|\Phi([a_i, b_i] \times [c_i, d_i])\big| < \varepsilon$$

holds.

Definition 2.4. We say that a function $u : \mathcal{D} \to \mathbb{R}$ is *absolutely continuous in the sense of Carathéodory* if the following conditions hold:

(a) the function of rectangles

$$\Phi_u([t_1, t_2] \times [x_1, x_2]) := u(t_1, x_1) - u(t_1, x_2) - u(t_2, x_1) + u(t_2, x_2) \ \text{ for } [t_1, t_2] \times [x_1, x_2] \subseteq \mathcal{D}$$

associated with u is absolutely continuous.

(b) the functions $u(\,\cdot\,, c) : [a, b] \to \mathbb{R}$ and $u(a,\,\cdot\,) : [c, d] \to \mathbb{R}$ are absolutely continuous.

Proposition 2.5 ([4, Theorem 3.1])**.** *The following assertions are equivalent:*

(1) *The function $u : \mathcal{D} \to \mathbb{R}$ is absolutely continuous in the sense of Carathéodory.*

(2) *The function $u : \mathcal{D} \to \mathbb{R}$ admits the integral representation*

$$u(t, x) = e + \int_a^t f(s)\,\mathrm{d}s + \int_c^x q(\eta)\,\mathrm{d}\eta + \iint\limits_{[a,t]\times[c,x]} p(s, \eta)\,\mathrm{d}s\,\mathrm{d}\eta \ \textit{ for } \ (t, x) \in \mathcal{D}, \tag{2.1}$$

where $e \in \mathbb{R}$, $f \in L([a, b]; \mathbb{R})$, $q \in L([c, d]; \mathbb{R})$, and $p \in L(\mathcal{D}; \mathbb{R})$.

(3) *The function $u : \mathcal{D} \to \mathbb{R}$ satisfies the following conditions:*

(a) *$u(\,\cdot\,, x) \in AC([a, b]; \mathbb{R})$ for every $x \in [c, d]$, $u(a,\,\cdot\,) \in AC([c, d]; \mathbb{R})$,*

(b) *$u'_{[1]}(t,\,\cdot\,) \in AC([c, d]; \mathbb{R})$ for almost all $t \in [a, b]$,*

(c) *$u''_{[1,2]} \in L(\mathcal{D}; \mathbb{R})$.*

(4) *The function $u : \mathcal{D} \to \mathbb{R}$ satisfies the following conditions:*

(A) *$u(t,\,\cdot\,) \in AC([c, d]; \mathbb{R})$ for every $t \in [a, b]$, $u(\,\cdot\,, c) \in AC([a, b]; \mathbb{R})$,*

(B) *$u'_{[2]}(\,\cdot\,, x) \in AC([a, b]; \mathbb{R})$ for almost all $x \in [c, d]$,*

(C) *$u''_{[2,1]} \in L(\mathcal{D}; \mathbb{R})$.*

Lemma 2.6 ([4, Proposition 3.5]). *Let a function u be defined by formula* (2.1)*, where $e \in \mathbb{R}$, $f \in L([a,b];\mathbb{R})$, $q \in L([c,d];\mathbb{R})$, and $p \in L(\mathcal{D};\mathbb{R})$. Then there exists a measurable set $E \subseteq [a,b]$ such that* $\operatorname{meas} E = b - a$ *and*

$$u'_{[1]}(t,x) = f(t) + \int_c^x p(t,\eta)\,\mathrm{d}\eta \quad \text{for } t \in E, \quad x \in [c,d].$$

Lemma 2.7 ([3, Remark 1.3.14]). *A function $g : [a,b] \to \mathbb{R}$ is Bochner integrable if and only if it is Lebesgue integrable and the two integrals of g have the same value.*

Lemma 2.8 ([3, Theorem 1.4.3]). *If $g \in B([a,b];\mathbb{X})$, then the function $\|g(\,\cdot\,)\|_{\mathbb{X}} : [a,b] \to \mathbb{R}$ is Lebesgue integrable.*

Definition 2.9. We say that an operator $G : C([a,b];\mathbb{X}) \to B([a,b];\mathbb{X})$ satisfies the *local Carathéodory condition* if for each $r > 0$ there exists a function $q_r \in L([a,b];\mathbb{R})$ such that

$$\|G(w)(t)\|_{\mathbb{X}} \le q_r(t) \quad \text{for a.e. } t \in [a,b] \text{ and all } w \in C([a,b];\mathbb{X}), \quad \|w\|_{C([a,b];\mathbb{X})} \le r.$$

Definition 2.10. We say that an operator $F : C(\mathcal{D};\mathbb{R}) \to L(\mathcal{D};\mathbb{R})$ satisfies the *local Carathéodory condition* if for each $r > 0$ there exists a function $\zeta_r \in L(\mathcal{D};\mathbb{R})$ such that

$$|F(z)(t,x)| \le \zeta_r(t,x) \quad \text{for a.e. } (t,x) \in \mathcal{D} \text{ and all } z \in C(\mathcal{D};\mathbb{R}), \quad \|z\|_{C(\mathcal{D};\mathbb{R})} \le r.$$

3 Hyperbolic functional differential equation

On the rectangle $\mathcal{D} = [a,b] \times [c,d]$, we consider the hyperbolic functional differential equation

$$\frac{\partial^2 u(t,x)}{\partial t\,\partial x} = F(u)(t,x), \tag{3.1}$$

where $F : C(\mathcal{D};\mathbb{R}) \to L(\mathcal{D};\mathbb{R})$ is a continuous operator satisfying the local Carathéodory condition (see Definition 2.10).

Definition 3.1. By a *solution* of equation (3.1) we understand a function $u : \mathcal{D} \to \mathbb{R}$ which is absolutely continuous in the sense of Carathéodory and satisfies equality (3.1) almost everywhere on $\mathcal{D}$.

Two main initial value problems for equation (3.1) are studied in the literature.

Darboux problem

The values of the solution u are prescribed on both characteristics $t = a$ and $x = c$, i.e., the initial conditions are

$$u(t,c) = \alpha(t) \quad \text{for } t \in [a,b], \qquad u(a,x) = \beta(x) \quad \text{for } x \in [c,d], \tag{3.2}$$

where $\alpha \in AC([a,b];\mathbb{R})$, $\beta \in AC([c,d];\mathbb{R})$ are such that $\alpha(a) = \beta(c)$.

The following statement follows from the proof of [5, Theorem 4.1].

Proposition 3.2. *The function u is a solution of problem* (3.1), (3.2) *if and only if it is a solution of the integral equation*

$$u(t,x) = -\alpha(a) + \alpha(t) + \beta(x) + \int_a^t \int_c^x F(u)(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s$$

in the space $C(\mathcal{D};\mathbb{R})$.

Cauchy problem

Let $\mathcal{H}$ be a curve, which is defined as the graph of a decreasing continuous (not absolutely continuous, in general) function $h : [a,b] \to [c,d]$ such that $h(a) = d$ and $h(b) = c$. The values of the solution u and its partial derivative $u'_{[2]}$ are prescribed on $\mathcal{H}$ as follows:

$$u(t,h(t)) = \gamma(t) \ \text{ for } \ t \in [a,b], \quad u'_{[2]}(h^{-1}(x),x) = \psi(x) \ \text{ for a.e. } \ x \in [c,d], \tag{3.3}$$

where $\gamma \in C([a,b];\mathbb{R})$, $\psi \in L([c,d];\mathbb{R})$ are such that

$$\text{the function } \ t \longmapsto \gamma(t) + \int\limits_{h(t)}^{d} \psi(\eta)\,\mathrm{d}\eta \ \text{ is absolutely continuous on } [a,b] \tag{3.4}$$

(in other words, the pair (γ,ψ) is h-consistent, see [2, Section 3]).

The following statement follows from [2, Lemmas 3.3 and 3.4].

Proposition 3.3. *The function u is a solution of problem* (3.1), (3.3) *if and only if it is a solution of the integral equation*

$$u(t,x) = \gamma(t) + \int\limits_{h(t)}^{x} \psi(\eta)\,\mathrm{d}\eta + \int\limits_{h^{-1}(x)}^{t} \int\limits_{h(s)}^{x} F(u)(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s$$

in the space $C(\mathcal{D};\mathbb{R})$.

4 Main results

In this section, we formulate main results of the paper, namely, Theorems 4.1 and 4.4 showing that both Darboux and Cauchy problems for the hyperbolic equation (3.1) can be rewritten as initial value problems for the abstract equation (1.1) in the Banach space $C([c,d];\mathbb{R})$. Consequently, the hyperbolic equation (3.1) can be regarded as a particular case of (1.1) with $\mathbb{X} = C([c,d];\mathbb{R})$.

Theorem 4.1. *Let $\alpha \in AC([a,b];\mathbb{R})$, $\beta \in AC([c,d];\mathbb{R})$ be such that $\alpha(a) = \beta(c)$ and let $F : C(\mathcal{D};\mathbb{R}) \to L(\mathcal{D};\mathbb{R})$ be a continuous operator satisfying the local Carathéodory condition.*

If u is a solution of problem (3.1), (3.2), *then the function v defined by the formula*

$$v(t)(x) := u(t,x) \ \textit{ for } \ t \in [a,b], \ \ x \in [c,d], \tag{4.1}$$

is a solution of the problem

$$\begin{gathered} v'(t) = G(v)(t), \\ v(a) = \beta \end{gathered} \tag{4.2}$$

in the Banach space $C([c,d];\mathbb{R})$, where

$$\left.\begin{aligned} G(w)(t) &:= \widetilde{w}(t) \ \textit{ for a.e. } \ t \in [a,b] \ \textit{ and all } w \in C([a,b];C([c,d];\mathbb{R})), \\ \widetilde{w}(t)(x) &:= \alpha'(t) + \int\limits_{c}^{x} F(z)(t,\eta)\,\mathrm{d}\eta \ \textit{ for a.e. } \ t \in [a,b] \ \textit{ and all } x \in [c,d], \\ z(t,x) &:= w(t)(x) \ \textit{ for } \ (t,x) \in \mathcal{D}. \end{aligned}\right\} \tag{4.3}$$

Conversely, if v is a solution of problem (1.1), (4.2) *with G given by* (4.3), *then the function u defined by the formula*

$$u(t,x) := v(t)(x) \ \textit{ for } \ (t,x) \in \mathcal{D} \tag{4.4}$$

is a solution of problem (3.1), (3.2).

Remark 4.2. It follows from Propositions 5.1, 5.2, and 5.9 below that the formulation of Theorem 4.1 is correct.

Remark 4.3. Theorem 4.1 can be easily extended to a "more general" Darboux problem for equation (3.1), where the values of the solution u are prescribed on characteristics $t = t_0$ and $x = x_0$, i.e., the initial conditions are

$$u(t, x_0) = \alpha(t) \text{ for } t \in [a, b], \quad u(t_0, x) = \beta(x) \text{ for } x \in [c, d],$$

where $t_0 \in [a, b]$, $x_0 \in [c, d]$, $\alpha \in AC([a, b]; \mathbb{R})$, $\beta \in AC([c, d]; \mathbb{R})$ are such that $\alpha(t_0) = \beta(x_0)$.

Theorem 4.4. *Let $h \in C([a, b]; \mathbb{R})$ be a decreasing function such that $h(a) = d$ and $h(b) = c$. Let, moreover, $\gamma \in C([a, b]; \mathbb{R})$ and $\psi \in L([c, d]; \mathbb{R})$ be such that condition (3.4) holds and $F : C(\mathcal{D}; \mathbb{R}) \to L(\mathcal{D}; \mathbb{R})$ be a continuous operator satisfying the local Carathéodory condition.*

If u is a solution of problem (3.1), (3.3), then the function v defined by formula (4.1) is a solution of the problem

$$\begin{gathered} v'(t) = G(v)(t), \\ v(t)(h(t)) = \gamma(t) \text{ for } t \in [a, b] \end{gathered} \tag{4.5}$$

in the Banach space $C([c, d]; \mathbb{R})$, where

$$\left.\begin{aligned} G(w)(t) &:= \widetilde{w}(t) \text{ for a.e. } t \in [a, b] \text{ and all } w \in C([a, b]; C([c, d]; \mathbb{R})), \\ \widetilde{w}(t)(x) &:= \frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma(t) + \int_{h(t)}^{d} \psi(\eta)\,\mathrm{d}\eta\right) + \int_{h(t)}^{x} F(z)(t, \eta)\,\mathrm{d}\eta \text{ for a.e. } t \in [a, b] \text{ and all } x \in [c, d], \\ z(t, x) &:= w(t)(x) \text{ for } (t, x) \in \mathcal{D}. \end{aligned}\right\} \tag{4.6}$$

Conversely, if v is a solution of problem (1.1), (4.5) with G given by (4.6), then the function u defined by formula (4.4) is a solution of problem (3.1), (3.3).

Remark 4.5. It follows from Propositions 5.1, 5.2, and 5.10 below that the formulation of Theorem 4.4 is correct.

5 Proofs of main results

5.1 Auxiliary statements

We first show the properties of the relationship between abstract functions and the functions of two variables given by formulae (4.1) and (4.4).

Proposition 5.1. *Let $u \in C(\mathcal{D}; \mathbb{R})$ and the function v be defined by formula (4.1). Then $v \in C([a, b]; C([c, d]; \mathbb{R}))$.*

Proof. It follows easily from the definitions of continuity. □

Proposition 5.2. *Let $v \in C([a, b]; C([c, d]; \mathbb{R}))$ and the function u be defined by formula (4.4). Then $u \in C(\mathcal{D}; \mathbb{R})$.*

Proof. Let $(t_0, x_0) \in \mathcal{D}$ be arbitrary and let $\{(t_n, x_n)\}_{n=1}^{+\infty}$ be a sequence of points from the rectangle $\mathcal{D}$ such that $(t_n, x_n) \to (t_0, x_0)$ as $n \to +\infty$. Then, clearly,

$$\lim_{n\to+\infty} t_n = t_0, \quad \lim_{n\to+\infty} x_n = x_0.$$

Let $\varepsilon > 0$ be arbitrary. Since $v \in C([a, b]; C([c, d]; \mathbb{R}))$ and $v(t_0) \in C([c, d]; \mathbb{R})$, there exists $n_0 \in \mathbb{N}$ such that

$$\|v(t_n) - v(t_0)\|_{C([c,d];\mathbb{R})} < \frac{\varepsilon}{2}, \quad |v(t_0)(x_n) - v(t_0)(x_0)| < \frac{\varepsilon}{2} \text{ for } n \ge n_0$$

which yields

$$|v(t_n)(x) - v(t_0)(x)| < \frac{\varepsilon}{2} \ \text{ for } \ x \in [c,d], \ \ n \geq n_0.$$

Consequently, we get

$$|u(t_n, x_n) - u(t_0, x_0)| \leq |v(t_n)(x_n) - v(t_0)(x_n)| + |v(t_0)(x_n) - v(t_0)(x_0)| \leq \varepsilon$$

for $n \geq n_0$ and thus, $\lim\limits_{n\to+\infty} u(t_n, x_n) = u(t_0, x_0)$. □

Proposition 5.3. *Let $u \in AC(\mathcal{D}; \mathbb{R})$. Then the function v defined by formula* (4.1) *is strongly absolutely continuous, i.e., $v \in AC([a,b]; C([c,d]; \mathbb{R}))$.*

Proof. It follows from Proposition 2.5 that the function u admits the integral representation (2.1), where $e \in \mathbb{R}$, $f \in L([a,b]; \mathbb{R})$, $q \in L([c,d]; \mathbb{R})$, and $p \in L(\mathcal{D}; \mathbb{R})$.

Let $\varepsilon > 0$ be arbitrary. Since the function

$$t \longmapsto |f(t)| + \int_c^d |p(t,\eta)| \,\mathrm{d}\eta$$

is Lebesgue integrable on $[a,b]$, there exists $\delta > 0$ such that

$$\int_E \bigg(|f(s)| + \int_c^d |p(s,\eta)| \,\mathrm{d}\eta \bigg) \mathrm{d}s < \varepsilon \ \text{ for } \ E \subseteq [a,b], \ \operatorname{meas} E < \delta. \tag{5.1}$$

Let $\{[a_k, b_k]\}_{k=1}^n$ be an arbitrary system of mutually non-overlapping subintervals of $[a,b]$ such that

$$\sum_{k=1}^n (b_k - a_k) < \delta.$$

By virtue of (2.1) and (4.1), it is clear that

$$v(b_k)(x) - v(a_k)(x) = \int_{a_k}^{b_k} \bigg(f(s) + \int_c^x p(s,\eta) \,\mathrm{d}\eta \bigg) \mathrm{d}s \ \text{ for } \ x \in [c,d], \ \ k = 1, \dots, n,$$

and thus, we get

$$\begin{aligned}\sum_{k=1}^n \|v(b_k) - v(a_k)\|_{C([c,d];\mathbb{R})} &= \sum_{k=1}^n \max \bigg\{ \bigg| \int_{a_k}^{b_k} \bigg(f(s) + \int_c^x p(s,\eta) \,\mathrm{d}\eta \bigg) \mathrm{d}s \bigg| : \ x \in [c,d] \bigg\} \\ &\leq \int_A \bigg(|f(s)| + \int_c^d |p(s,\eta)| \,\mathrm{d}\eta \bigg) \mathrm{d}s, \end{aligned} \tag{5.2}$$

where $A := \bigcup\limits_{k=1}^n [a_k, b_k]$. Since $\operatorname{meas} A = \sum\limits_{k=1}^n (b_k - a_k) < \delta$, it follows from (5.1) and (5.2) that

$$\sum_{k=1}^n \|v(b_k) - v(a_k)\|_{C([c,d];\mathbb{R})} < \varepsilon.$$

□

Lemma 5.4. *Let $q \in L(\mathcal{D}; \mathbb{R})$ be such that*

$$q(t,x) \geq 0 \ \textit{ for a.e. } \ (t,x) \in \mathcal{D}. \tag{5.3}$$

Then there exists a measurable set $E \subseteq [a,b]$ *such that* $\operatorname{meas} E = b-a$ *and for each* $t \in E$ *the condition*

$$\lim_{\delta\to 0} \frac{1}{\delta} \int_t^{t+\delta} \left(\int_c^x q(s,\eta)\,\mathrm{d}\eta \right) \mathrm{d}s = \int_c^x q(t,\eta)\,\mathrm{d}\eta \quad \textit{uniformly on } [c,d] \tag{5.4}$$

holds.

Proof. It follows from Lemma 2.6 that there exists a measurable set $E \subseteq \,]a,b[$ such that $\operatorname{meas} E = b-a$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_a^t \int_c^x q(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s = \int_c^x q(t,\eta)\,\mathrm{d}\eta \quad \text{for } \; t \in E, \quad x \in [c,d],$$

i.e.,

$$\lim_{\delta\to 0} \frac{1}{\delta} \int_t^{t+\delta} \left(\int_c^x q(s,\eta)\,\mathrm{d}\eta \right) \mathrm{d}s = \int_c^x q(t,\eta)\,\mathrm{d}\eta \quad \text{for } \; (t,x) \in E \times [c,d]. \tag{5.5}$$

Let $t_0 \in E$ and $\varepsilon > 0$ be arbitrary. Since $q(t_0,\,\cdot\,) \in L([c,d];\mathbb{R})$, there exists $\zeta > 0$ such that

$$\int_{c_1}^{d_1} q(t_0,\eta)\,\mathrm{d}\eta < \frac{\varepsilon}{2} \quad \text{for } \; c_1, d_1 \in [c,d], \quad |d_1 - c_1| < \zeta. \tag{5.6}$$

It is clear that there is a collection $x_1, x_2, \dots, x_n \in [c,d]$ such that $c = x_1 < x_2 < \dots < x_n = d$ and

$$\max\{x_{k+1} - x_k : k = 1, \dots, n-1\} < \zeta.$$

Condition (5.5) yields that for each $k \in \{1, \dots, n\}$, there exists $\zeta_k > 0$ such that

$$\left| \frac{1}{\delta} \int_{t_0}^{t_0+\delta} \left(\int_c^{x_k} q(s,\eta)\,\mathrm{d}\eta \right) \mathrm{d}s - \int_c^{x_k} q(t_0,\eta)\,\mathrm{d}\eta \right| < \frac{\varepsilon}{2} \quad \text{for } \; 0 < |\delta| < \zeta_k. \tag{5.7}$$

Put $\zeta_0 := \min\{\zeta_k : k = 1, \dots, n\}$ and let $x_0 \in [c,d]$ be arbitrary. It is clear that there exists $m \in \{1, \dots, n-1\}$ such that $x_m \le x_0 \le x_{m+1}$. According to assumption (5.3), the function

$$x \longmapsto \frac{1}{\delta} \int_{t_0}^{t_0+\delta} \left(\int_c^x q(s,\eta)\,\mathrm{d}\eta \right) \mathrm{d}s \quad \text{is non-decreasing on } [c,d]$$

and thus, by virtue of (5.6) and (5.7), for any $\delta \in \mathbb{R}$ satisfying $0 < |\delta| < \zeta_0$, we get

$$\begin{aligned}
&\frac{1}{\delta} \int_{t_0}^{t_0+\delta} \left(\int_c^{x_0} q(s,\eta)\,\mathrm{d}\eta \right) \mathrm{d}s - \int_c^{x_0} q(t_0,\eta)\,\mathrm{d}\eta \\
&\qquad \le \frac{1}{\delta} \int_{t_0}^{t_0+\delta} \left(\int_c^{x_{m+1}} q(s,\eta)\,\mathrm{d}\eta \right) \mathrm{d}s - \int_c^{x_{m+1}} q(t_0,\eta)\,\mathrm{d}\eta + \int_{x_0}^{x_{m+1}} q(t_0,\eta)\,\mathrm{d}\eta \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

and

$$\begin{aligned}
&\int_c^{x_0} q(t_0,\eta)\,\mathrm{d}\eta - \frac{1}{\delta} \int_{t_0}^{t_0+\delta} \left(\int_c^{x_0} q(s,\eta)\,\mathrm{d}\eta \right) \mathrm{d}s \\
&\qquad \le \int_{x_m}^{x_0} q(t_0,\eta)\,\mathrm{d}\eta + \int_c^{x_m} q(t_0,\eta)\,\mathrm{d}\eta - \frac{1}{\delta} \int_{t_0}^{t_0+\delta} \left(\int_c^{x_m} q(s,\eta)\,\mathrm{d}\eta \right) \mathrm{d}s \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

However, it means that

$$\left|\frac{1}{\delta}\int_{t_0}^{t_0+\delta}\left(\int_c^{x_0} q(s,\eta)\,\mathrm{d}\eta\right)\mathrm{d}s-\int_c^{x_0} q(t_0,\eta)\,\mathrm{d}\eta\right|<\varepsilon \ \text{ for } \ 0<|\delta|<\zeta_0.$$

Since x_0 is arbitrary and ζ_0 does not depend on x_0, we have

$$\left|\frac{1}{\delta}\int_{t_0}^{t_0+\delta}\left(\int_c^{x} q(s,\eta)\,\mathrm{d}\eta\right)\mathrm{d}s-\int_c^{x} q(t_0,\eta)\,\mathrm{d}\eta\right|<\varepsilon \ \text{ for } \ 0<|\delta|<\zeta_0, \ \ x\in[c,d],$$

i.e., the desired condition (5.4) holds for every $t\in E$. □

Proposition 5.5. *Let $u\in AC(\mathcal{D};\mathbb{R})$. Then the function v defined by formula* (4.1) *is differentiable a.e. on $[a,b]$ and*

$$v'(t)=u'_{[1]}(t,\,\cdot\,) \ \text{ for a.e. } \ t\in[a,b]. \tag{5.8}$$

Proof. It follows from Proposition 2.5 that the function u admits the integral representation (2.1), where $e\in\mathbb{R}$, $f\in L([a,b];\mathbb{R})$, $q\in L([c,d];\mathbb{R})$, and $p\in L(\mathcal{D};\mathbb{R})$.

By virtue of Lemma 2.6, there exists a measurable set $E_1\subseteq[a,b]$ such that $\operatorname{meas}E_1=b-a$ and

$$u'_{[1]}(t,x)=f(t)+\int_c^x p(t,\eta)\,\mathrm{d}\eta \ \text{ for } \ t\in E_1, \ \ x\in[c,d], \tag{5.9}$$

$$\lim_{\delta\to0}\frac{1}{\delta}\int_t^{t+\delta} f(s)\,\mathrm{d}s=\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t f(s)\,\mathrm{d}s=f(t) \ \text{ for } \ t\in E_1. \tag{5.10}$$

Moreover, it follows from Lemma 5.4 with $q:=\frac{|p|+p}{2}$ and $q:=\frac{|p|-p}{2}$ that there exists a measurable set $E_2\subseteq[a,b]$ such that $\operatorname{meas}E_2=b-a$ and for every $t\in E_2$, relation (5.4) holds.

Put $E=E_1\cap E_2$ and let $t_0\in E$ be arbitrary. In view of (4.1) and (5.9), from (2.1) we get

$$\begin{aligned}
&\left|\frac{v(t_0+\delta)(x)-v(t_0)(x)}{\delta}-u'_{[1]}(t_0,x)\right|\\
&\qquad\le\left|\frac{1}{\delta}\int_{t_0}^{t_0+\delta} f(s)\,\mathrm{d}s+\frac{1}{\delta}\int_{t_0}^{t_0+\delta}\left(\int_c^x p(s,\eta)\,\mathrm{d}\eta\right)\mathrm{d}s-f(t_0)-\int_c^x p(t_0,\eta)\,\mathrm{d}\eta\right|\\
&\qquad\qquad\le\left|\frac{1}{\delta}\int_{t_0}^{t_0+\delta} f(s)\,\mathrm{d}s-f(t_0)\right|+\left|\frac{1}{\delta}\int_{t_0}^{t_0+\delta}\left(\int_c^x p(s,\eta)\,\mathrm{d}\eta\right)\mathrm{d}s-\int_c^x p(t_0,\eta)\,\mathrm{d}\eta\right|
\end{aligned}$$

for $x\in[c,d]$ and $\delta\neq0$ small enough which, together with (5.10) and (5.4) with $t:=t_0$, guarantees that

$$\lim_{\delta\to0}\left\|\frac{v(t_0+\delta)-v(t_0)}{\delta}-u'_{[1]}(t_0,\,\cdot\,)\right\|_{C([c,d];\mathbb{R})}=0.$$

However, it means that the abstract function v is differentiable at t_0 and, moreover, $v'(t_0)=u'_{[1]}(t_0,\,\cdot\,)$. To conclude the proof it is sufficient to mention that $t_0\in E$ was arbitrary and $\operatorname{meas}E=b-a$. □

Now we provide several statements concerning Bochner integrable abstract functions and their primitives.

Lemma 5.6. *Let $g\in B([a,b];\mathbb{X})$ and $F(t):=\int_a^t g(s)\,\mathrm{d}s$ for $t\in[a,b]$. Then $F\in AC([a,b];\mathbb{X})$, F is differentiable a.e. on $[a,b]$, and*

$$F'(t)=g(t) \ \text{ for a.e. } \ t\in[a,b]. \tag{5.11}$$

Proof. The assertion of the lemma follows from Theorems 7.4.9, 7.4.11, and 5.3.4 stated in [3]. □

Lemma 5.7 ([3, Theorem 7.4.13 and 5.3.4])**.** *Let $F \in AC([a,b];\mathbb{X})$ be differentiable a.e. on $[a,b]$ and condition* (5.11) *hold. Then $g \in B([a,b];\mathbb{X})$ and*

$$F(t) = F(a) + \int_a^t g(s)\,\mathrm{d}s \quad \text{for} \quad t \in [a,b]. \tag{5.12}$$

Proposition 5.8. *Let $f \in L([a,b];\mathbb{R})$ and $p \in L(\mathcal{D};\mathbb{R})$. For a.e. $t \in [a,b]$, we put*

$$g(t)(x) := f(t) + \int_c^x p(t,\eta)\,\mathrm{d}\eta \quad \text{for} \quad x \in [c,d]. \tag{5.13}$$

Then $g \in B([a,b];C([c,d];\mathbb{R}))$ and for each $t \in [a,b]$, the equality

$$\Big(\int_a^t g(s)\,\mathrm{d}s\Big)(x) = \int_a^t f(s)\,\mathrm{d}s + \int_a^t \int_c^x p(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s \quad \text{for} \quad x \in [c,d]^{\,2} \tag{5.14}$$

holds.

Proof. Observe that the abstract function $g : [a,b] \to C([c,d];\mathbb{R})$ is defined a.e. on $[a,b]$. Put

$$u(t,x) := \int_a^t f(s)\,\mathrm{d}s + \int_a^t \int_c^x p(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s \quad \text{for} \quad (t,x) \in \mathcal{D} \tag{5.15}$$

and define the function v by formula (4.1). It follows from Proposition 2.5 that $u \in AC(\mathcal{D};\mathbb{R})$ and, in view of Lemma 2.6, we have

$$u'_{[1]}(t,x) = f(t) + \int_c^x p(t,\eta)\,\mathrm{d}\eta \quad \text{for a.e. } t \in [a,b] \text{ and all } x \in [c,d]. \tag{5.16}$$

On the other hand, Proposition 5.5 yields that the abstract function $v : [a,b] \to C([c,d];\mathbb{R})$ is differentiable a.e. on $[a,b]$ and condition (5.8) holds which, together with (5.13) and (5.16), guarantees that

$$v'(t) = g(t) \quad \text{for a.e.} \quad t \in [a,b]. \tag{5.17}$$

Moreover, according to Proposition 5.3, $v \in AC([a,b];C([c,d];\mathbb{R}))$ and thus, it follows from (5.17) and Lemma 5.7 that $g \in B([a,b];C([c,d];\mathbb{R}))$ and

$$v(t) = v(a) + \int_a^t g(s)\,\mathrm{d}s \quad \text{for} \quad t \in [a,b].$$

However, in view of (4.1) and (5.15), it means that for each $t \in [a,b]$, equality (5.14) holds. □

At the end of this section, we provide two statements guaranteeing that formulations of Theorems 4.1 and 4.4 are correct.

Proposition 5.9. *Let $\alpha \in AC([a,b];\mathbb{R})$ and $F : C(\mathcal{D};\mathbb{R}) \to L(\mathcal{D};\mathbb{R})$ be a continuous operator satisfying the local Carathéodory condition* (*see Definition* 2.10)*. Then the operator G defined by formula* (4.3) *maps the set $C([a,b];C([c,d];\mathbb{R}))$ into the set $B([a,b];C([c,d];\mathbb{R}))$, it is continuous and satisfies the local Carathéodory condition* (*see Definition* 2.9)*.*

[2] Observe that the integral on the left-hand side of the equality is Bochner one, whereas both integrals on its righ-hand side are Lebesgue ones.

Proof. Let $w \in C([a,b];C([c,d];\mathbb{R}))$ be arbitrary and put $z(t,x) := w(t)(x)$ for $(t,x) \in \mathcal{D}$. Observe that, in view of Proposition 5.2, we have $z \in C(\mathcal{D};\mathbb{R})$. It follows from Proposition 5.8 with $f := \alpha'$ and $p := F(z)$ that $G(w) \in B([a,b];C([c,d];\mathbb{R}))$ and thus, the operator G maps $C([a,b];C([c,d];\mathbb{R}))$ into $B([a,b];C([c,d];\mathbb{R}))$.

Now let $v_n, v \in C([a,b];C([c,d];\mathbb{R}))$, $n \in \mathbb{N}$, be such that $\lim\limits_{n\to+\infty} \|v_n - v\|_{C([a,b];C([c,d];\mathbb{R}))} = 0$. Put

$$u_n(t,x) := v_n(t)(x), \quad u(t,x) := v(t)(x) \ \text{ for } \ (t,x) \in \mathcal{D}, \quad n \in \mathbb{N}. \tag{5.18}$$

Then, by virtue of Proposition 5.2, we have $u_n, u \in C(\mathcal{D};\mathbb{R})$ for $n \in \mathbb{N}$ and, moreover, it is not difficult to verify that

$$\lim_{n\to+\infty} \|u_n - u\|_{C(\mathcal{D};\mathbb{R})} = 0.$$

Since the operator F is supposed to be continuous, the latter relation yields

$$\lim_{n\to+\infty} \iint\limits_{\mathcal{D}} |F(u_n)(s,\eta) - F(u)(s,\eta)| \,\mathrm{d}s\,\mathrm{d}\eta = 0. \tag{5.19}$$

According to (4.3) and (5.18), it is clear that

$$\begin{aligned}\|G(v_n)(t) - G(v)(t)\|_{C([c,d];\mathbb{R})} &= \max\left\{\left|\int_c^x F(u_n)(t,\eta)\,\mathrm{d}\eta - \int_c^x F(u_n)(t,\eta)\,\mathrm{d}\eta\right| : \ x \in [c,d]\right\} \\ &\le \int_c^d |F(u_n)(t,\eta) - F(u_n)(t,\eta)|\,\mathrm{d}\eta \ \text{ for a.e. } \ t \in [a,b], \quad n \in \mathbb{N},\end{aligned}$$

whence we get

$$\begin{aligned}&\|G(v_n) - G(v)\|_{B([a,b];C([c,d];\mathbb{R}))} \\ &\qquad = \int_a^b \|G(v_n)(t) - G(v)(t)\|_{C([c,d];\mathbb{R})}\,\mathrm{d}t \le \iint\limits_{\mathcal{D}} |F(u_n)(s,\eta) - F(u)(s,\eta)|\,\mathrm{d}s\,\mathrm{d}\eta \ \text{ for } \ n \in \mathbb{N}.\end{aligned}$$

However, the latter inequality and (5.19) guarantee that $\lim\limits_{n\to+\infty} \|G(v_n) - G(v)\|_{B([a,b];C([c,d];\mathbb{R}))} = 0$, i.e., the operator G is continuous.

Finally, let $r > 0$ be arbitrary and $\zeta_r \in L(\mathcal{D};\mathbb{R})$ be the function appearing in the Carathéodory condition for the operator F (see Definition 2.10). Let, moreover, $w \in C([a,b];C([c,d];\mathbb{R}))$ be an arbitrary function such that $\|w\|_{C([a,b];C([c,d];\mathbb{R}))} \le r$ and put $z(t,x) := w(t)(x)$ for $(t,x) \in \mathcal{D}$. In view of Proposition 5.2, we have $z \in C(\mathcal{D};\mathbb{R})$ and, moreover, it is not difficult to verify that $\|z\|_{C(\mathcal{D};\mathbb{R})} \le r$.

Then

$$\begin{aligned}\|G(w)(t)\|_{C([c,d];\mathbb{R})} &= \max\left\{\left|\alpha'(t) + \int_c^x F(z)(t,\eta)\,\mathrm{d}\eta\right| : \ x \in [c,d]\right\} \\ &\le |\alpha'(t)| + \int_c^d |F(z)(t,\eta)|\,\mathrm{d}\eta \le |\alpha'(t)| + \int_c^d \zeta_r(t,\eta)\,\mathrm{d}\eta\end{aligned}$$

for a.e. $t \in [a,b]$. Since the function

$$t \longmapsto |\alpha'(t)| + \int_c^d \zeta_r(t,\eta)\,\mathrm{d}\eta \ \text{ is Lebesgue integrable on } [a,b],$$

the operator G satisfies the local Carathéodory condition with the function $q_r = |\alpha'| + \int_c^d \zeta_r(\,\cdot\,,\eta)\,\mathrm{d}\eta$ (see Definition 2.9). $\square$

Proposition 5.10. *Let $h \in C([a,b];\mathbb{R})$ be a decreasing function such that $h(a) = d$ and $h(b) = c$. Let, moreover, $\gamma \in C([a,b];\mathbb{R})$ and $\psi \in L([c,d];\mathbb{R})$ be such that condition* (3.4) *holds and $F : C(\mathcal{D};\mathbb{R}) \to L(\mathcal{D};\mathbb{R})$ be a continuous operator satisfying the local Carathéodory condition* (*see Definition* 2.10). *Then the operator G defined by formula* (4.6) *maps $C([a,b];C([c,d];\mathbb{R}))$ into $B([a,b];C([c,d];\mathbb{R}))$, it is continuous and satisfies the local Carathéodory condition* (*see Definition* 2.9).

Proof. Let $w \in C([a,b];C([c,d];\mathbb{R}))$ be arbitrary. Put $z(t,x) := w(t)(x)$ for $(t,x) \in \mathcal{D}$ and

$$H := \big\{(s,\eta) \in \mathcal{D} :\ a \le s \le b,\ c \le \eta \le h(s)\big\}.$$

Observe that, in view of Proposition 5.2, we have $z \in C(\mathcal{D};\mathbb{R})$. Since $F(z) \in L(\mathcal{D};\mathbb{R})$, it is easy to see that

$$\iint\limits_H F(z)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta = \int\limits_a^b \bigg(\int\limits_c^{h(s)} F(z)(s,\eta)\,\mathrm{d}\eta\bigg)\,\mathrm{d}s$$

and thus,

$$\text{the function } t \longmapsto \int\limits_c^{h(t)} F(z)(t,\eta)\,\mathrm{d}\eta \text{ is Lebesgue integrable on } [a,b]. \tag{5.20}$$

Now we put

$$\varphi(t) := \frac{\mathrm{d}}{\mathrm{d}t}\bigg(\gamma(t) + \int\limits_{h(t)}^d \psi(\eta)\,\mathrm{d}\eta\bigg). \tag{5.21}$$

Clearly, $\varphi \in L([a,b];\mathbb{R})$ because we assume that condition (3.4) holds. Therefore, it follows from Proposition 5.8 with $f := \varphi - \int\limits_c^{h(\cdot)} F(z)(\,\cdot\,,\eta)\,\mathrm{d}\eta$ and $p := F(z)$ that $G(w) \in B([a,b];C([c,d];\mathbb{R}))$ and thus, the operator G maps $C([a,b];C([c,d];\mathbb{R}))$ into $B([a,b];C([c,d];\mathbb{R}))$.

Analogously to the proof of Proposition 5.9, we show that the operator G is continuous and satisfies the local Carathéodory condition with the function $q_r = \varphi + \int\limits_c^d \zeta_r(\,\cdot\,,\eta)\,\mathrm{d}\eta$ (see Definition 2.9), where $\zeta_r \in L(\mathcal{D};\mathbb{R})$ is the function appearing in the Carathéodory condition for the operator F (see Definition 2.10). ☐

5.2 Proofs of Theorems 4.1 and 4.4

Proof of Theorem 4.1. Let u be a solution of problem (3.1), (3.2) and let the function v be defined by formula (4.1). In view of Proposition 3.2, it follows from Lemma 2.6 that

$$u'_{[1]}(t,x) = \alpha'(t) + \int\limits_c^x F(u)(t,\eta)\,\mathrm{d}\eta \ \text{ for a.e. } t \in [a,b] \text{ and all } x \in [c,d]. \tag{5.22}$$

On the other hand, Propositions 5.3 and 5.5 yield that the abstract function $v : [a,b] \to C([c,d];\mathbb{R})$ is strongly absolutely continuous, differentiable a.e. on $[a,b]$, and satisfies condition (5.8). Hence, from (5.8) and (5.22) we get

$$v'(t) = \alpha'(t) + \int\limits_c^{\cdot} F(u)(t,\eta)\,\mathrm{d}\eta = G(v)(t) \ \text{ for a.e. } t \in [a,b],$$

where the operator G is defined by formula (4.3). Moreover, $v(a) = u(a,\,\cdot\,) = \beta$ and thus, the function v is a solution of problem (1.1), (4.2) in the Banach space $C([c,d];\mathbb{R})$.

Conversely, assume that v is a solution of problem $(1.1), (4.2)$ with G given by (4.3) and define the function u by formula (4.4). Since the function v is strongly absolutely continuous, according to Proposition 5.2, we have $u \in C(\mathcal{D};\mathbb{R})$. It follows immediately from Lemma 5.7 that

$$v(t) = v(a) + \int_a^t G(v)(s)\,\mathrm{d}s \quad \text{for} \quad t \in [a,b], \tag{5.23}$$

i.e.,

$$v(t) = \beta + \int_a^t g(s)\,\mathrm{d}s \quad \text{for} \quad t \in [a,b], \tag{5.24}$$

where the function $g : [a,b] \to C([c,d];\mathbb{R})$ is for a. a. $t \in [a,b]$ defined by formula (5.13) with $f := \alpha'$ and $p := F(u)$. Therefore, by virtue of Proposition 5.8, we get

$$\left(\int_a^t g(s)\,\mathrm{d}s\right)(x) = \int_a^t \alpha'(s)\,\mathrm{d}s + \int_a^t \int_c^x F(u)(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s \quad \text{for} \quad (t,x) \in \mathcal{D}$$

which, together with (4.4) and (5.24), yields that

$$u(t,x) = -\alpha(a) + \alpha(t) + \beta(x) + \int_a^t \int_c^x F(u)(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s \quad \text{for} \quad (t,x) \in \mathcal{D}.$$

Consequently, according to Proposition 3.2, the function u is a solution of problem $(3.1), (3.2)$. □

Proof of Theorem 4.4. Define the function φ by formula (5.21). It is clear that $\varphi \in L([a,b];\mathbb{R})$ because we assume that condition (3.4) holds.

Let u be a solution of problem $(3.1), (3.3)$ and let the function v be defined by formula (4.1). In view of Proposition 3.3, it follows from Lemma 2.6 that

$$u'_{[1]}(t,x) = \varphi(t) + \int_{h(t)}^x F(u)(t,\eta)\,\mathrm{d}\eta \quad \text{for a.e.} \ t \in [a,b] \text{ and all } x \in [c,d]. \tag{5.25}$$

On the other hand, Propositions 5.3 and 5.5 yield that the abstract function $v : [a,b] \to C([c,d];\mathbb{R})$ is strongly absolutely continuous, differentiable a.e. on $[a,b]$, and satisfies condition (5.8). Hence, from (5.8) and (5.25) we get

$$v'(t) = \varphi(t) + \int_{h(t)}^{\cdot} F(u)(t,\eta)\,\mathrm{d}\eta = G(v)(t) \quad \text{for a.e.} \quad t \in [a,b],$$

where the operator G is defined by formula (4.6). Moreover, $v(t)(h(t)) = u(t,h(t)) = \gamma(t)$ for $t \in [a,b]$ and thus, the function v is a solution of problem $(1.1), (4.5)$ in the Banach space $C([c,d];\mathbb{R})$.

Conversely, assume that v is a solution of problem $(1.1), (4.5)$ with G given by (4.6) and define the function u by formula (4.4). Since the function v is strongly absolutely continuous, according to Proposition 5.2, we have $u \in C(\mathcal{D};\mathbb{R})$. Analogously to the proof of Proposition 5.10, we show that

$$\text{the function} \quad t \longmapsto \int_c^{h(t)} F(u)(t,\eta)\,\mathrm{d}\eta \quad \text{is Lebesgue integrable on } [a,b].$$

It follows immediately from Lemma 5.7 that condition (5.23) holds, i.e.,

$$v(t) = v(a) + \int_a^t g(s)\,\mathrm{d}s \quad \text{for} \quad t \in [a,b], \tag{5.26}$$

where the function $g : [a,b] \to C([c,d];\mathbb{R})$ is for a.a. $t \in [a,b]$ defined by formula (5.13) with $f := \varphi - \int\limits_c^{h(\,\cdot\,)} F(u)(\,\cdot\,,\eta)\,\mathrm{d}\eta$ and $p := F(u)$. Therefore, by virtue of Proposition 5.8, we get

$$\begin{aligned}\Big(\int\limits_a^t g(s)\,\mathrm{d}s\Big)(x) &= \int\limits_a^t \Big(\varphi(s) - \int\limits_c^{h(s)} F(u)(s,\eta)\,\mathrm{d}\eta\Big)\,\mathrm{d}s + \int\limits_a^t \int\limits_c^x F(u)(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s \\ &= \gamma(t) + \int\limits_{h(t)}^d \psi(\eta)\,\mathrm{d}\eta - \gamma(a) + \int\limits_a^t \int\limits_{h(s)}^x F(u)(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s\end{aligned}$$

for $(t,x) \in \mathcal{D}$ which, together with (4.4) and (5.26), yields that

$$u(t,x) = u(a,x) + \gamma(t) - \gamma(a) + \int\limits_{h(t)}^d \psi(\eta)\,\mathrm{d}\eta + \int\limits_a^t \int\limits_{h(s)}^x F(u)(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s \quad \text{for} \quad (t,x) \in \mathcal{D}. \tag{5.27}$$

It follows from the initial condition (4.5) that $u(h^{-1}(x),x) = \gamma(h^{-1}(x))$ for $x \in [c,d]$. Therefore, substituting $h^{-1}(x)$ for t in equality (5.27), we get

$$u(a,x) = \gamma(a) - \int\limits_x^d \psi(\eta)\,\mathrm{d}\eta - \int\limits_a^{h^{-1}(x)} \int\limits_{h(s)}^x F(u)(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s \quad \text{for} \quad x \in [c,d].$$

Hence, (5.27) implies

$$u(t,x) = \gamma(t) + \int\limits_{h(t)}^x \psi(\eta)\,\mathrm{d}\eta + \int\limits_{h^{-1}(x)}^t \int\limits_{h(s)}^x F(u)(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s \quad \text{for} \quad (t,x) \in \mathcal{D}.$$

Consequently, according to Proposition 3.3, the function u is a solution of problem (3.1), (3.3). $\square$

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(Received 11.10.2017)

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