Memoirs on Differential Equations and Mathematical Physics

Volume 74, 2018, 113–124

D. E. Limanska, G. E. Samkova

ON THE EXISTENCE OF ANALYTIC SOLUTIONS OF CERTAIN TYPES OF SYSTEMS, PARTIALLY RESOLVED RELATIVELY TO THE DERIVATIVES IN THE CASE OF A POLE

Abstract. For the systems of ordinary differential equations which are partially resolved relatively to the derivatives in the case of a pole, the theorems on the existence of at least one analytic in the complex domain solution of the Cauchy problem with an additional condition are established. Moreover, the asymptotic behavior of these solutions in this domain is studied.

2010 Mathematics Subject Classification. 39A45.

Key words and phrases. Ordinary differential equation, pole, Cauchy's problem, complex domain, singularity, asymptotic behavior, analytic solution.

რეზიუმე. ჩვეულებრივი დიფერენციალური განტოლებების სისტემებისთვის, რომლებიც ნაწილობრივ ამოხსნადია წარმოებულების მიმართ პოლუსის შემთხვევაში, დამტკიცებულია კომპლექსურ არეში ერთი მაინც ანალიზური ამონახსნის არსებობის თეორემები კოშის ამოცანისთვის დამატებითი პირობით. გარდა ამისა, შესწავლილია ასეთი ამონახსნების ასიმპტოტური ყოფაქცევა ამ არეში.

Introduction

R. Fuchs, Ch. Beriot, J. Bouquet, A. Lyapunov, H. Poincare, P. Painleve are the founders of the theory that investigates the behavior of solutions of systems of ordinary differential equations in the neighborhood of the singularity.

A separate class of problems in this area is the study of the existence and asymptotic behavior of solutions of systems of differential equations that are not resolved relatively to the derivatives. Certain types of systems not resolved relatively to the derivatives in a complex domain were investigated by such scientists as M. Jwano [4], O Song Guk, Pak Ponk, Chol Permissible [12], V. Gromak and many others.

One of the methods studying the systems of differential equations that are not resolved relatively to the derivatives in the real-valued domain was suggested by R. Grabovskaya [3] and J. Diblic [1, 2]. Later, this method in the case of a complex domain was developed by G. Samkova [7,8], N. Sharay [10], E. Michalenko, D. Limanska [5, 6] and others. The present article is a continuation of the research devoted to the systems of differential equations that are not resolved relatively to the derivatives in a complex domain.

Let us consider the system of ordinary differential equations

$$A(z)Y' = B(z)Y + f(z, Y, Y'), \tag{0.1}$$

where the matrices $A, B : D_1 \to \mathbb{C}^{p\times n}$, $D_1 = \{z \in \mathbb{C} : \ |z| < R_1, \ R_1 > 0\}$, the matrices $A(z)$, $B(z)$ are analytic in the domain D_{10}, $D_{10} = D_1 \setminus \{0\}$, the pencil of matrices $A(z)\lambda - B(z)$ is singular as $z \to 0$, the vector-function $f : D_1 \times G_1 \times G_2 \to \mathbb{C}^p$, where domains $G_k \subset \mathbb{C}^n$, $0 \in G_k$, $k = 1, 2$, the function $f(z, Y, Y')$ is analytic in the domain $D_{10} \times G_{10} \times G_{20}$, $G_{k0} = G_k \setminus \{0\}$, $k = 1, 2$.

The main goal of our paper is to establish the existence and to study the asymptotic behavior of solutions of the system of differential equations (0.1) in the domain with the point $z = 0$ on its border, under the conditions that $p < n$, the matrix $A(z)$ is analytic in the domain D_1 and $\operatorname{rank} A(z) = p$ in this domain.

1 On some singular Cauchy problem for a system of ordinary differential equations, not resolved relatively to the derivatives

Let us consider the system of differential equations

$$z^l Y_1' = z^l P(z)Y_1 + F(z, Y_1, Y_1'), \tag{1.1}$$

where $l \in \mathbb{Z}$, $Y_1 = \operatorname{col}(Y_{11}(z), \dots, Y_{1p}(z))$, $Y_1 : D_1 \to \mathbb{C}^p$, the matrix $P(z)$ is analytic in the domain D_1, $F : D_1 \times G_{11} \times G_{21} \to \mathbb{C}^p$, $G_{j1} \subset \mathbb{C}^p$, $j = 1, 2$, $F(z, Y_1, Y_1')$ is analytic vector-function in the domain $D_1 \times G_{11} \times G_{21}$, $F(0, 0, 0) = 0$.

We study the questions of the existence of analytic solutions of system (1.1) that satisfy the initial condition

$$Y_1(z) \to 0 \ \text{ for } \ z \to 0, \ \ z \in D_{10}, \tag{1.2}$$

and the additional condition

$$Y_1'(z) \to 0 \ \text{ for } \ z \to 0, \ \ z \in D_{10}. \tag{1.3}$$

According to the method of analytic continuation of solutions [3], system (1.1) will be investigated over two sets of curves. We analytically continue solutions from the curve of the first set to some domain by using the curves of the second set.

1.1 Introduction of some intermediary notations

For arbitrary fixed $t_1 > 0$, $v_1, v_2 \in \mathbb{R}$, $v_1 < v_2$, let us introduce the following intermediary sets:

$$\check{I} = \big\{(t, v) \in \mathbb{R}^2 : \ t \in (0, t_1), \ v \in (v_1, v_2)\big\},$$
$$L_{v_0}(t_1) = \big\{(t, v) \in \mathbb{R}^2 : \ t \in (0, t_1), \ v = v_0 \in (v_1, v_2)\big\},$$

v_0 is a fixed number.

For arbitrary fixed $t_0 \in (0, t_1)$, $O_{t_1}(t_0) = \{(t, v) \in \mathbb{R}^2 : \ t = t_0, \ v \in (v_1, v_2)\}$.

For $z = z(t, v) = te^{iv}$, let us assign for set $\check{I} \subset \mathbb{R}^2$ the set $I \subset \mathbb{C}$, $I = \{z = te^{iv} \in \mathbb{C} : \ t \in (0, t_1), \ v \in (v_1, v_2)\}$.

Definition 1.1. Let $p, g : \check{I} \to [0, +\infty)$. We say that the function $p(t, v)$ possesses property Q_1 relative to the function $g(t, v)$ for $v = v_0 \in (v_1, v_2)$, if the function $p(t, v_0)$ is of higher-order of smallness relative to the function $g(t, v_0)$ as $t \to +0$.

Definition 1.2. Let $p, g : \check{I} \to [0, +\infty)$. We say that the function $p(t, v)$ possesses property Q_2 relative to the function $g(t, v)$, if there exist $C_1 \geq 0$, $C_2 \geq 0$ such that in the set $\check{I}$ the inequalities

$$C_1 \cdot g(t, v) \leq p(t, v) \leq C_2 \cdot g(t, v)$$

hold.

Let us introduce the following intermediary vector-functions:

$$\varphi^{(0)}(z) = (\varphi_1^{(0)}(z), \dots, \varphi_p^{(0)}(z)), \ \ \varphi^{(0)} : I \to \mathbb{C}^p,$$
$$\psi^{(0)}(t, v) = (\psi_1^{(0)}(t, v), \dots, \psi_p^{(0)}(t, v)), \ \ \psi_j^{(0)} : \check{I} \to [0; +\infty), \ \ j = \overline{1, p}.$$

For $z = z(t, v) = te^{iv}$, we have

$$\psi_j^{(0)}(t, v) = |\varphi_j^{(0)}(z(t, v))|, \ \ j = \overline{1, p}.$$

Definition 1.3. We say that the analytic on the set I vector-function $\varphi^{(0)}(z)$ possesses the property T_0, if for any $z \in I$, for the counterpart vector-functions $\psi_j^{(0)}(t, v)$ the conditions

$$\psi_j^{(0)}(t, v) > 0, \ \ (\psi_j^{(0)}(t, v))'_t > 0, \ \ (\psi_j^{(0)}(t, v))'_v \geq 0,$$
$$\psi_j^{(0)}(+0, v) = 0, \ \ (\psi_j^{(0)}(+0, v))'_t = 0, \ \ j = \overline{1, p} \ \text{ uniformly in } \ v \in (v_1, v_2)$$

are fulfilled.

1.2 System (1.1) on the set $L_{v_0}(t_1)$

Let us consider system (1.1) over the interval $L_{v_0}(t_1)$ for an arbitrary fixed $v_0 \in (v_1, v_2)$.

For $z = z(t, v_0) = te^{iv_0}$, in system (1.1) we write each vector-function and matrix in the algebraic form and separate real and imaginary parts. Introduce the following designations:

$$Y_1(z(t, v_0)) = \widetilde{Y}_1(t), \ \ \widetilde{Y}_1(t) = \widetilde{Y}_{11}(t) + i\widetilde{Y}_{12}(t); \ \ \widetilde{Y}_{1j}(t) = \mathrm{col}(\widetilde{Y}_{1j1}(t), \dots, \widetilde{Y}_{1jp}(t)), \ \ j = 1, 2,$$
$$P(z(t, v_0)) = \|\widetilde{p}_{jk}(t)\|_{j,k=1}^p = \widetilde{P}_1(t) + i\widetilde{P}_2(t), \ \ \widetilde{P}_s(t) = \|\widetilde{p}_{jks}(t)\|_{j,k=1}^p, \ \ s = 1, 2,$$

where

$$\widetilde{p}_{jk}(t) = \widetilde{p}_{jk1}(t) + i\widetilde{p}_{jk2}(t), \ \ j, k = \overline{1, p},$$
$$F\big(z(t, v_0), Y_1(z(t, v_0)), Y_1'(z(t, v_0))\big) = \widetilde{F}(t, \widetilde{Y}_1, \widetilde{Y}_1'),$$
$$\widetilde{F}\big(t, \widetilde{Y}_1, \widetilde{Y}_1'\big) = \mathrm{col}\left(\widetilde{F}_1(t, \widetilde{Y}_1, \widetilde{Y}_1'), \dots, \widetilde{F}_p(t, \widetilde{Y}_1, \widetilde{Y}_1')\right),$$
$$\widetilde{F}_j(t, \widetilde{Y}_1, \widetilde{Y}_1') = \widetilde{F}_{1j}(t, \widetilde{Y}_1, \widetilde{Y}_1') + i\widetilde{F}_{2j}(t, \widetilde{Y}_1, \widetilde{Y}_1'), \ \ j = \overline{1, p}.$$

Due to the fact that for each $v \in [v_1, v_2]$ we have the equality

$$\widetilde{Y}_1'(t) = \big(Y_1(z(t, v))\big)'_t = \frac{dY_1}{dz} \cdot \frac{dz}{dt} = Y_1'(z) \cdot e^{iv},$$

then for $z = z(t, v_0) = te^{iv_0}$ system (1.1) takes the form

$$t^l(\widetilde{Y}_{11}' + i\widetilde{Y}_{12}') = t^l(\widetilde{P}_1 + i\widetilde{P}_2)(\widetilde{Y}_{11} + i\widetilde{Y}_{12})e^{iv_0} + e^{(1-l)iv_0}\big(\mathrm{Re}\,\widetilde{F}(t, \widetilde{Y}_1, \widetilde{Y}_1') + i\,\mathrm{Im}\,\widetilde{F}(t, \widetilde{Y}_1, \widetilde{Y}_1')\big). \quad (1.4)$$

Let us introduce the matrices and the vector-function

$$\widetilde{P}(t)=\begin{pmatrix}\widetilde{P}_1(t) & -\widetilde{P}_2(t)\\ \widetilde{P}_2(t) & \widetilde{P}_1(t)\end{pmatrix},$$

$$\widetilde{Q}_1(v_0)=\begin{pmatrix}\cos(v_0)E & -\sin(v_0)E\\ \sin(v_0)E & \cos(v_0)E\end{pmatrix},\quad \widetilde{Q}_2(v_0)=\begin{pmatrix}\cos((l-1)v_0)E & \sin((l-1)v_0)E\\ -\sin((l-1)v_0)E & \cos((l-1)v_0)E\end{pmatrix},$$

$$\widetilde{f}\big(t,\widetilde{Y}_{11},\widetilde{Y}_{12},\widetilde{Y}'_{11},\widetilde{Y}'_{12}\big)=\operatorname{col}\big(\widetilde{F}_{11}\cdots\widetilde{F}_{1p}\widetilde{F}_{21}\cdots\widetilde{F}_{2p}\big),$$

where E is the $p\times p$ identity matrix.

Equating the real and imaginary parts of the vector-functions from the left– and right-hand sides of system (1.4), system (1.4) reduces to

$$t^l\begin{pmatrix}\widetilde{Y}'_{11}(t)\\ \widetilde{Y}'_{12}(t)\end{pmatrix}=t^l\widetilde{P}(t)\widetilde{Q}_1(v_0)\begin{pmatrix}\widetilde{Y}_{11}(t)\\ \widetilde{Y}_{12}(t)\end{pmatrix}+\widetilde{Q}_2(v_0)\widetilde{f}\big(t,\widetilde{Y}_{11},\widetilde{Y}_{12},\widetilde{Y}'_{11},\widetilde{Y}'_{12}\big). \tag{1.5}$$

This implies that system (1.1) over the interval $L_{v_0}(t_1)$ for an arbitrary fixed $v_0\in(v_1,v_2)$ reduces to the system of real differential equations (1.5).

1.3 System (1.1) on the set $O_{t_1}(t_0)$

Let us consider system (1.1) over the arc of circle $O_{t_1}(t_0)$ for an arbitrary fixed $t_0\in(0,t_1)$.

For $z=z(t,v_0)=te^{iv_0}$, in system (1.1) we write each vector-function and matrix in the algebraic form and separate real and imaginary parts. Let us introduce the following designations:

$$Y_1(z(t_0,v))=\widehat{Y}_1(v),\ \ \widehat{Y}_1(v)=\widehat{Y}_{11}(v)+i\widehat{Y}_{12}(v);$$
$$\widehat{Y}_{1j}(v)=\operatorname{col}\big(\widehat{Y}_{1j1}(v),\dots,\widehat{Y}_{1jp}(v)\big),\ \ j=1,2,$$
$$P(z(t_0,v))=\|\widehat{p}_{jk}(v)\|_{j,k=1}^p=\widehat{P}_1(v)+i\widehat{P}_2(v),\ \ \widehat{P}_s(v)=\|\widehat{p}_{jks}(v)\|_{j,k=1}^p,\ \ s=1,2,$$

where

$$\widehat{p}_{jk}(v)=\widehat{p}_{jk1}(v)+i\widehat{p}_{jk2}(v),\ \ j,k=\overline{1,p},$$
$$F\big(z(t_0,v),Y_1(z(t_0,v)),Y_1'(z(t_0,v))\big)=\widehat{F}(v,\widehat{Y}_1,\widehat{Y}_1'),$$
$$\widehat{F}(v,\widehat{Y}_1,\widehat{Y}_1')=\operatorname{col}\big(\widehat{F}_1(v,\widehat{Y}_1,\widehat{Y}_1'),\dots,\widehat{F}_p(v,\widehat{Y}_1,\widehat{Y}_1')\big),$$
$$\widehat{F}_j(v,\widehat{Y}_1,\widehat{Y}_1')=\widehat{F}_{1j}(v,\widehat{Y}_1,\widehat{Y}_1')+i\widehat{F}_{2j}(v,\widehat{Y}_1,\widehat{Y}_1'),\ \ j=\overline{1,p}.$$

Due to the fact that for each $t\in(0,t_1)$ we have the equality

$$\widehat{Y}_1'(v)=\big(Y_1(z(t,v))\big)'_t=\frac{dY_1}{dz}\cdot\frac{dz}{dv}=Y_1'(z)\cdot ite^{iv},$$

then for $z=z(t_0,v)=t_0e^{iv}$, system (1.1) reduces to the form

$$t_0^{l-1}(\widehat{Y}'_{11}+i\widehat{Y}'_{12})=it_0^l(\widehat{P}_1+i\widehat{P}_2)(\widehat{Y}_{11}+i\widehat{Y}_{12})e^{iv}+e^{(1-l)iv}\big(\operatorname{Re}\widehat{F}(v,\widehat{Y}_1,\widehat{Y}_1')+i\operatorname{Im}\widehat{F}(v,\widehat{Y}_1,\widehat{Y}_1')\big). \tag{1.6}$$

Let us introduce matrices and the vector-function

$$\widehat{P}(v)=\begin{pmatrix}\widehat{P}_1(v) & -\widehat{P}_2(v)\\ \widehat{P}_2(v) & \widehat{P}_1(v)\end{pmatrix},$$

$$\widehat{Q}_1(v)=\begin{pmatrix}-\sin(v)E & -\cos(v)E\\ \cos(v)E & -\sin(v)E\end{pmatrix},\quad \widehat{Q}_2(v)=\begin{pmatrix}\sin((l-1)v)E & -\cos((l-1)v)E\\ \cos((l-1)v)E & \sin((l-1)v)E\end{pmatrix},$$

$$\widehat{f}\big(v,\widehat{Y}_{11},\widehat{Y}_{12},\widehat{Y}'_{11},\widehat{Y}'_{12}\big)=\operatorname{col}(\widehat{F}_{11}\cdots\widehat{F}_{1p}\widehat{F}_{21}\cdots\widehat{F}_{2p}),$$

where E is the $p\times p$ identity matrix.

Equating the real and imaginary parts of the vector-functions from the left– and right-hand sides of system (1.6), system (1.6) reduces to

$$t_0^{l-1}\begin{pmatrix}\widehat{Y}_{11}'(v)\\ \widehat{Y}_{12}'(v)\end{pmatrix}=t_0^l\widehat{P}(v)\widehat{Q}_1(v)\begin{pmatrix}\widehat{Y}_{11}(v)\\ \widehat{Y}_{12}(v)\end{pmatrix}+\widehat{Q}_2(v)\widehat{f}\big(v,\widehat{Y}_{11},\widehat{Y}_{12},\widehat{Y}_{11}',\widehat{Y}_{12}'\big). \tag{1.7}$$

This implies that system (1.1) over the arc of the circle $O_{t_1}(t_0)$ for an arbitrary fixed $t_0\in(0,t_1)$ reduces to the system of real differential equations (1.7).

1.4 On some classes of systems of form (1.1)

Definition 1.4. We say that the matrix $P(z)$ possesses property S_{2l} relative to the vector-function $\varphi^{(0)}(z)$, if the following conditions are fulfilled:

(1) For each $v_0\in(v_1,v_2)$, the functions $(\psi_j^{(0)}(t,v))_t'$ possess property Q_1 relative to the functions $|\widetilde{p}_{jj}(t)|\psi_j^{(0)}(t,v)$, $j=\overline{1,p}$, for $v=v_0\in(v_1,v_2)$.

(2) The functions $t^l(\psi_j^{(0)}(t,v))_v'$ possess property Q_2 relative to the functions $t^{l-1}|\widehat{p}_{jj}(v)|\psi_j^{(0)}(t,v)$, $j=\overline{1,p}$.

(3) For each $v_0\in(v_1,v_2)$, the functions $|\widetilde{p}_{jk}(t)|\psi_k^{(0)}(t,v)$ possess property Q_1 relative to the functions $|\widetilde{p}_{jj}(t)|\psi_j^{(0)}(t,v)$, $j,k=\overline{1,p}$, $j\neq k$, for $v=v_0\in(v_1,v_2)$.

(4) The functions $t^l|\widehat{p}_{jk}(v)|\psi_k^{(0)}(t,v))$ possess property Q_2 relative to the functions $t^{l-1}(\psi_j^{(0)}(t,v))_v'$, $j,k=\overline{1,p}$, $j\neq k$.

Let us define the sets

$$\widetilde{\Omega}\big(\delta,\varphi^{(0)}(z(t,v_0))\big)=\Big\{(t,\widetilde{Y}_{11},\widetilde{Y}_{12}):\ t\in(0,t_1),\ \widetilde{Y}_{11j}^2+\widetilde{Y}_{12j}^2<\delta_j^2(\psi_j^{(0)}(t,v_0))^2,\ j=\overline{1,p}\Big\},$$

v_0 is fixed on (v_1,v_2),

$$\widehat{\Omega}\big(\sigma,\varphi^{(0)}(z(t_0,v))\big)=\Big\{(v,\widehat{Y}_{11},\widehat{Y}_{12}):\ v\in(v_1,v_2),\ \widehat{Y}_{11j}^2+\widehat{Y}_{12j}^2<\sigma_j^2(\psi_j^{(0)}(t_0,v))^2,\ j=\overline{1,p}\Big\},$$

t_0 is fixed on $(0,t_1)$, where $\delta=(\delta_1,\dots,\delta_p)$, $\sigma=(\sigma_1,\dots,\sigma_p)$, $\delta_j,\sigma_j\in\mathbb{R}\setminus\{0\}$, $j=\overline{1,p}$.

Definition 1.5. We say that the vector-function $F(z,Y_1,Y_1')$ possesses property M_{2l} relative to the vector-function $\varphi^{(0)}(z)$, if the following conditions hold:

(1) For each $v_0\in(v_1,v_2)$, when $(t,\widetilde{Y}_{11},\widetilde{Y}_{12})\in\widehat{\Omega}(\sigma,\varphi^{(0)}(z(t,v_0)))$, the functions $\widetilde{F}_{kj}(t,\widetilde{Y}_{11},\widetilde{Y}_{12},\widetilde{Y}_{11}',\widetilde{Y}_{12}')$ possess property Q_1 relative to the vector-functions $t^l|\widetilde{p}_{jj}(t)|\psi_j^{(0)}(t,v)$, $j=\overline{1,p}$, $k=1,2$, for $v=v_0\in(v_1,v_2)$.

(2) For each $(v,\widehat{Y}_{11},\widehat{Y}_{12})\in\widehat{\Omega}(\sigma,\varphi^{(0)}(z(t_0,v)))$ the functions $\widehat{F}_{kj}(v,\widehat{Y}_{11},\widehat{Y}_{12},\widehat{Y}_{11}',\widehat{Y}_{12}')$ possess property Q_2 relative to vector-functions $t^l|\widehat{p}_{jj}(v)|\psi_j^{(0)}(t,v)$, $j=\overline{1,p}$, $k=1,2$.

Let us introduce intermediary functions $\widetilde{\alpha}_{jk}(t)$, $\widehat{\alpha}_{jk}(v)$, $j,k=\overline{1,p}$,

$$\begin{aligned}\cos(\widetilde{\alpha}_{jk}(t))&=\frac{\widetilde{p}_{jk1}(t)}{\sqrt{(\widetilde{p}_{jk1}(t))^2+(\widetilde{p}_{jk2}(t))^2}}\,,\\ \sin(\widetilde{\alpha}_{jk}(t))&=\frac{\widetilde{p}_{jk2}(t)}{\sqrt{(\widetilde{p}_{jk1}(t))^2+(\widetilde{p}_{jk2}(t))^2}}\,,\end{aligned}\qquad j,k=\overline{1,p}, \tag{1.8}$$

$$\begin{aligned}\cos(\widehat{\alpha}_{jk}(t))&=\frac{\widehat{p}_{jk1}(t)}{\sqrt{(\widehat{p}_{jk1}(t))^2+(\widehat{p}_{jk2}(t))^2}}\,,\\ \sin(\widehat{\alpha}_{jk}(t))&=\frac{\widehat{p}_{jk2}(t)}{\sqrt{(\widehat{p}_{jk1}(t))^2+(\widehat{p}_{jk2}(t))^2}}\,,\end{aligned}\qquad j,k=\overline{1,p}. \tag{1.9}$$

Without loss of generality, let us suppose that $t_1 \le R_1$ and introduce the domains $\Lambda_{+.k}(t_1)$, $k \in \{+,-\}$ defined as follows:

$$\Lambda_{+.+}(t_1) = \Big\{(t,v):\ \cos\big((l-1)v + \widetilde{\alpha}_{jj}(t)\big) > 0,\ \sin\big((l-1)v + \widehat{\alpha}_{jj}(v)\big) > 0,$$
$$j = \overline{1,p},\ t \in (0,t_1),\ v \in (v_1,v_2)\Big\};$$
$$\Lambda_{+.-}(t_1) = \Big\{(t,v):\ \cos((l-1)v + \widetilde{\alpha}_{jj}(t)) > 0,\ \sin((l-1)v + \widehat{\alpha}_{jj}(v)) < 0,$$
$$j = \overline{1,p},\ t \in (0,t_1),\ v \in (v_1,v_2)\Big\}.$$

Definition 1.6. We say that system (1.1) belongs to the class $C_{+.k}$, $k \in \{+,-\}$, if for the matrix $P(z) = P(te^{iv})$ the condition $(t,v) \in \Lambda_{+.k}(t_1)$, $k \in \{+,-\}$ is true.

1.5 On the existence of a solution of problem (1.1), (1.2), (1.3)

Let us introduce the domains $G_{+.k}(t_1) = \{z = z(t,v):\ 0 < |z| < t_1,\ (t,v) \in \Lambda_{+.k}(t_1)\}$, $k \in \{+,-\}$.

Theorem 1.1. *For system* (1.1)*, let the following conditions be fulfilled:*

(1) *The matrix $P(z)$ is analytic in the domain D_1 and possesses property S_{2l} relative to the analytic vector-function $\varphi^{(0)}(z)$.*

(2) *The vector-function $F(z,Y_1,Y_1')$ is analytic in the domain $D_1 \times G_{11} \times G_{21}, F(0,0,0) = 0$ and possesses property M_{2l} relative to the analytic vector-function $\varphi^{(0)}(z)$.*

(3) *System* (1.1) *belongs to one of the classes $C_{+.k}$, $k \in \{+,-\}$.*

Then for each $k \in \{+,-\}$ and for some $t^* \in (0,t_1)$ there exist analytic solutions $Y_1(z)$ of system (1.1) *that satisfy the initial condition $Y_1(z_0) = Y_{10}$ for $z_0 \in G_{+.k}(t^*)$, $Y_{10} \in \{Y_1:\ |Y_{1j}(z_0)| < \delta_j|\varphi_j^{(0)}(z_0)|,\ \delta_j > 0,\ j = \overline{1,p}\}$. These solutions are analytic in the domain $D_1 \cap G_{+.k}(t^*)$ and satisfy the inequalities*

$$|Y_{1j}(z)|^2 < \delta_j^2|\varphi_j^{(0)}(z)|^2,\ \ j = \overline{1,p}. \tag{1.10}$$

Proof. (1) Let us consider system (1.1) over the interval $L_{v_0}(t_1)$ for an arbitrary fixed $v_0 \in (v_1,v_2)$.

We introduce the sets

$$\widetilde{\Omega}_j\big(\delta,\varphi^{(0)}(z(t,v_0))\big) = \Big\{(t,\widetilde{Y}_{11},\widetilde{Y}_{12}):\ \widetilde{Y}_{11j}^2 + \widetilde{Y}_{12j}^2 < \delta_j^2(\psi_j^{(0)}(t,v_0))^2,\ t \in (0,t_1)\Big\},\ \ j = \overline{1,p}.$$

Thus the set $\widetilde{\Omega}(\delta,\varphi^{(0)}(z(t,v_0)))$ can be considered as intersection of the sets $\widetilde{\Omega}_j$ of the form

$$\widetilde{\Omega}\big(\delta,\varphi^{(0)}(z(t,v_0))\big) = \bigcap_{j=1}^{p} \widetilde{\Omega}_j\big(\delta,\varphi^{(0)}(z(t,v_0))\big).$$

A part of the boundary of the set $\widetilde{\Omega}_j$, $j \in \{1,2,\dots,p\}$, will be denoted by

$$\partial\widetilde{\Omega}_j\big(\delta,\varphi^{(0)}(z(t,v_0))\big) = \Big\{(t,\widetilde{Y}_{11},\widetilde{Y}_{12}):\ \widetilde{Y}_{11j}^2 + \widetilde{Y}_{12j}^2 = \delta_j^2(\psi_j^{(0)}(t,v_0))^2,$$
$$\widetilde{Y}_{11k}^2 + \widetilde{Y}_{12k}^2 < \delta_k^2(\psi_j^{(0)}(t,v_0))^2,\ k = \overline{1,p},\ k \ne j,\ t \in (0,t_1)\Big\}.$$

Assume

$$\widetilde{\Phi}_j(t,\widetilde{Y}(t)) = \widetilde{Y}_{11j}^2(t) + \widetilde{Y}_{12j}^2(t) - \delta_j^2(\psi_j^{(0)}(t,v_0))^2,\ \ j \in \{1,2,\dots,p\}.$$

Then the outward normal vector for the surface $\partial(\widetilde{\Omega}_j)(\delta,\psi(z(t,v_0)))$, for the fixed $j \in \{1,\dots,p\}$, will take the form

$$\frac{\overline{N}_j}{2} = (-\delta_j^2\psi_j^{(0)}(t,v_0))\Big((\psi_j^{(0)}(t,v_0))_t',0,\dots,0,\widetilde{Y}_{11j},0,\dots,0,\widetilde{Y}_{12j},0,\dots,0\Big).$$

Let $\overline{T}$ be a slope-field vector of system (1.5) at an arbitrary fixed point $(t^*, \widetilde{Y}_{11}(t^*), \widetilde{Y}_{12}(t^*)) \in \partial\widetilde{\Omega}_j(\delta, \varphi^{(0)}(z(t, v_0)))$, $j \in \{1, \ldots, p\}$.

Consider the dot product

$$\begin{aligned}\Big(t^l\overline{T}, \frac{\overline{N}_j}{2}\Big) = {} & -t^l\delta_j^2\psi_j^{(0)}(t, v_0)(\psi_j^{(0)}(t, v_0))'_t \\ & + t^l\Big(\widetilde{p}_{jj1}(t)\cos((l-1)v_0) - \widetilde{p}_{jj2}(t)\sin((l-1)v_0)\Big)\delta_j^2\big(\psi_j^{(0)}(t, v_0)\big)^2 \\ & + t^l\sum_{k=1,\ k\neq j}^{p}\Big(\widetilde{p}_{jk1}(t)\cos\big(((l-1)v_0) - \widetilde{p}_{jk2}(t)\sin((l-1)v_0)\big)\big(\widetilde{Y}_{11k}\widetilde{Y}_{11j} + \widetilde{Y}_{12k}\widetilde{Y}_{12j}\big)\Big) \\ & + t^l\sum_{k=1,\ k\neq j}^{p}\Big(\widetilde{p}_{jk1}(t)\sin\big(((l-1)v_0) + \widetilde{p}_{jk2}(t)\cos((l-1)v_0)\big)\big(\widetilde{Y}_{11k}\widetilde{Y}_{12j} - \widetilde{Y}_{12k}\widetilde{Y}_{11j}\big)\Big) \\ & + \big(\widetilde{F}_{1j}\cos((l-1)v_0) + \widetilde{F}_{2j}\sin((l-1)v_0)\big)\widetilde{Y}_{11j} + \big(-\widetilde{F}_{1j}\sin((l-1)v_0) + \widetilde{F}_{2j}\cos((l-1)v_0)\big)\widetilde{Y}_{12j}, \quad j = \overline{1, p}.\end{aligned}$$

Since by condition the matrix $P(z)$ possesses property S_{2l} and the vector-function $F(z, Y_1, Y_1')$ possesses property M_{2l} relative to the vector-function $\varphi^{(0)}(z)$, we have

$$\Big(t^l\overline{T}, \frac{\overline{N}_j}{2}\Big) \sim \sqrt{(\widetilde{p}_{jj1}(t))^2 + (\widetilde{p}_{jj2}(t))^2}\,\big(\cos((l-1)v_0 + \widetilde{\alpha}_{jj}(t))\big), \quad j = \overline{1, p},$$

as $t \to +0$, where the functions $\widetilde{\alpha}_{jj}(t)$ are defined by equalities (1.8).

According to the fact that system (1.1) pertains to one of the classes $C_{+.k}(t, v)$, $k \in \{+, -\}$, there exists t^* such that for $t \in (0, t^*)$ the inequality $(t^l\overline{T}, \frac{\overline{N}_j}{2}) > 0$, $j = \overline{1, p}$, holds true. Thus, for $t \in (0, t^*)$, $\partial\widetilde{\Omega}(\delta, \varphi^{(0)}(z(t, v_0)))$ is the surface without contact for system (1.5). Moreover, the integral curve enters the domain $\widetilde{\Omega}(\delta, \varphi^{(0)}(z(t, v_0)))$ as the variable t decreases.

According to the topological principle of T. Wazewski [13], at least one smooth integral curve of system (1.5) goes through every point of the set $\widetilde{\Omega}(\delta, \varphi^{(0)}(z(t, v_0))) \cup \partial\widetilde{\Omega}(\delta, \varphi^{(0)}(z(t, v_0))) \cap (t = t^*)$. All integral curves of this system going through the points $\widetilde{\Omega}(\delta, \varphi^{(0)}(z(t, v_0))) \cup \partial\widetilde{\Omega}(\delta, \varphi^{(0)}(z(t, v_0))) \cap (t = t^*)$, remain in the domain $\widetilde{\Omega}(\delta, \varphi^{(0)}(z(t, v_0)))$ for $(t, v_0) \in \Lambda_{+.k}(t^*)$, $k \in \{+, -\}$, $v_0 \in (v_1, v_2)$. Moreover, the inequalities

$$|Y_{1sj}(z(t, v_0))|^2 < \delta_j^2\big(\psi_j^{(0)}(t, v_0)\big)^2, \quad j = \overline{1, p}, \quad s = 1, 2, \tag{1.11}$$

are fulfilled for $(t, v_0) \in \Lambda_{+.k}(t^*)$, $k \in \{+, -\}$.

(2) Consider system (1.1) over the arc of circle $O_{t_1}(t_0)$ for an arbitrary fixed $t_0 \in (0, t_1)$.

Let us introduce the sets

$$\widehat{\Omega}_j\big(\sigma, \varphi^{(0)}(z(t_0, v))\big) = \Big\{(v, \widehat{Y}_{11}, \widehat{Y}_{12}) : \ \widehat{Y}_{11j}^2 + \widehat{Y}_{12j}^2 < \sigma_j^2(\psi_j^{(0)}(t_0, v))^2, \ v \in (v_1, v_2)\Big\}, \quad j = \overline{1, p}.$$

Thus the set $\widehat{\Omega}(\sigma, \varphi^{(0)}(z(t_0, v)))$ can be considered as the intersection of sets $\widehat{\Omega}_j$ of the form

$$\widehat{\Omega}(\sigma, \varphi^{(0)}(z(t_0, v))) = \bigcap_{j=1}^{p}\widehat{\Omega}_j\big(\sigma, \varphi^{(0)}(z(t_0, v))\big).$$

A part of the boundary of the set $\widehat{\Omega}_j$, $j \in \{1, 2, \ldots, p\}$ is denoted by

$$\begin{aligned}\partial\widehat{\Omega}_j\big(\sigma, \varphi^{(0)}(z(t_0, v))\big) = \Big\{(v, \widehat{Y}_{11}, \widehat{Y}_{12}) : \ & \widehat{Y}_{11j}^2 + \widehat{Y}_{12j}^2 = \sigma_j^2(\psi_j^{(0)}(t_0, v))^2, \\ & \widehat{Y}_{11k}^2 + \widehat{Y}_{12k}^2 < \sigma_k^2(\psi_j^{(0)}(t_0, v))^2, \ k = \overline{1, p}, \ k \neq j, \ t \in (0, t_1)\Big\}.\end{aligned}$$

Let $\overline{T}$ be a slope-field vector of system (1.7) at an arbitrary fixed point $(t^*, \widehat{Y}_{11}(t^*), \widehat{Y}_{12}(t^*)) \in \partial\widehat{\Omega}_j(\sigma, \varphi(z(t_0, v)))$, for the fixed $j \in \{1, \dots, p\}$,

$$\begin{aligned}\Big(t_0^{l-1}\overline{T}, \frac{\overline{N}_j}{2}\Big) &= -t_0^{l-1}\sigma_j^2\psi_j^{(0)}(t_0,v)(\psi_j^{(0)}(t_0,v))'_v \\ &\quad + t_0^l\Big(\widehat{p}_{jj1}(v)\cos((l-1)v) - \widehat{p}_{jj2}(v)\sin((l-1)v)\Big)\sigma_j^2(\psi_j^{(0)}(t_0,v))^2 \\ &\quad + t_0^l \sum_{k=1,\ k\neq j}^{p} \Big(\widehat{p}_{jk1}(v)\cos\big(((l-1)v) - \widehat{p}_{jk2}(v)\sin((l-1)v)\big)\big(\widehat{Y}_{11k}\widehat{Y}_{12j} - \widehat{Y}_{12k}\widehat{Y}_{11j}\big)\Big) \\ &\quad + t_0^l \sum_{k=1,\ k\neq j}^{p} \Big(-\widehat{p}_{jk1}(v)\sin\big(((l-1)v) - \widehat{p}_{jk2}(v)\cos((l-1)v)\big)\big(\widehat{Y}_{12k}\widehat{Y}_{11j} + \widehat{Y}_{12k}\widehat{Y}_{12j}\big)\Big) \\ &\quad + \big(\widehat{F}_{1j}\sin((l-1)v) + \widehat{F}_{2j}\cos((l-1)v)\big)\widehat{Y}_{11j} + \big(\widehat{F}_{1j}\cos((l-1)v) + \widehat{F}_{2j}\sin((l-1)v)\big)\widehat{Y}_{12j}, \quad j = \overline{1,p}.\end{aligned}$$

Since by the condition the matrix $P(z)$ possesses property S_{2l} and the vector-function $F(z, Y_1, Y_1')$ possesses property M_{2l} relative to the vector-function $\varphi^{(0)}(z)$, we have

$$\Big(t_0^{l-1}\overline{T}, \frac{\overline{N}_j}{2}\Big) \sim \sqrt{(\widehat{p}_{jj1}(v))^2 + (\widehat{p}_{jj2}(v))^2}\,\big(\sin((l-1)v) + \widehat{\alpha}_{jj}(v)\big), \quad j = \overline{1,p},$$

as $t \to +0$, $v \in (v_1, v_2)$, where the functions $\widehat{\alpha}_{jj}(v)$ are defined by equalities (1.9). Thus

$$\operatorname{sign}\Big(t_0^{l-1}\overline{T}, \frac{\overline{N}_j}{2}\Big) = \operatorname{sign}\big(\sin((l-1)v) + \widehat{\alpha}_{jj}(v)\big), \quad j = \overline{1,p}, \quad v \in (v_1, v_2).$$

Without loss of generality, we suppose that for each fixed $t_0 \in (0, t^*)$, $\partial\widehat{\Omega}(\sigma, \varphi^{(0)})(z(t_0, v)) \in \Lambda_{+.k}(t^*)$, $k \in \{+, -\}$ is the surface without contact for system (1.7).

According to the fact that system (1.1) belongs to one of the classes $C_{+.k}(t, v)$, $k \in \{+, -\}$, any integral curve of system (1.7) going through the point of the set $\widehat{\Omega}(\sigma, \varphi^{(0)}(z(t_0, v))) \cap (v = v_0)$, $v_0 \in (v_1, v_2)$, remains in the domain $\widehat{\Omega}(\sigma, \varphi^{(0)}(z(t_0, v)))$ under the condition that variable v decreases if $(t_0, v_0) \in \Lambda_{+.+}(t^*)$, and v increases if $(t_0, v_0) \in \Lambda_{+.-}(t^*)$.

Moreover, the inequalities

$$|Y_{1sj}(z(t_0, v))|^2 < \sigma_j^2\big(\psi_j^{(0)}(t_0, v)\big)^2, \quad j = \overline{1,p}, \quad s = 1, 2, \tag{1.12}$$

hold true for $(t_0, v) \in \Lambda_{+.k}(t^*)$, $k \in \{+, -\}$.

(3) Let us use the method of analytic continuation of solutions for the problems that are solved relatively to the derivatives, i.e., the method suggested by R. Grabovskaya [3] and developed by G. Samkova [7, 8] for the problems that are not solved relatively to the derivatives and also used by D. Limanska and G. Samkova [6] in the proof of the third point of Theorem 2 [6].

Let us suppose that for vectors $\delta, \sigma \in \mathbb{C}^p$, $\delta_j \neq 0$, $\sigma_j \neq 0$, $j = \overline{1,p}$, the inequalities

$$(\delta_j)^2 < (\sigma_j)^2, \quad j = \overline{1,p}, \tag{1.13}$$

are true.

In the proof of item (1) of the theorem, we have got the fact that there are infinitely many continuously differentiable solutions of system (1.5) over the interval $v_0 \in (v_1, v_2)$ for $t \in (0, t^*)$, and these solutions satisfy inequality (1.11). We denote a set of such solutions by $\{Y_1(z(t, v_0))\}$.

Any solution $Y_1(z(t, v_0))$ from the set $\{Y_1(z(t, v_0))\}$ is analytically continuable from the interval $L_{v_0}(t_1)$, where $(t, v) \in \Lambda_{+.k}(t^*)$, for fixed $v_0 \in (v_1, v_2)$, to the domain containing this interval, with preservation of inequalities (1.12).

From the proof of item 2 of the theorem it follows that if inequalities (1.13) are fulfilled, then the solution $Y_1(z(t, v))$ for fixed $v = v_0$ can be continued from the interval $L_{v_0}(t_1)$ over the curves $O_{t_1}(t_0)$ to the set $\widehat{\Omega}(\sigma, \varphi^{(0)}(z(t^*, v)))$ for $t \in (0, |z(t_0, v)|\,]$. We denote the obtained analytic continuation by $Y_1(z)$. The set of solutions of system (1.4) is $\{Y_1(z)\}$.

As a result, the solutions $Y_1(z)$ of system (1.1) are analytically continuable to the domain $G_{+.k}(t^*) \times \{Y : |Y_{1j}| < \delta_j|\varphi_j^{(0)}(z(t_0, v))|, \ j = \overline{1,p}\}$, and, moreover, in this domain solutions $Y_1(z)$ satisfy inequality (1.10). □

2 The main results for system (0.1)

Let us consider the system of ordinary differential equations (0.1) under the conditions that $p < n$, $A(z)$ is an analytic matrix in the domain D_1, and $\operatorname{rank} A(z) = p$ for $z \in D_1$. Let us introduce the function $Y = \operatorname{col}\begin{pmatrix} Y_1 & Y_2 \end{pmatrix}$, $Y_1 = \operatorname{col}(Y_{11}(z), \ldots, Y_{1p}(z))$, $Y_2 = \operatorname{col}(Y_{21}(z), \ldots, Y_{2n-p}(z))$, $Y_1 : D_1 \to \mathbb{C}^p$, $Y_2 : D_1 \to \mathbb{C}^{n-p}$. Without loss of generality, we assume that the matrices $A(z)$, $B(z)$ and the vector-function $f(z, Y, Y')$ take the forms

$$A(z) = \begin{pmatrix} A_1(z) & A_2(z) \end{pmatrix}, \quad B(z) = \begin{pmatrix} B_1(z) & B_2(z) \end{pmatrix}, \quad f(z, Y, Y') = f^*(z, Y_1, Y_2, Y_1', Y_2'),$$

$A_1 : D_1 \to \mathbb{C}^{p\times p}$, $A_2 : D_1 \to \mathbb{C}^{p\times(n-p)}$, $B_1 : D_1 \to \mathbb{C}^{p\times p}$, $B_2 : D_1 \to \mathbb{C}^{p\times(n-p)}$, $\det A_1(z) \neq 0$ for $z \in D_1$, $f^* : D_1 \times G_{11} \times G_{12} \times G_{21} \times G_{22} \to C^p$, $G_{j1} \times G_{j2} = G_j$, $G_{j1} \subset C^p$, $G_{j2} \subset C^{n-p}$, $j = 1, 2$.

Due to the above-said, system (0.1) can be written as

$$Y_1' = A_1^{-1}(z)B_1(z)Y_1 + A_1^{-1}(z)B_2(z)Y_2 - A_1^{-1}(z)A_2(z)Y_2' + A_1^{-1}(z)f^*(z, Y_1, Y_2, Y_1', Y_2') \tag{2.1}$$

Suppose that the matrices $A_1^{-1}(z)B_1(z)$, $A_1^{-1}(z)A_2(z)$, $A_1^{-1}(z)B_2(z)$ are analytic in the domain D_{10} and have removable singularity at the point $z = 0$.

Let us introduce

$$\begin{gathered} P(z) = A_1^{-1}(z)B_1(z), \\ F^*(z, Y_1, Y_2, Y_1', Y_2') = A_1^{-1}(z)B_2(z)Y_2 - A_1^{-1}(z)A_2(z)Y_2' + A_1^{-1}f^*(z, Y_1, Y_2, Y_1', Y_2'), \end{gathered} \tag{2.2}$$

then system (1.1) can be written as

$$Y_1' = P(z)Y_1 + F^*(z, Y_1, Y_2, Y_1', Y_2'), \tag{2.3}$$

where $P(z)$ is the matrix, analytic in the domain D_{10} having removable singularity at the point $z = 0$, and $P : D_{10} \times \mathbb{C}^{p\times p}$, $F^*(z, Y_1, Y_2, Y_1', Y_2')$ is the vector-function, analytic in the domain $D_{10} \times G_{110} \times G_{120} \times G_{210} \times G_{220}$, $G_{jk0} = G_{jk} \setminus \{0\}$, $j, k = 1, 2$. Therefore, the vector-function $F^*(z, Y_1, Y_2, Y_1', Y_2')$ has isolated singularity at the point $(0, 0, 0, 0, 0)$. This means that according to the theorem on the isolated singularity of the function of several complex variables, the point $(0, 0, 0, 0, 0)$ is a removable singular point of that function.

Let us define the vector-function $F^*(z, Y_1, Y_2, Y_1', Y_2')$ at the point $(0, 0, 0, 0, 0)$ in such a way that it becomes analytic in the domain $D_1 \times G_{11} \times G_{12} \times G_{21} \times G_{22}$. Without loss of generality, assume that $F^*(0, 0, 0, 0, 0) = 0$.

By H_r^{n-p} we basically mean a class of $(n-p)$-dimensional analytic in the domain D_{10} functions that have pole of r-order at the point $z = 0$.

Let us consider system (2.3) for an arbitrary fixed vector-function $Y_2 \in H_r^{n-p}$. Then the function $Y_2 = Y_2(z)$ can be written as

$$Y_2(z) = z^{-r}Y_2^*(z), \tag{2.4}$$

where $r \in \mathbb{N}$, $Y_2^*(z)$ is an analytic vector-function in the domain D_1 such that $Y_2^*(0) \neq 0$. Moreover, the function $Y_2^*(z)$ is represented as a convergent power series for $z \in D_1$. Therefore, (2.4) in the domain D_{10} takes the form

$$Y_2(z) = \sum_{k=0}^{\infty} C_k z^{k-r},$$

where $C_k \in \mathbb{C}^{n-p}$, $k = 0, 1, 2, \ldots, {}_0 \neq 0$.

Since $C_0 \neq 0$, the vector-function $Y_2'(z)$ has a pole of $r+1$-order at the point $z = 0$.

Since the vector-function $F^*(z, Y_1, Y_2, Y_1', Y_2')$ is analytic in the domain $D_1 \times G_{11} \times G_{12} \times G_{21} \times G_{22}$ and $F^*(0, 0, 0, 0, 0) = 0$, we get that $F^*(z, Y_1, Y_2, Y_1', Y_2')$ can be represented as a convergent power series

$$F^*(z, Y_1, Y_2, Y_1', Y_2') = \sum_{a+|j|+|k|+|b|+|d|=1}^{\infty} C_a jkbd z^a Y_1^j Y_2^k (Y_1')^b (Y_2')^d$$

near the point (0,0,0,0,0), where $C_{ajkbd} \in \mathbb{C}^p$, $j = (j_1, \dots, j_p)$, $(Y_1)^j = (Y_{11})^{j_1} \cdots (Y_{1p})^{j_p}$, $|j| = j_1 + \dots + j_p$, $k = (k_1, \dots, k_{n-p})$, $(Y_2)^k = (Y_{21})^{k_1} \cdots (Y_{2(n-p)})^{k_{n-p}}$, $|k| = k_1 + \dots + k_{n-p}$, $b = (b_1, \dots, b_p)$, $(Y_1')^b = (Y_{11}')^{b_1} \cdots (Y_{1p}')^{b_p}$, $|b| = b_1 + \dots + b_p$, $d = (d_1, \dots, d_{n-p})$, $(Y_2')^d = (Y_{21}')^{d_1} \cdots (Y_{2(n-p)}')^{d_{n-p}}$, $|d| = d_1 + \dots + d_{n-p}$.

Assume that there exist $q \in \mathbb{N}$ and $s \in \mathbb{N}$ such that

(1) for some $a_0 \in \mathbb{N}$, $j_0 = (j_{01}, \dots, j_{0p})$, $j_{1h} \in \mathbb{N} \cup \{0\}$, $b_0 = (b_{01}, \dots, b_{0p})$, $b_{0h} \in \mathbb{N} \cup \{0\}$, $h = \overline{1,p}$, we have $C_{a_0 j_0 k b_0 d} \neq 0$ for $|k| = q$, $|d| = s$;

(2) for any $h, m \in \mathbb{N}$ and $u = 1, 2, \dots, n-p$, $c = 1, 2, \dots, n-p$, we have $C_{aj(k+he_u)b(d+me_c)} = 0$,

where e_u is the $(n-p)$-dimensional uth orthogonal unit vector, and e is the $(n-p)$-dimensional cth orthogonal unit vector.

Consequently, the summands in the power series expansion of function F^* in the neighbourhood of the point $(0,0,0,0,0)$, containing the maximum powers of vector-functions Y_2 and Y_2' with non-zero coefficients, take the form

$$\begin{aligned} C_{ajkbd} z^a Y_1^j Y_2^k (Y_1')^b (Y_2')^d &= C_{ajkbd} z^a Y_1^j (z^{-r} Y_2^*)^k (Y_1')^b (z^{-r} Y_2^{*\prime} - r z^{-r-1} Y_2^*)^d \\ &= C_{ajkbd} z^{a-rq-(r+1)s} Y_1^j (Y_2^*)^k (Y_1')^b (z Y_2^{*\prime} - r Y_2^*)^d, \end{aligned}$$

for $a = 0, 1, 2 \dots$, $|j| = 0, 1, 2, \dots$, $|b| = 0, 1, 2, \dots$, $|k| = q$, $|d| = s$ and, at least, if $a = a_0$, $j = j_0$, $b = b_0$.

Two logical cases are possible:

(1) $a - rq - (r+1)s \geq 0$. Then for an arbitrary fixed function $Y_2 \in H_r^{n-p}$, we have $F^*(z, Y_1, Y_2, Y_1', Y_2') = F(z, Y_1, Y_1')$, where $F(z, Y_1, Y_1')$ is analytic at the point (0,0,0), and system (2.1) is reduced to the system

$$Y_1' = P(z) Y_1 + F(z, Y_1, Y_1'). \tag{2.5}$$

According to Theorem 1.1 of [6, p. 22], the sufficient conditions for the existence of analytic solutions of the Cauchy problem (2.5), (1.2) with the additional condition (1.3) are found.

(2) $a - rq - (r+1)s < 0$. Let us introduce $l = rq + (r+1)s - a$, then the vector-function $F^*(z, Y_1, Y_2, Y_1', Y_2')$ may take the form

$$\begin{aligned} F^*(z, Y_1, Y_2, Y_1', Y_2') &= z^{-l} \sum_{a+|j|+|k|+|b|+|d|=0}^{\infty} C_{ajkbd} z^a Y_1^j (Y_2^*)^k (Y_1')^b (z Y_2^{*\prime} - r Y_2^*)^d \\ &= z^{-l} \dot{F}(z, Y_1, Y_2^*, Y_1', Y_2^{*\prime}), \end{aligned}$$

where $\dot{F}(z, Y_1, Y_2^*, Y_1', Y_2^{*\prime})$ is the analytic vector-function in the domain $D_1 \times G_{11} \times G_{12} \times G_{21} \times G_{22}$. Without loss of generality, we assume that $\dot{F}(z, Y_1, Y_2^*, Y_1', Y_2^{*\prime}) = F(z, Y_1, Y_1')$, and $F(0,0,0) = 0$.

According to (2.2), system (0.1) takes form (1.1). Let us consider the problem on the existence and asymptotic behavior of the solutions of system (0.1) that satisfy the initial condition (1.2) and the additional condition (1.3).

Theorem 2.1. *Let $p < n$, $A(z)$ be an analytic matrix in the domain D_1,* rank $A(z) = p$ *for $z \in D_1$. Moreover, let system* (0.1) *take form* (2.3)*, and for $Y_2 \in H_r^{n-p}$, conditions* (1)–(3) *of Theorem* 1.1 *be true for the associate system* (1.1).

Then for each $k \in \{+, -\}$, some $t^* \in (0, t_1)$ and for each $Y_2 \in H_r^{n-p}$ there exist analytic solutions $Y(z) = (Y_{11}(z), \dots, Y_{1p}(z), Y_{21}(z), \dots, Y_{2n-p}(z))$ of system (1.1)*. The first p-elements of these solutions are analytic in the domain $D_1 \cap G_{+.k}(t^*)$ and satisfy inequality* (1.10).

Proof. According to Theorem 1.1, the solution $Y_1(z)$ of system (1.1) is analytically continuable on $G_{+.k}(t^*) \times \{Y : |Y_{1j}| < \delta_j |\varphi_j(z)|, \ j = \overline{1,p}\}$. Moreover, the solution satisfies inequality (1.10) in this domain. Therefore, system (0.1), for an arbitrary fixed function $Y_2 \in H_r^{n-p}$, has solutions $Y = (Y_1(z), Y_2(z))$, the first p-elements of which are analytic in the domain $G_{+.k}(t^*) \times \{Y : |Y_{1j}| < \delta_j |\varphi_j(z)|, \ j = \overline{1,p}\}$ and satisfy inequality (1.10) for $z \in D_1 \cap G_{+.k}(t^*)$. □

References

[1] J. Diblík, Asymptotic properties of solutions of the system of first-order equations and n-order equations which is not resolved relatively to the highest derivative. *Thesis of the Candidate of Physical and Mathematical Sciences (Odessa)*, 1979.

[2] J. Diblík, One theorem on asymptotic behaviour of solutions of a certain system of quasilinear differential equations not solved with respect to derivatives. *Riv. Mat. Univ. Parma*, IV. Ser. **13** (1987), 413–419.

[3] R. G. Grabovskaja, The asymptotic behavior of the solution of a system of two first order nonlinear differential equations. (Russian) *Differencial'nye Uravnenija* **11** (1975), no. 4, 639–644.

[4] M. Iwano, On an n-parameter family of solutions of a nonlinear n-system with an irregular type singularity. *Ann. Mat. Pura Appl. (4)* **140** (1985), 57–132.

[5] D. Limanska, On behavior of solutions to some differential systems partially solved with respect to derivatives in the case of a pole. *Nonlinear Oscillations* **20** (2017), no. 1, 113–126.

[6] D. Limanska and G. Samkova, Behavior of the solutions of some systems of differential equations which are partially resolved relatively to the derivatives. (Russian) *Visnyk Odesk. Nats. Univers. Mat. i Mekh.* **19** (2014), no. 1(21), 16–28.

[7] G. Samkova, Existence and asymptotic behavior of analytic solutions of the certain singular differential systems which is not resolved relatively to derivatives. (Russian) *Differencial'nye Uravnenija* **27** (1991), no. 11, 2012–2013.

[8] G. E. Samkova and N. V. Sharaĭ, On the investigation of a semi-explicit system of differential equations in the case of a variable matrix pencil. (Russian) *Nelīnīĭnī Koliv.* **5** (2002), no. 2, 224–236; translation in *Nonlinear Oscil. (N. Y.)* **5** (2002), no. 2, 215–226.

[9] A. M. Samojlenko, M. I. Shkil' and V. L. Yakovets', *Linear systems of differential equations with degenerations.* (Ukrainian) Vyshcha Shkola, Kiev, 2000.

[10] N. V. Sharaj and G. Ye. Samkova, Asymptotic of solutions of some semi-explicit systems of differential equations. (Ukrainian) *Nauk. Visn. Chernivets'kogo Univ.*, Mat. **314-315** (2006), 181–188.

[11] N. I. Shkil', I. I. Starun and V. P. Yakovets, *Asymptotic integration of linear systems of ordinary differential equations. Textbook.* (Russian) Vyshcha Shkola, Kiev, 1989.

[12] O Song Guk, Pak Ponk, Chol Permissible, Boundary condition of a system of linear ordinary differential equations in a closed angle domain of complex plane. *Kwahagwon Thongbo Bull. Acad. Sci. DPR Korea* **2001**, no. 3, 2–4.

[13] T. Ważewski, Sur l'évaluation du domaine d'existence des fonctions implicites réelles ou complexes. (French) *Ann. Soc. Polon. Math.* **20** (1947), 81–120 (1948).

(Received 14.12.2017)

Authors' address:

Odessa I. I. Mechnikov National University, 2 Dvoryanskaya St., Odessa 65082, Ukraine.
E-mail: liman.diana@gmail.com; samkovagalina@i.ua