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SOLVABILITY OF A NONLOCAL PROBLEM BY A NOVEL CONCEPT OF FUNDAMENTAL FUNCTION

Abstract. Cauchy function, Green function and Riemann function are the several of the fundamental functions used frequently in the expression of a fundamental solution in the literature. In order to construct such functions, various ideas can be considered. The lesser-known one of these ideas is contained in the papers [1–4] by Seyidali S. Akhiev. Inspired by these papers, the solvability of some problems [12, 14, 15, 17–19] has been investigated. In this work, a novel kind of adjoint problem for a generally nonlocal problem, and also Green's functional via the solvability of that adjoint problem are constructed [21]. By means of the obtained Green's functional, an integral representation for the solution of the nonlocal problem is established.[1]

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Key words and phrases. Green's function, nonlocal condition, adjoint problem, uncoupled linear system.

რეზიუმე. კოშის ფუნქცია, გრინის ფუნქცია და რიმანის ფუნქცია ძირითადი ფუნქციებია, რომლებიც ლიტერატურაში ხშირად გამოიყენება ფუნდამენტური ამონახსნის წარმოსადგენად. ამ ფუნქციების ასაგებად არსებობს რამდენიმე მიდგომა. მათ შორის ერთ-ერთი ნაკლებად ცნობილი მოყვანილია ს. ს. ახიევის ნაშრომებში [1–4]. ამ სტატიებზე დაყრდნობით გამოკვლეულ იქნა ზოგიერთი ამოცანის ამოხსნადობა [12, 14, 15, 17–19]. ნაშრომში ზოგადი არალოკალური ამოცანისთვის აგებულია ახალი ტიპის შეუღლებული ამოცანა, რომლის ამოხსნადობაზე დაყრდნობით აგებულია გრინის ფუნქციონალი [21]. მიღებული გრინის ფუნქციონალის საშუალებით დადგენილია არალოკალური ამოცანის ამონახსნის ინტეგრალური წარმოდგენა.

[1]Reported on Conference "Differential Equation and Applications", September 4-7, 2017, Brno

1 Introduction

There are various papers related to the investigations on the differential systems involving general boundary conditions [7,8,20,23]. To the best of our knowledge, there is no paper on the construction of Green's functional for an uncoupled system of linear ordinary differential equations with the exception the abstract of conference [13]. This work deals with the construction of Green's functional for such a system with a general nonlocal condition. The main aim at this dealing is to identify the Green function for the above-said system.

The rest of the work is organized as follows. In Section 2, the problem considered throughout the work is stated in detail. In Section 3, the solution space and its adjoint space are introduced. In Section 4, the adjoint operator, adjoint system and solvability conditions for the completely nonhomogeneous problem are given. In Section 5, Green's functional is defined. In the last section, the conclusions are emphasized.

2 Statement of the problem

Let $\mathbb{R}$ be the space of all real numbers, consider a bounded open interval $G=(0,1)$ in $\mathbb{R}$. The problem under consideration is stated as follows:

$$(V_1U)(x) \equiv U'(x) + A(x)U(x) = Z^1(x), \quad x \in G=(0,1), \tag{2.1}$$

$$V_0U \equiv aU(0) + \int_0^1 g(\xi)U'(\xi)\,d\xi = Z^0, \tag{2.2}$$

where $U(x) = \begin{bmatrix} u_1(x) \\ u_2(x) \end{bmatrix}$, $Z^1(x) = \begin{bmatrix} z_1^1(x) \\ z_2^1(x) \end{bmatrix}$, $A(x) = \begin{bmatrix} A_1(x) & 0 \\ 0 & A_2(x) \end{bmatrix}$, $g(\xi) = \begin{bmatrix} g_1(\xi) & 0 \\ 0 & g_2(\xi) \end{bmatrix}$ are 2-vectors and 2-square matrices defined on G, respectively; $Z^0 = \begin{bmatrix} z_1^0 \\ z_2^0 \end{bmatrix}$ and $a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$ are 2-vector and 2-square matrix with real entries, respectively. The symbol $'$ denotes the ordinary derivative of order one. Here $A_1(x), A_2(x), z_1^1(x), z_2^1(x) \in L_p(G)$ with $1 \le p < \infty$ and $g_1(\xi), g_2(\xi) \in L_q(G)$ $(\frac{1}{p}+\frac{1}{q}=1)$. $L_p(G)$ with $1 \le p < \infty$ denotes the space of Lebesgue p-integrable functions on G. $L_\infty(G)$ denotes the space of measurable and essentially bounded functions on G, and $W_p^{(1)}(G)$ with $1 \le p \le \infty$ denotes the space of all functions $u(x) \in L_p(G)$ having derivative $du/dx \in L_p(G)$ [12,16,19]. The space $W_p^{(1)}(G)$ is equipped with the norm

$$\|u\|_{W_p^{(1)}(G)} = \sum_{k=0}^{1} \left\| \frac{d^k u}{dx^k} \right\|_{L_p(G)}.$$

The characteristic feature of this problem is that, instead of an ordinary boundary condition, it involves a more comprehensive nonlocal boundary condition. The stated problem is investigated for a solution vector U such that its entries u_1 and u_2 belong to the space $W_p^{(1)}(G)$.

Problem (2.1), (2.2) is a linear problem which can be considered as an operator equation

$$VU = Z \tag{2.3}$$

with the linear operator $V=(V_1, V_0)$ and $Z=(Z^1(x), Z^0)$.

From the considerations given above, we have that V is bounded from $W_p^{(1)}(G)^2$ into the Banach space $E_p^2 \equiv L_p(G)^2 \times \mathbb{R}^2$ of the elements $Z=(Z^1(x), Z^0)$ with

$$\|z_1\|_{E_p} = \|z_1^1(x)\|_{L_p(G)} + |z_1^0|, \quad \|z_2\|_{E_p} = \|z_2^1(x)\|_{L_p(G)} + |z_2^0|, \ \ 1 \le p \le \infty.$$

If, for a given $Z \in E_p^2$, problem (2.1), (2.2) has a unique solution $U \in W_p^{(1)}(G)^2$ with $\|u_1\|_{W_p^{(1)}(G)} \le c_0\|z_1\|_{E_p}$ and $\|u_2\|_{W_p^{(1)}(G)} \le c_1\|z_2\|_{E_p}$, then this problem is called a well-posed problem, where c_0 and c_1 are constants independent of z_1 and z_2, respectively. Problem (2.1), (2.2) is well-posed if and only if $V: W_p^{(1)}(G)^2 \to E_p^2$ is a (linear) homeomorphism.

3 Adjoint space of the solution space

Problem (2.1), (2.2) is investigated by means of a novel concept of the adjoint problem which is introduced in [2,5]. Some isomorphic decompositions of the solution space $W_p^{(1)}(G)^2$ and its adjoint space $W_p^{(1)}(G)^{2*}$ are employed. Some of the principal features concerning with the solution space can be given as follows: any function $u \in W_p^{(1)}(G)$ can be represented as

$$u(x) = u(\alpha) + \int_{\alpha}^{x} u'(\xi)\, d\xi, \tag{3.1}$$

where α is a given point in $\overline{G}$ which is the set of closure points for G [12,16,19]. Furthermore, the trace or the value operator $D_0 u = u(\gamma)$ is bounded and surjective from $W_p^{(1)}(G)$ onto $\mathbb{R}$ for a given point γ of $\overline{G}$. In addition, the value $u(\alpha)$ and the derivative $u'(x)$ are unrelated elements of the function $u \in W_p^{(1)}(G)$ such that for any real number ν_0 and any function $\nu_1 \in L_p(G)$, there exists one and only one $u \in W_p^{(1)}(G)$ such that $u(\alpha) = \nu_0$ and $u'(x) = \nu_1(x)$. Therefore, there exists a linear homeomorphism between $W_p^{(1)}(G)^2$ and E_p^2. In other words, the space $W_p^{(1)}(G)^2$ has the isomorphic decomposition $W_p^{(1)}(G)^2 = L_p(G)^2 \times \mathbb{R}^2$. The structure of the adjoint space is determined by the following theorem.

Theorem 3.1 ([1,2,4,12,16,19]). *If $1 \le p < \infty$, then any linear bounded functional $F \in W_p^{(1)}(G)^{2*}$ can be represented as*

$$F(U) = \begin{bmatrix} F^1(u_1) \\ F^2(u_2) \end{bmatrix} = \begin{bmatrix} \displaystyle\int_0^1 u_1'(x)\varphi_1^1(x)\, dx + u_1(0)\varphi_0^1 \\ \displaystyle\int_0^1 u_2'(x)\varphi_1^2(x)\, dx + u_2(0)\varphi_0^2 \end{bmatrix} \tag{3.2}$$

with a unique element $\varphi = (\varphi_1(x), \varphi_0) \in E_q^2$, where $\frac{1}{p} + \frac{1}{q} = 1$. Any linear bounded functional $F \in W_\infty^{(1)}(G)^{2*}$ can be represented as

$$F(U) = \begin{bmatrix} F^1(u_1) \\ F^2(u_2) \end{bmatrix} = \begin{bmatrix} \displaystyle\int_0^1 u_1'(x)\, d\varphi_1^1 + u_1(0)\varphi_0^1 \\ \displaystyle\int_0^1 u_2'(x)\, d\varphi_1^2 + u_2(0)\varphi_0^2 \end{bmatrix} \tag{3.3}$$

with a unique element $\varphi = (\varphi_1(e), \varphi_0) \in \widehat{E}_1 = (BA(\Sigma, \mu))^2 \times \mathbb{R}^2$, where μ is Lebesgue measure on $\mathbb{R}$, Σ is σ-algebra of the μ-measurable subsets $e \subset G$ and $BA(\Sigma, \mu)$ is the space of all bounded additive functions $\varphi_1(e)$ defined on Σ with $\varphi_1(e) = 0$ when $\mu(e) = 0$ [9]. The inverse is also valid, that is, if $\varphi \in E_q^2$, then (3.2) is bounded on $W_p^{(1)}(G)^{2*}$ for $1 \le p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $\varphi \in \widehat{E}_1$, then (3.3) is bounded on $W_\infty^{(1)}(G)^{2*}$.

Proof. The operator $NU \equiv (U'(x), U(0)) : W_p^{(1)}(G)^2 \to E_p^2$ is bounded and has a bounded inverse N^{-1} represented by

$$U(x) = (N^{-1}h)(x) \equiv \int_0^x h_1(\xi)\, d\xi + h_0, \quad h = (h_1(x), h_0) \in E_p^2.$$

The kernel $\operatorname{Ker} N$ of N is trivial and the image $\operatorname{Im} N$ of N is equal to E_p^2. Hence, there exists a bounded adjoint operator $N^* : E_p^{2*} \to W_p^{(1)}(G)^{2*}$ with $\operatorname{Ker} N^* = \{0\}$ and $\operatorname{Im} N^* = W_p^{(1)}(G)^{2*}$. In other words, for a given $F \in W_p^{(1)}(G)^{2*}$, there exists a unique $\psi \in E_p^{2*}$ such that

$$F = N^*\psi \ \text{ or } \ F(U) = \psi(NU), \ \ U \in W_p^{(1)}(G)^2. \tag{3.4}$$

If $1 \le p < \infty$, then $E_p^{2*} = E_q^2$ in the sense of an isomorphism [9]. Hence, the functional ψ can be represented by

$$\psi(h) = \int_0^1 \varphi_1(x)h_1(x)\,dx + \varphi_0 h_0, \ \ h \in E_p^2, \tag{3.5}$$

with a unique element $\varphi = (\varphi_1(x), \varphi_0) \in E_q^2$. Due to expressions (3.4) and (3.5), any $F \in W_p^{(1)}(G)^{2*}$ can uniquely be written by (3.2). For a given $\varphi \in E_q^2$, the functional F written by (3.2) is bounded on $W_p^{(1)}(G)^2$. Hence, (3.2) is a general form for the functional $F \in W_p^{(1)}(G)^{2*}$.

The proof is complete due to the fact that the case $p = \infty$ can likewise be shown [4,12,16,19]. □

Theorem 3.1 guarantees that $W_p^{(1)}(G)^{2*} = E_q^2$ for all $1 \le p < \infty$, and $W_\infty^{(1)}(G)^{2*} = E_\infty^{2*} = \widehat{E}_1$. The space E_1 can also be considered as a subspace of the space $\widehat{E}_1$ [4,12,16,19].

4 Adjoint operator, adjoint system and solvability conditions

In this section, an explicit form for the adjoint operator V^* of V is investigated. To this end, any $f = (f_1(x), f_0) \in E_q^2$ is taken as a linear bounded functional on E_p^2 and also we assume

$$f(VU) \equiv \int_0^1 f_1(x)(V_1U)(x)\,dx + f_0(V_0U), \ \ U \in W_p^{(1)}(G)^2. \tag{4.1}$$

By substituting expressions (2.1) and (2.2), and expression (3.1) for all entries of $U \in W_p^{(1)}(G)^2$ (for $\alpha = 0$) into (4.1), we have

$$f(VU) \equiv \begin{bmatrix} \displaystyle\int_0^1 f_1^1(x)\big\{u_1'(x) + A_1(x)u_1(x)\big\}\,dx + f_0^1\left(a_1u_1(0) + \int_0^1 g_1(\xi)u_1'(\xi)\,d\xi\right) \\ \displaystyle\int_0^1 f_1^2(x)\big\{u_2'(x) + A_2(x)u_2(x)\big\}\,dx + f_0^2\left(a_2u_2(0) + \int_0^1 g_2(\xi)u_2'(\xi)\,d\xi\right) \end{bmatrix}.$$

Hence, we obtain

$$\begin{aligned} f(VU) \equiv \int_0^1 f_1(x)(V_1U)(x)\,dx + f_0(V_0U) &= \int_0^1 (w_1f)(\xi)U'(\xi)\,d\xi + (w_0f)U(0) \\ &\equiv (wf)(U) \ \ \forall\, f \in E_q^2, \ \ \forall\, U \in W_p^{(1)}(G)^2, \ \ 1 \le p \le \infty, \end{aligned} \tag{4.2}$$

where

$$w_1 = \begin{bmatrix} w_1^1 \\ w_1^2 \end{bmatrix}, \quad w_0 = \begin{bmatrix} w_0^1 \\ w_0^2 \end{bmatrix},$$

$$\begin{aligned} (w_1^1f^1)(\xi) &= f_1^1(\xi) + \int_\xi^1 f_1^1(s)A_1(s)\,ds + f_0^1g_1(\xi), \quad w_0^1f^1 = \int_0^1 f_1^1(x)A_1(x)\,dx + f_0^1a_1, \\ (w_1^2f^2)(\xi) &= f_1^2(\xi) + \int_\xi^1 f_1^2(s)A_2(s)\,ds + f_0^2g_2(\xi), \quad w_0^2f^2 = \int_0^1 f_1^2(x)A_2(x)\,dx + f_0^2a_2. \end{aligned} \tag{4.3}$$

The operators w_1^1, w_0^1, w_1^2 and w_0^2 are linear and bounded from the space E_q of the pairs $f = (f_1(x), f_0)$ into the spaces $L_q(G), \mathbb{R}, L_q(G)$ and $\mathbb{R}$, respectively. Therefore, the operator $w = (w_1, w_0) : E_q^2 \to E_q^2$ represented by $wf = (w_1 f, w_0 f)$ is linear and bounded. By (4.2) and Theorem 3.1, the operator w is an adjoint operator for the operator V, when $1 \le p < \infty$, in other words, $V^* = w$. When $p = \infty$, $w : E_1^2 \to E_1^2$ is bounded; in this case, the operator w is the restriction of the adjoint operator $V^* : E_\infty^{2*} \to W_\infty^{(1)}(G)^{2*}$ of V onto $E_1^2 \subset E_\infty^{2*}$.

Equation (2.3) can always be transformed into the following equivalent equation

$$VSh = Z \tag{4.4}$$

with an unknown $h = (h_1, h_0) \in E_p^2$ by the transformation $U = Sh$, where $S = N^{-1}$. If $U = Sh$, then $U'(x) = h_1(x)$, $U(0) = h_0$. Hence, (4.2) can be rewritten as

$$\begin{aligned} f(VSh) &\equiv \int_0^1 f_1(x)(V_1 Sh)(x)\,dx + f_0(V_0 Sh) \\ &= \int_0^1 (w_1 f)(\xi) h_1(\xi)\,d\xi + (w_0 f) h_0 \equiv (wf)(h) \;\; \forall\, f \in E_q^2, \;\; \forall\, h \in E_p^2, \;\; 1 \le p \le \infty. \end{aligned}$$

Therefore, one of the operators VS and w becomes an adjoint operator for the other one. Consequently, the equation

$$wf = \varphi \tag{4.5}$$

with an unknown function $f = (f_1(x), f_0) \in E_q^2$ and a given function $\varphi = (\varphi_1(x), \varphi_0) \in E_q^2$ can be considered as an adjoint equation of (4.4) (or of (2.3)) for all $1 \le p \le \infty$, where

$$\varphi_1 = \begin{bmatrix} \varphi_1^1 \\ \varphi_1^2 \end{bmatrix}, \quad \varphi_0 = \begin{bmatrix} \varphi_0^1 \\ \varphi_0^2 \end{bmatrix}.$$

Equation (4.5) can be written in explicit form as the system of equations

$$\begin{aligned} (w_1^1 f^1)(\xi) &= \varphi_1^1(\xi), \;\; \xi \in G, \\ w_0^1 f^1 &= \varphi_0^1, \\ (w_1^2 f^2)(\xi) &= \varphi_1^2(\xi), \;\; \xi \in G, \\ w_0^2 f^2 &= \varphi_0^2. \end{aligned} \tag{4.6}$$

By expressions (4.3), the first and third equations in (4.6) are the integral equations for $f_1^1(\xi), f_1^2(\xi)$, respectively, and include f_0^1, f_0^2, respectively, as parameters; on the other hand, the second and fourth equations in (4.6) are the algebraic equations for the unknowns f_0^1, f_0^2, respectively, and they include some integral functionals defined on $f_1^1(\xi), f_1^2(\xi)$, respectively. In other words, (4.6) is a system of four integro-algebraic equations. This system called the adjoint system for (4.4) (or (2.3)) is constructed by using (4.2) which is actually a formula of integration by parts in a nonclassical form. The traditional type of an adjoint problem is defined by the classical Green's formula of integration by parts [22], therefore, has a sense only for some restricted class of problems [4, 12, 16, 19].

The following theorem concerning with the solvability of the problem can be derived.

Theorem 4.1 ([4, 12, 16, 19])**.** *If $1 < p < \infty$, then $VU = 0$ has either only the trivial solution or a finite number of linearly independent solutions in $W_p^{(1)}(G)^2$:*

(1) *If $VU = 0$ has only the trivial solution in $W_p^{(1)}(G)^2$, then also $wf = 0$ has only the trivial solution in E_q^2. Then the operators $V : W_p^{(1)}(G)^2 \to E_p^2$ and $w : E_q^2 \to E_q^2$ become linear homeomorphisms.*

(2) *If* $VU = 0$ *has* m *linearly independent solutions* $U_1, U_2, \ldots, U_m$ *in* $W_p^{(1)}(G)^2$, *then* $wf = 0$ *has also* m *linearly independent solutions*

$$f^{\star\,1\,\star} = \left(f_1^{\star\,1\,\star}(x), f_0^{\star\,1\,\star}\right), \ldots, f^{\star\,m\,\star} = \left(f_1^{\star\,m\,\star}(x), f_0^{\star\,m\,\star}\right)$$

in E_q^2. *In this case,* (2.3) *and* (4.5) *have solutions* $U \in W_p^{(1)}(G)^2$ *and* $f \in E_q^2$ *for the given* $Z \in E_p^2$ *and* $\varphi \in E_q^2$ *if and only if the conditions*

$$\int_0^1 f_1^{\star\,i\,\star}(\xi) Z^1(\xi)\, d\xi + f_0^{\star\,i\,\star} Z^0 = 0, \;\; i = 1, \ldots, m,$$

and

$$\int_0^1 \varphi_1(\xi) U_i'(\xi)\, d\xi + \varphi_0 U_i(0) = 0, \;\; i = 1, \ldots, m,$$

are satisfied, respectively.

5 Green's functional

Consider the equation in the form of a functional identity

$$(wf)(U) = U(x) \;\; \forall\, U \in W_p^{(1)}(G)^2, \tag{5.1}$$

where $f = (f_1(\xi), f_0) \in E_q^2$ is an unknown pair and $x \in \overline{G}$ is a parameter [4, 12, 16, 19].

Definition 5.1 ([4, 12, 16, 19]). Let $f(x) = (f_1(\xi, x), f_0(x)) \in E_q^2$ be a pair with parameter $x \in \overline{G}$. If $f = f(x)$ is a solution of (5.1) for a given $x \in \overline{G}$, then $f(x)$ is called Green's functional of V (or of (2.3)).

Theorem 5.1 ([4, 12, 16, 19]). *If Green's functional* $f(x) = (f_1(\xi, x), f_0(x))$ *of* V *exists, then any solution* $U \in W_p^{(1)}(G)^2$ *of* (2.3) *can be represented by*

$$U(x) = \int_0^1 f_1(\xi, x) Z^1(\xi)\, d\xi + f_0(x) Z^0.$$

Additionally, $\operatorname{Ker} V = \{0\}$.

6 Conclusion

The proposed approach principally differs from the known classical construction methods of Green's function, it is based on the use of the structural properties of the space of solutions instead of the classical Green's formula of integration by parts, and it has a natural property which can be easily applied to a very wide class of linear and some nonlinear boundary value problems involving linear nonlocal nonclassical multi-point conditions with also integral-type terms. Because of these properties, it is one of the scarce methods which are aimed at the derivation of a solution to such problems by reducing to an integral equation in general. The proposed approach can successfully be employed also for the functional differential problems resulting from the addition of some delayed, loaded (forced) or neutral terms to the main operator as long as its linearity is conserved [6]. The work emphasizes as a significant result that the unique solvability of the stated problem arises in the unique solvability of the stated adjoint systems of integro-algebraic equations.

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