МАТЕМАТИЧКИ ВЕСНИК
52 (2000), 119–123

UDK 515.122.2
оригинални научни рад
research paper

A NOTE ON SUBPARACOMPACT SPACES

D. Buhagiar and S. Lin

Abstract. Some results on subparacompact spaces have only been obtained for the class of regular spaces. The aim of this note is to obtain some properties of subparacompact spaces, and also some applications to generalized metric spaces, without requiring the regularity of the spaces involved.

Subparacompactness is one of the important covering properties in General Topology. D. K. Burke [2] and H. J. K. Junnila [7] have made many contributions to the characterizations and applications of subparacompact spaces. Some results on subparacompact spaces have only been obtained for the class of regular spaces. In particular, we have the following two results:

(1) A regular inverse image of a subparacompact space under a perfect map is a subparacompact space [2].

(2) Every regular, strong Σ-space is a subparacompact space [4,9].

In [1], D. Buhagiar and T. Miwa investigated four proerties on continuous maps without requiring the spaces to satisfy any separation axioms. In particular, they proved that the inverse image of a subparacompact space under a subparacompact map is a subparacompact space. In this paper we use a technique from[1] about subparacompact maps obtaining some further properties of subparacompact spaces, and also some applications to generalized metric spaces, without requiring the regularity of the spaces involved.

In this paper all spaces are Hausdorff (T_2) and all maps are continuous. For a space X, if $\mathcal{U}$ and $\mathcal{V}$ are families of subsets of X, and G is a subset of X, then we denote the family $\{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$ by $\mathcal{U} \wedge \mathcal{V}$ and the family $\{U \cap G : U \in \mathcal{U}\}$ by $\mathcal{U}_G$. If X is a space and $A \subset X$ then by $\mathrm{cl}_X A$ we understand the *closure* of A in X.

AMS Subject Classification: 54 D 20, 54 E 20, 54 E 18, 54 C 10

Keywords and phrases: Subparacompact space, semi-stratifiable space, (strong) Σ- [Σ^*-] space, isocompact space.

The research of the second author was supported by the NSF of Fujian Province under grant number A97025.

LEMMA 1. *Let K be a compact subset of a space X. If $\mathcal{U}$ is an open in X cover of K, then there exists an open subset O in X such that $K \subset O$ and $\mathcal{U}_O$ has a finite refinement closed in O.*

Proof. Since $\mathcal{U}$ is an open in X cover of K, there exists a finite subfamily $\{ U_i : i \leqslant n \}$ of $\mathcal{U}$ such that $K \subset \bigcup \{ U_i : i \leqslant n \}$. For every $x \in K$, there exists $i(x) \leqslant n$ satisfying $x \in U_{i(x)}$. Let $F(x) = K \setminus U_{i(x)}$. Since the space X is T_2 and $F(x)$ is a compact subset of X, there exist open subsets $V(x)$ and $W(x)$ in X such that $x \in V(x)$, $F(x) \subset W(x)$ and $V(x) \cap W(x) = \emptyset$. Let $G(x) = U_{i(x)} \cup W(x)$, then $G(x)$ is open in X and $K \subset G(x)$. Take a finite subset $\{ x_j : j \leqslant m \}$ of K satisfying $K \subset \bigcup \{ V(x_j) : j \leqslant m \}$ and let $O = \left(\bigcap_{j \leqslant m} G(x_j) \right) \cap \left(\bigcup_{j \leqslant m} V(x_j) \right) \cap \left(\bigcup_{i \leqslant n} U_i \right)$. Then, the set O is open in X and contains K. For every $j \leqslant m$, let $P_j = \mathrm{cl}_O(V(x_j) \cap O)$, then we have that P_j is closed in O, $O = \bigcup_{j \leqslant m} P_j$, and

$$P_j \setminus U_{i(x_j)} \subset (G(x_j) \setminus U_{i(x_j)}) \cap \mathrm{cl}_X(V(x_j)) \subset W(x_j) \cap \mathrm{cl}_X(V(x_j)) = \emptyset.$$

Consequently, $P_j \subset U(i(x_j))$ and $\{ P_j : j \leqslant m \}$ is a closed refinement of $\mathcal{U}_O$ in O. ∎

Recall that a space X is said to be *subparacompact* if every open cover of X has a σ-discrete closed refinement. This condiditon was introduced by L. F. McAuley [8] as *F_σ-screenable.* A map $f \colon X \to Y$ is said to be *perfect* if it is closed, onto and $f^{-1}(y)$ is compact for every $y \in Y$.

THEOREM 2. *Let $f \colon X \to Y$ be a perfect map. If Y is a subparacompact space, then X is also a subparacompact space.*

Proof. Let $\mathcal{U}$ be an open cover of X. For every $y \in Y$, $f^{-1}(y)$ is compact in X and therefore, by Lemma 1, there exists an open subset $O(y)$ in X such that $f^{-1}(y) \subset O(y)$ and $\mathcal{U}_{O(y)}$ has a finite refinement closed in $O(y)$. Since the map f is closed, there exists an open subset $H(y)$ in Y such that $y \in H(y)$ and $f^{-1}(H(y)) \subset O(y)$. Thus, $\mathcal{U}_{f^{-1}(H(y))}$ has a finite closed refinement $\mathcal{F}_y$ in $f^{-1}(H(y))$. Let $\mathcal{H} = \{ H(y) : y \in Y \}$. Since the space Y is subparacompact, $\mathcal{H}$ has a σ-discrete closed refinement $\mathcal{W} = \bigcup_{i \in \mathbf{N}} \mathcal{W}_i$, here each $\mathcal{W}_i = \{ W_{i,y} : y \in Y \}$ is discrete and closed in Y, and $W_{i,y} \subset H(y)$. For every $i \in \mathbf{N}$, let

$$\mathcal{P}_i = \{ f^{-1}(W_{i,y}) \cap F : y \in Y \text{ and } F \in \mathcal{F}_y \}.$$

To complete the proof of the theorem, it suffices to show that $\mathcal{P} = \bigcup_{i \in \mathbf{N}} \mathcal{P}_i$ is a σ-discrete closed refinement of $\mathcal{U}$ in X. The fact that $\mathcal{P}$ is a refinement of $\mathcal{U}$ is evident. Since $f^{-1}(\mathcal{W}_i)$ is discrete in X and $\mathcal{F}_y$ is finite for every $y \in Y$, we have that $\mathcal{P}_i$ is σ-discrete in X for every $i \in \mathbf{N}$. For every $y \in Y$ and every $F \in \mathcal{F}_y$, let

$$L_x = \begin{cases} X \setminus f^{-1}(W_{i,y}), & \text{if } x \in X \setminus f^{-1}(W_{i,y}); \\ f^{-1}(H(y)) \setminus F, & \text{if } x \in f^{-1}(W_{i,y}) \setminus F, \end{cases}$$

whenever $x \in X \setminus (f^{-1}(W_{i,y}) \cap F)$. Then L_x is an open neighbourhood of x in X and $L_x \cap (f^{-1}(W_{i,y}) \cap F) = \emptyset$. This shows that $f^{-1}(W_{i,y}) \cap F$ is closed in X and consequently, X is a subparacompact space. ∎

Semi-stratifiable spaces were introduced and studied by G. D. Creede [3]. Recall that a space X is *semi-stratifiable* if there exists a function G which assigns to each $n \in \mathbf{N}$ and closed set $H \subset X$, an open set $G(n, H)$ containing H and satisfying

(i) $H = \bigcap_{n \in \mathbf{N}} G(n, H)$;

(ii) $H \subset K \implies G(n, H) \subset G(n, K)$.

As an application of Theorem2, we obtain a pull-back theorem for semi-stratifiable spaces.

COROLLARY 3. *Let $f\colon X \to Y$ be a perfect map. If the space X has a G_δ-diagonal, and Y is a semi-stratifiable space, then X is also semi-stratifiable.*

Proof. Since the space Y is semi-stratifiable, it is a subparacompact space [4, Theorem 5.11]. Hence, by Theorem 2, X is also a subparacompact space. Since X is a subparacompact space with a G_δ-diagonal, X is a $\sigma^\sharp$-space (i.e., a space with a σ-closure preserving closed p-network) [5, Proposition 4.1]. Since a semi-stratifiable space is a β-space (i.e. a space (X, τ) having a function $g\colon \mathbf{N} \times X \to \tau$ satisfying (i) $x \in g(n, x)$, and (ii) if $x \in g(n, x_n)$, then the set $\{x_n : n \in \mathbf{N}\}$ has a cluster point in X [6]) and an inverse image of a β-space under a perfect map is a β-space, the space X is a $\sigma^\sharp$-space and a β-space. Consequently, X is a semi-stratifiable space [5, Theorem 5.2]. ∎

Remember that a space X is said to be a (*strong*) Σ-*space* [Σ^*-*space*] if there is a σ-locally finite [σ-hereditarily closure-preserving] family $\mathcal{P}$ of closed subsets of X, and a cover $\mathcal{C}$ of X consisting of closed countably compact (compact) subsets, such that, whenever $C \in \mathcal{C}$ and $C \subset U$ for some open subset U of X, then $C \subset P \subset U$ for some $P \in \mathcal{P}$. The family $\mathcal{P}$ is called a $\pmod{k}$-*network* (*with respect to* $\mathcal{C}$). It is convenient to consider the following modification of $\pmod{k}$-networks. A family $\mathcal{P}$ of subsets of X is called an *almost* $\pmod{k}$-*network* [9] for X if every $x \in X$ is in some compact $C(x) \subset X$ such that, whenever $C(x) \subset U$ for some open subset U of X, then $x \in P \subset U$ for some $P \in \mathcal{P}$.

LEMMA 4. *Every strong Σ^*-space is a subparacompact space.*

Proof. Let X be a strong Σ^*-space, and let $\mathcal{P}$ be a σ-hereditarily closure-preserving closed $\pmod{k}$-network with respect to a cover $\mathcal{C}$ of compact subsets of X. If $\mathcal{U}$ is an open cover of X, for every $x \in X$, there exists a compact subset $C(x) \in \mathcal{C}$ with $x \in C(x)$. By Lemma 1, there is an open subset $O(x)$ in X satisfying $C(x) \subset O(x)$ and $\mathcal{U}_{O(x)}$ has a finite closed refinement $\mathcal{F}(x)$ in $O(x)$. Then, $C(x) \subset P(x) \subset O(x)$ for some $P(x) \in \mathcal{P}$. Let $\mathcal{P}' = \{\, P(x) : x \in X \,\}$, and for each $P \in \mathcal{P}'$ choose a point x_P such that $P(x_P) = P$. Then, since $\mathcal{P}$ is σ-hereditarily closure-preserving, the family $\mathcal{F} = \{\, P \cap F : P \in \mathcal{P}' \text{ and } F \in \mathcal{F}(x_P) \,\}$ is a σ-closure-preserving closed refinement of $\mathcal{U}$ in X, and hence, X is a subparacompact space. ∎

The proof of the following lemma can be fonud in [9].

LEMMA 5. *Let $\mathcal{P}$ be a point-countable family of closed subsets of a space X which is closed under finite intersections. If $\mathcal{P}$ is an almost $\pmod{k}$-network of*

X, there is a cover $\mathcal{C}$ of compact subsets of X such that $\mathcal{P}$ is a $\pmod{k}$-network with respect to $\mathcal{C}$.

THEOREM 6. *The following are equivalent for a space X:*

(1) X is a strong Σ-space;

(2) X is a subparacompact space;

(3) X is an isocompact Σ-space;

(4) X has a σ-discrete closed $\pmod{k}$-network with respect to a cover of compact subsets.

Proof. We only need to show the implication (1) $\Longrightarrow$ (4). Let $\bigcup_{i\in\mathbf{N}}\mathcal{P}_i$ be a σ-locally finite closed $\pmod{k}$-network with respect to a cover $\mathcal{C}$ of compact subsets of X, here each $\mathcal{P}_i$ is locally finite in X. Since X is a subparacompact space, for every $i \in \mathbf{N}$, tehre is a σ-discrete closed cover $\mathcal{F}_i$ of X such that each element of $\mathcal{F}_i$ intersects only finitely many elements of $\mathcal{P}_i$. Thus, $\mathcal{P}_i \wedge \mathcal{F}_i$ is σ-discrete and closed in X. Let $\mathcal{P} = \{\bigcap\mathcal{F} : \mathcal{F}$ is a finite subfamily of $\bigcup_{i\in\mathbf{N}}(\mathcal{P}_i \wedge \mathcal{F}_i)\}$. Then $\mathcal{P}$ is a σ-discrete family of closed subsets of X which is closed under finite intersections. For every $x \in C \in \mathcal{C}$ and every open subset $U \supset C$ of X, there exist $i \in \mathbf{N}$, $P \in \mathcal{P}_i$ and $F \in \mathcal{F}_i$ such that $x \in F$ and $C \subset P \subset U$. Therefore, $P \cap F \in \mathcal{P}$ and $x \in P \cap F \subset U$, which shows that $\mathcal{P}$ is an almost $\pmod{k}$-network of X. By Lemma 5, $\mathcal{P}$ is a σ-discrete closed $\pmod{k}$-network with respect to a cover of compact subsets of X. ■

REMARK 1. H. J. Junilla [7] proved the equivalence (1) $\Longleftrightarrow$ (4) in Theorem 6 for the class of regular spaces.

THEOREM 7. *Let $f\colon X \to Y$ be a closed map with Lindelöf fibres. If X is a strong Σ-space, then so is Y.*

Proof. Since X is a strong Σ-space, by Lemma 5 we have that X is subparacompact. Therefore, since f is closed, Y is also subparacompact [2]. Suppose that $\bigcup_{i\in\mathbf{N}}\mathcal{P}_i$ is a σ-discrete closed $\pmod{k}$-network with respect to a cover $\mathcal{K}$ of compact subsets of X. For every $i \in \mathbf{N}$, tehre is an open cover $\mathcal{U}_i$ of X such that each element of $\mathcal{U}_i$ intersects at most one element of $\mathcal{P}_i$. Since $f^{-1}(y)$ is Lindelöf, for every $y \in Y$, there exists a countable subfamily $\mathcal{U}_i(y)$ of $\mathcal{U}_i$ such that $f^{-1}(y) \subset \bigcup\mathcal{U}_i(y)$. Thus, $f^{-1}(V_i(y)) \subset \bigcup\mathcal{U}_i(y)$ for some neighbouhood $V(y)$ of y in Y. Let $\mathcal{V}_i = \{V_i(y) : y \in Y\}$. Since $\mathcal{V}_i$ is an open cover of Y, $\mathcal{V}_i$ has a σ-discrete closed refinement $\mathcal{F}_i = \{F_\alpha : \alpha \in \mathcal{A}_i\}$. Now, for each $\alpha \in \mathcal{A}_i$, there exists $y_\alpha \in Y$ satisfying $F_\alpha \subset V_i(y_\alpha)$. Let $\mathcal{U}_i(y_\alpha) = \{U_\alpha(j) : j \in \mathbf{N}\}$ and let $\mathcal{B}_{ij} = \{f^{-1}(F_\alpha) \cap U_\alpha(j) : \alpha \in \mathcal{A}_i\}$. Then $\mathcal{B}_i = \bigcup_{j\in\mathbf{N}}\mathcal{B}_{ij}$ is a cover of X. Also, let $\mathcal{C}_i = (\mathcal{P}_i \wedge \mathcal{B}_i)^- = \{\mathrm{cl}_X(P \cap B) : P \in \mathcal{P}_i, B \in \mathcal{B}_i\}$. Since $\mathcal{P}_i$ is closed and discrete in X, each of its elements is a union of some subfamily of $\mathcal{C}_i$. Let $\mathcal{P}_i = \{P_\gamma : \gamma \in \Gamma_i\}$, then $\mathcal{C}_i = \bigcup_{j\in\mathbf{N}}\mathcal{C}_{ij}$, where $\mathcal{C}_{ij} = \{\mathrm{cl}_X(P_\gamma \cap f^{-1}(F_\alpha) \cap U_\alpha(j)) : \alpha \in \mathcal{A}_i, \gamma \in \Gamma_i\}$, so that $f(\mathcal{C}_{ij}) = \{\mathrm{cl}_X(f(P_\gamma \cap U_\alpha(j)) \cap F_\alpha) : \alpha \in \mathcal{A}_i, \gamma \in \Gamma_i\}$. Since each $U_\alpha(j)$ intersects at most one element of $\mathcal{P}_i$ and $\mathcal{F}_i$ is σ-discrete and closed in X, we have that $f(\mathcal{C}_{ij})$ is σ-discrete and closed in Y. To complete the proof, by Lemma 5, it is sufficient to show that $\bigcup_{i,j\in\mathbf{N}} f(\mathcal{C}_{ij})$ is an almost $\pmod{k}$-network for Y.

For every $y \in Y$, there exists $x \in K$ for some $K \in \mathcal{K}$, satisfying $y = f(x)$. For an open subset $W \supset f(K)$ in Y, there exists $\gamma \in \Gamma_i$ with $K \subset P_\gamma \subset f^{-1}(W)$. Since P_γ is a union of some subfamily of $\mathcal{C}_i$, there exist $\alpha \in \mathcal{A}_i$ and $j \in \mathbf{N}$ such that

$$x \in \mathrm{cl}_X(P_\gamma \cap f^{-1}(F_\alpha) \cap U_\alpha(j)) \subset P_\gamma \subset f^{-1}(W),$$

so that

$$y \in \mathrm{cl}_X(f(P_\gamma \cap U_\alpha(j)) \cap F_\alpha) \subset f(P_\gamma) \subset W.$$

Consequently, $\bigcup_{i,j\in\mathbf{N}} f(\mathcal{C}_{ij})$ is a σ-discrete closed, almost $\pmod{k}$-network for Y, and therefore, Y is a strong Σ-space. ■

REMARK 2. E. Michael [9] proved that strong Σ-spaces are preserved by σ-locally finite maps for the class of regular spaces. This is not true for the class of Hausdorff spaces. Indeed, let Y be the space with a point irrational extension topology ([10], Example 69). Then Y is a Hausdorff, second countable space which is not subparacompact. Since Y is second countable, there exists a separable metric space X and a map $f\colon X \to Y$. Then, X is a strong Σ-space and f is a σ-locally finite map, while Y is not a strong Σ-space. ■

REFERENCES

[**1**] D. Buhagiar and T. Miwa, *Covering properties on maps*, Q & A in General Toplogy **16**, 1 (1998), 53–66.

[**2**] D. K. Burke, *On subparacompact spaces*, Proc. Amer. Math. Soc. **23** (1969), 655–663.

[**3**] G. D. Creede, *Concerning semi-stratifiable spaces*, Pacific J. Math. **32** (1970), 47–54.

[**4**] G. Gruenhage, *Generalized metric spaces*, Handbook of Set-Theoretic Topology, North-Holland, 1984, pp. 423–501.

[**5**] R. E. Hodel, *Moore spaces and $w\delta$-spaces*, Pacific J. Math. **38** (1971), 641–652.

[**6**] R. E. Hodel, *Spaces defined by sequences of open covers which guarantee that certain sequences have cluster points*, Duke Math. J. **39** (1972), 253–263.

[**7**] H. J. K. Junilla, *Three covering properties*, Surveys in General Toplogy, Academic Press, 1980, pp. 195–245.

[**8**] L. F. McAuley, *A note on complete colectionwise normality and paracompactness*, Proc. Amer. Math. Soc. **9** (1958), 796–799.

[**9**] E. Michael, *σ-locally finite maps*, Proc. Amer. Math. Soc. **65** (1977), 159–164.

[**10**] L. A. Steen and J. A. Seebach, Jr., *Counterexamples in Topology*, Springer-Verlag, New York, 1978.

(received 22.05.2000.)

Department of Mathematics, Faculty of Science, University of Malta, Msida MSD.06, Malta

E-mail: dbuha@maths.um.edu.mt

Department of Mathematics, Ningde Teachers' College, Ningde, Fujian 352100, P.R.China