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SOME CHARACTERIZATIONS OF THE LORENTZIAN SPHERICAL TIMELIKE AND NULL CURVES

Miroslava Petrović-Torgašev and Emilija Šućurović

Abstract. In [5] and [6] the authors have characterized the Lorentzian spherical spacelike curves in the Minkowski 3-space E_1^3. In this paper, we shall characterize the Lorentzian spherical timelike and null curves in the same space.

1. Introduction

In the Euclidean space E^3 a spherical unit speed curves and their characterizations are given in [3], [9] and [10]. In [5] and [6] the authors have characterized the Lorentzian spherical spacelike curves in the Minkowski 3-space E_1^3. In this paper, we shall characterize the Lorentzian spherical timelike and null curves in the same space.

2. Preliminaries

The Minkowski 3-space E_1^3 is the Euclidean 3-space E^3 provided with the Lorentzian inner product

$$g(a,b) = -a_1b_1 + a_2b_2 + a_3b_3,$$

where $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$.

An arbitrary vector $a = (a_1, a_2, a_3)$ in E_1^3 can have one of three Lorentzian causal characters: it is *spacelike* if $g(a,a) > 0$ or $a = 0$, *timelike* if $g(a,a) < 0$ and *null (lightlike)* if $g(a,a) = 0$ and $a \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in E_1^3 is locally *spacelike, timelike* or *null (lightlike)*, if all of its velocity vectors $\alpha'(s)$ are respectively *spacelike, timelike* or *null*, for each $s \in I \subset R$. Recall that the pseudo-norm of an arbitrary vector $a \in E_1^3$ is given by

$$\| a \| = \sqrt{| g(a,a) |},$$

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and that the velocity v of the curve α is given by $v = \|\alpha'(s)\|$. Therefore, α is a unit speed curve if and only if $g(\alpha'(s), \alpha'(s)) = \pm 1$.

The Lorentzian sphere of center $m = (m_1, m_2, m_3)$ and radius $r \in R^+$ in the space E_1^3 is defined by

$$S_1^2 = \{a = (a_1, a_2, a_3) \in E_1^3 \mid g(a - m, a - m) = r^2\}.$$

The vectors $a, b \in E_1^3$ are orthogonal if and only if $g(a, b) = 0$.

Denote by $\{T(s), N(s), B(s)\}$ the moving Frenet frame along the curve $\alpha = \alpha(s)$ parameterized by a pseudo-arclength parameter s, i.e. $g(\alpha'(s), \alpha'(s)) = \pm 1$. In particular, null curve $\alpha(s)$ in E_1^3 is parameterized by a pseudo-arclength s if $g(\alpha''(s), \alpha''(s)) = 1$. Let $T(s) = \alpha'(s)$, $N(s) = \alpha''(s)/\|\alpha''(s)\|$ and $B(s)$ be the tangent, the principal normal and the binormal vector of the curve $\alpha(s)$ respectively. If α is a timelike curve, i.e. if T is a timelike vector, then the Frenet formulae read:

$$T' = \kappa N, \quad N' = \kappa T + \tau B, \quad B' = -\tau N,$$
$$g(T,T) = -1, \quad g(N,N) = g(B,B) = 1, \quad g(T,N) = g(T,B) = g(N,B) = 0.$$

On the other hand, if α is a null curve, i.e. if T is a null vector, then the Frenet formulae read:

$$T' = \kappa N, \quad N' = \tau T - \kappa B, \quad B' = -\tau N,$$
$$g(T,T) = g(B,B) = 0, \quad g(N,N) = 1, \quad g(T,N) = g(N,B) = 0, \quad g(T,B) = 1$$

where κ takes only two values: $\kappa = 0$ when α is a straight null line or $\kappa = 1$ in all other cases. The functions $\kappa = \kappa(s)$ and $\tau = \tau(s)$ are called the curvature and the torsion of α respectively [8].

3. The Lorentzian spherical timelike curves

THEOREM 3.1. *Let $\alpha(s)$ be a plane unit speed timelike curve with a curvature $\kappa = \kappa(s)$. Then α lies on the Lorentzian sphere of center m and radius $r \in R^+$ in E_1^3 if and only if $\kappa =$ constant $\neq 0$ and*

$$\alpha - m = (1/\kappa)\,N \pm \sqrt{r^2 - (1/\kappa)^2}B.$$

Proof. Let us first suppose that α lies on the Lorentzian sphere of center m and radius $r \in R^+$. Then $g(\alpha - m, \alpha - m) = r^2$, for each $s \in I \subset R$. By differentiation with respect to s of the previous relation, we find that

$$g(T, \alpha - m) = 0. \tag{3.1}$$

Further, the differentiation with respect to s of (3.1) gives

$$g(T', \alpha - m) + g(T,T) = 0,$$
$$\kappa\, g(N, \alpha - m) = 1,$$

where we have used the corresponding Frenet formula. It follows that $\kappa \neq 0$ for each $s \in I \subset R$ and that

$$g(N, \alpha - m) = 1/\kappa. \tag{3.2}$$

Next, decompose the vector $\alpha - m$ as

$$\alpha - m = aT + bN + cB, \tag{3.3}$$

where $a = a(s)$, $b = b(s)$ and $c = c(s)$ are arbitrary functions. Then the relations (3.1) and (3.2) imply that

$$g(T, \alpha - m) = -a = 0, \quad g(N, \alpha - m) = b = 1/\kappa, \quad g(B, \alpha - m) = c.$$

Further, the differentiation of (3.2) with respect to s gives

$$g(N', \alpha - m) + g(N, \alpha') = (1/\kappa)'.$$

By assumption α is a plane curve. Hence $\tau = 0$ and using the corresponding Frenet formula we get that $\kappa g(T, \alpha - m) = (1/\kappa)'$. Then the relation (3.1) implies $(1/\kappa)' = 0$ and thus $1/\kappa = \text{constant} \in R$, i.e. $\kappa = \text{constant} \in R$. Since $\kappa \neq 0$ for each s, it follows that $\kappa = \text{constant} \neq 0$. Further, the substitution of the coefficients a, b and c in (3.3) gives

$$\alpha - m = (1/\kappa)\, N + cB.$$

Now it is easy to see that $g(\alpha - m, \alpha - m) = (1/\kappa)^2 + c^2 = r^2$, so it follows that $c = \pm\sqrt{r^2 - (1/\kappa)^2}$. Consequently,

$$\alpha - m = (1/\kappa)\, N \pm \sqrt{r^2 - (1/\kappa)^2} B.$$

Conversely, if $\kappa = \text{constant} \neq 0$ and

$$\alpha - m = (1/\kappa)\, N \pm \sqrt{r^2 - (1/\kappa)^2} B,$$

$m \in E_1^3$ is an arbitrary vector and $r \in R^+$, we shall prove that $m = \text{constant}$. Since

$$m = \alpha - (1/\kappa)\, N \pm \sqrt{r^2 - (1/\kappa)^2} B,$$

by differentiation with respect to s of the previous equation and using the corresponding Frenet formulae we get $m' = 0$. It follows that $m = \text{constant}$ and that $g(\alpha - m, \alpha - m) = r^2$. Therefore, α lies on the Lorentzian sphere of center m and radius r. ■

REMARK. In [8] a classification of all W-curves (i.e. a curves for which a curvature and a torsion are constants) in space E_1^3 is given. Since α is a curve with $\kappa = \text{constant} \neq 0$ and $\tau = 0$, by that classification it is a part of an orthogonal hyperbola.

THEOREM 3.2. *Let $\alpha(s)$ be a unit speed timelike curve in E_1^3 with a curvature $\kappa(s) \neq 0$ and a torsion $\tau(s) \neq 0$ for each $s \in I \subset R$. Then α lies on the Lorentzian sphere of radius $r \in R^+$ if and only if*

$$(1/\kappa)^2 + ((1/\tau)(1/\kappa)')^2 = r^2.$$

Proof. Let us first suppose that α lies on the Lorentzian sphere of center m and radius r. Then $g(\alpha - m, \alpha - m) = r^2$. By three differentiations with respect to s of the previous equation and using the corresponding Frenet formulae, we get

$$g(B, \alpha - m) = (1/\tau)(1/\kappa)'.$$

Next, decompose the vector $\alpha - m$ as

$$\alpha - m = aT + bN + cB, \tag{3.4}$$

where $a = a(s)$, $b = b(s)$ and $c = c(s)$ are arbitrary functions. Then

$$g(T, \alpha - m) = -a = 0, \quad g(N, \alpha - m) = b = 1/\kappa, \quad g(B, \alpha - m) = c = (1/\tau)(1/\kappa)'.$$

Therefore, substitution of the coefficients a, b and c in (3.4) gives

$$\alpha - m = (1/\kappa)\, N + (1/\tau)(1/\kappa)' B.$$

Thus

$$g(\alpha - m, \alpha - m) = r^2 = (1/\kappa)^2 + ((1/\tau)(1/\kappa)')^2.$$

Conversely, if

$$(1/\kappa)^2 + ((1/\tau)(1/\kappa)')^2 = r^2, \tag{3.5}$$

where $r \in R^+$, we may consider the vector $m \in E_1^3$ of the form

$$m = \alpha - (1/\kappa)\, N - (1/\tau)(1/\kappa)' B. \tag{3.6}$$

We shall prove that $m =$ constant. By differentiation with respect to s of the previous equation, we have that

$$\begin{aligned} m' &= T - (1/\kappa)' N - (1/\kappa)(\kappa T + \tau B) - ((1/\tau)(1/\kappa)')' B + (1/\tau)(1/\kappa)'(\tau N) \\ &= (-\tau/\kappa - ((1/\tau)(1/\kappa)')')\, B. \end{aligned} \tag{3.7}$$

By differentiation with respect to s of the assumption (3.5), we have

$$(2/\kappa)(1/\kappa)' + (2/\tau)(1/\kappa)'((1/\tau)(1/\kappa)')' = 0$$

and thus

$$(\tau/\kappa) + ((1/\tau)(1/\kappa)')' = 0. \tag{3.8}$$

Substituting the last relation in (3.7), we find that $m' = 0$ for each $s \in I \subset R$ and thus $m =$ constant. The relation (3.6) implies that

$$g(\alpha - m, \alpha - m) = (1/\kappa)^2 + ((1/\tau)(1/\kappa)')^2 = r^2.$$

Hence α lies on the Lorentzian sphere of center m and radius r. ∎

THEOREM 3.3. *Let $\alpha(s)$ be a unit speed timelike curve, with a curvature $\kappa(s) \neq 0$ and a torsion $\tau(s) \neq 0$ for each $s \in I \subset R$. Then α lies on a Lorentzian sphere in E_1^3 if and only if*

$$(\tau/\kappa) = -((1/\tau)(1/\kappa)')'.$$

Proof. Let us first assume that α is a curve lying on the Lorentzian sphere of radius $r \in R^+$. Then by the Theorem 3.2 it follows that the relation (3.5) holds, so differentiation with respect s of the relation (3.5) implies the relation (3.8).

Conversely, suppose that the equation (3.8) holds for each $s \in I \subset R$. Since (3.8) is the differential of the equation

$$(1/\kappa)^2 + ((1/\tau)(1/\kappa)')^2 = c = \text{constant} > 0,$$

we may take $c = r^2$, $r \in R^+$. Finally, by Theorem 3.2 it follows that image of the curve α lies on a Lorentzian sphere of radius r. ■

THEOREM 3.4. *A unit speed timelike curve $\alpha(s)$ with $\kappa(s) \neq 0$ and $\tau(s) \neq 0$ for each $s \in I \subset R$ lies on a Lorentzian sphere in E_1^3 if and only if $\kappa(s) > 0$ and there is a differentiable function $f(s)$ such that $f\tau = (1/\kappa)'$ and $f' + \tau/\kappa = 0$.*

Proof. Let us first assume that $\alpha(s)$ is a curve lying on the Lorentzian sphere. Then by the Theorem 3.3 we have that $\tau/\kappa = -((1/\tau)(1/\kappa)')'$. Next, define the differentiable function $f = f(s)$ by

$$f = (1/\tau)(1/\kappa)'.$$

Consequently, $f' = -\tau/\kappa$. Since $\kappa(s) = \|T'\| \geq 0$ and $\kappa(s) \neq 0$ for each $s \in I \subset R$, it follows that $\kappa(s) > 0$.

Conversely, assume that α is a curve for which $\kappa > 0$ for each $s \in I \subset R$ and that there is a differentiable function $f(s)$ such that $f\tau = (1/\kappa)'$ and $f' = -\tau/\kappa$. Next, since $f = (1/\tau)(1/\kappa)'$, we have that

$$((1/\tau)(1/\kappa)')' = -\tau/\kappa.$$

Hence by the Theorem 3.3 it follows that α lies on a Lorentzian sphere. ■

THEOREM 3.5. *A unit speed timelike curve $\alpha(s)$ with $\kappa(s) \neq 0$ and $\tau(s) \neq 0$ lies on a Lorentzian sphere in E_1^3 if and only if there are constants $A, B \in R$ such that the equation*

$$\kappa\Big(A\cos\big(\int_0^s \tau(s)\,ds\big) + B\sin\big(\int_0^s \tau(s)\,ds\big)\Big) = 1.$$

holds for each $s \in I \subset R$.

Proof. Let us first suppose that $\alpha(s)$ is a curve lying on a Lorentzian sphere. Then by the Theorem 3.4 there is a differentiable function $f(s)$ such that $f\tau = (1/\kappa)'$ and $f' = -\tau/\kappa$. Next, define the C^2 function $\theta(s)$ and the C^1 functions $g(s)$ and $h(s)$ by $\theta(s) = \int_0^s \tau(s)\,ds$,

$$g(s) = (1/\kappa)\cos\theta - f(s)\sin\theta, \quad h(s) = (1/\kappa)\sin\theta + f(s)\cos\theta. \tag{3.9}$$

Differentiation with respect to s of the functions θ, g and h easily gives $\theta'(s) = \tau(s)$, $g'(s) = h'(s) = 0$ and therefore $g(s) = A$, $h(s) = B$, so the relation (3.9) becomes

$$(1/\kappa)\cos\theta - f(s)\sin\theta = A, \quad (1/\kappa)\sin\theta + f(s)\cos\theta = B.$$

Multiplying the first of the previous equations with $\cos\theta$ and the second with $\sin\theta$ and adding, we find that $1/\kappa = A\cos\theta + B\sin\theta$. Thus the equation

$$\kappa\Big(A\cos\big(\int_0^s \tau(s)\,ds\big) + B\sin\big(\int_0^s \tau(s)\,ds\big)\Big) = 1,$$

is satisfied.

Conversely, let A and B be the real constants, such that the equation

$$\kappa\Big(A\cos\big(\int_0^s \tau(s)\,ds\big)+B\sin\big(\int_0^s \tau(s)\,ds\big)\Big)=1 \tag{3.10}$$

holds for each $s\in I\subset R$. Then obviously $\kappa(s)\neq 0$ and therefore $\kappa(s)=\|T'\|>0$ for each s. The differentiation with respect to s of the relation (3.10) gives

$$\tau\Big(-A\sin\big(\int_0^s \tau(s)\,ds\big)+B\cos\big(\int_0^s \tau(s)\,ds\big)\Big)=(1/\kappa)'. \tag{3.11}$$

Next, define the differentiable function $f(s)$ by

$$f(s)=-A\sin\big(\int_0^s \tau(s)\,ds\big)+B\cos\big(\int_0^s \tau(s)\,ds\big). \tag{3.12}$$

Then the relations (3.11) and (3.12) give $(1/\kappa)'=\tau f$, that is $f=(1/\tau)(1/\kappa)'$. By differentiation with respect to s of (3.12) and using (3.10), we find that

$$f'=-\tau\Big(A\cos\big(\int_0^s \tau(s)\,ds\big)+B\sin\big(\int_0^s \tau(s)\,ds\big)\Big)=-\tau/\kappa.$$

Therefore, by the Theorem 3.4 it follows that $\alpha(s)$ lies on a Lorentizan sphere. ■

4. The Lorentzian spherical null curves

THEOREM 4.1. *There are no null curves $\alpha(s)$ lying on the Lorentzian sphere in E_1^3.*

Proof. Assume that $\alpha(s)$ is a null curve lying on the Lorentzian sphere of center $m\in E_1^3$ and radius $r\in R^+$. Then we have

$$g(\alpha-m,\alpha-m)=r^2, \tag{4.1}$$

for each $s\in I\subset R$. If α is a straight null line with the equation $\alpha(s)=p+sq$, $p,q\in E_1^3$, then by differentiation with respect to s of the relation (4.1) we get $g(p+sq-m,q)=0$ and therefore $g(q,p)=g(q,m)=$ constant. It follows that $p=m$ and consequently $\alpha-m=sq$. But then $g(\alpha-m,\alpha-m)=0$, which is a contradiction. On the other hand, if α is not a straight null line, by differentiation with respect to s of the relation (4.1), we find that

$$g(T,\alpha-m)=0. \tag{4.2}$$

By differentiation with respect to s of the relation (4.2), we get

$$g(T',\alpha-m)+g(T,T)=0,\qquad \kappa g(N,\alpha-m)=0,$$

and since in this case we have $\kappa=1$ for each $s\in I\subset R$, it follows that

$$g(N,\alpha-m)=0. \tag{4.3}$$

By differentiation of (4.3) and using the corresponding Frenet formula, we find that

$$\tau g(T, \alpha - m) - \kappa g(B, \alpha - m) = 0,$$

which together with the relation (4.2) gives $-\kappa g(B, \alpha - m) = 0$, and consequently

$$g(B, \alpha - m) = 0. \tag{4.4}$$

Next, decompose the vector $\alpha - m$ as

$$\alpha - m = aT + bN + cB, \tag{4.5}$$

where $a = a(s)$, $b = b(s)$ and $c = c(s)$ are arbitrary functions. Then by the relations (4.2), (4.3) and (4.4), we have that

$$g(T, \alpha - m) = c = 0, \quad g(N, \alpha - m) = b = 0, \quad g(B, \alpha - m) = a = 0.$$

Therefore, the equation (4.5) implies that $\alpha = m$, which is a contradiction. ∎

REFERENCES

[**1**] M. Barros, B. Y. Chen, *Spherical submanifolds which are of 2-type via the second standard immersion of the shpere*, Nagoya Math. J., **108** (1987), 77–91.

[**2**] G. S. Birman, K. Nomizu, *Trigonometry in Lorentzian geometry*, Amer. Math. Month. **91(9)** (1984), 543–549.

[**3**] R. S. Milman, G. D. Parker, *Elements of Differential Geometry*, Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1977.

[**4**] B. O'Neill, *Semi-Riemannian Geometry*, Academic Press, New York, 1983.

[**5**] U. Pekmen, S. Pasali, *Some characterizations of Lorentzian spherical space-like curves*, Mathematica Moravica **3** (1999), 33–37.

[**6**] M. Petrović-Torgašev, E. Šućurović, *Some characterizations of Lorentzian spherical spacelike curves with the timelike and the null principal normal*, Mathematica Moravica **4** (2000), 83–92.

[**7**] I. Van de Woestyne, *Minimal surface of the 3-dimensional Minkowski space*, Geometry and Topology of Submanifolds, II, World Scientific, Singapore, (1990), 344–369.

[**8**] J. Walrave, *Curves and surfaces in Minkowski space*, Ph. thesis, K. U. Leuven, Fac. of Science, Leuven, 1995.

[**9**] Y. C. Wong, *A global formulation of the condition for a curve to lie in a sphere*, Monatschefte fur Mathematik **67** (1963), 363–365.

[**10**] Y. C. Wong, *On an explicit characterization of spherical curves*, Proc. Amer. Math. Soc. **34** (1972), 239–242.

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Faculty of Science, Radoja Domanovića 12, 34000 Kragujevac, Yugoslavia

E-mail: mirapt@uis0.uis.kg.ac.yu, emilija@knez.uis.kg.ac.yu