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GENERALIZED BINOMIAL LAW AND REGULARLY VARYING MOMENTS

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Abstract. In this paper we demonstrate a method for estimating asymptotic behavior of the regularly varying moments $E(K_\rho(X_n))$, $(n \to \infty)$ in the case of generalized Binomial Law. Here $K_\rho(x)$ is from the class of regularly varying functions in the sense of Karamata. We prove that

$$E(K_\rho(X_n)) \sim K_\rho(E(X_n)),\ \rho > 0, \quad E(X_n) \to \infty \quad (n \to \infty),$$

i.e., that the asymptotics of the first moment determines the behavior of all other moments.

1. Introduction

1.1. We shall consider a polynomial $P_n(c) := \sum_{k\le n} p_{nk}c^k$ with non-positive zeros and a random variable X_n defined as follows:

$$P\{X_n = k\} = \frac{p_{nk}c^k}{P_n(c)}, \qquad k \le n; \quad k, n \in N \cup \{0\}.$$

We call this a generalized Binomial Law with parameter $c > 0$, since for

$$P_n(c) = (1+c)^n, \quad c/(1+c) := p; \quad 1/(1+c) := q,$$

we obtain the well-known Binomial Law.

Define also, in the usual way, the first moment $E(X_n)$ and variance $D^2(X_n)$:

$$E(X_n) := \frac{1}{P_n(c)} \sum_{k\le n} kp_{nk}c^k; \qquad D^2(X_n) := \frac{1}{P_n(c)} \sum_{k\le n} (k - E(X_n))^2 p_{nk}c^k.$$

The aim of this paper is to determine the asymptotic behavior of the moments generalized in the following way.

Let $K_\rho(x) := x^\rho \ell(x),\ x > 0;\ K_\rho(0) := 0$ be a regularly varying function of index $\rho \in R$ in the sense of Karamata. Then

$$E(K_\rho(X_n)) := \frac{1}{P_n(c)} \sum_{k\le n} k^\rho \ell(k) p_{nk}c^k, \quad \rho \in R.$$

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We shall prove the following

THEOREM A. *For the generalized Binomial Law, defined above, we have*

$$E(K_\rho(X_n)) \sim K_\rho(E(X_n)), \quad E(X_n) \to \infty \quad (n \to \infty).$$

for each $\rho \in R^+$.

Therefore, for this class of distributions, it is particularly simple to determine the asymptotic behavior of its moments.

1.2. Karamata's class K_ρ plays here an important role. We say that $c \in K_\rho$ if it can be represented in the form $c(x) := x^\rho \ell(x)$, $x > 0$, $\rho \in R$, where ρ is the index of regular variation and $\ell(x) \in K_0$ is a slowly varying function, i.e., positive, measurable and satisfying $\ell(tx) \sim \ell(x)$, $\forall t > 0$ $(x \to \infty)$. Some examples of $\ell(x)$ are:

$$1,\ \log^a x,\ \log^b(\log x),\ \exp\left(\frac{\log x}{\log\log x}\right),\ \exp(\log^c x), \quad a, b \in R;\ 0 < c < 1.$$

According to [2], a sequence (c_n), $c_0 = 0$ is regularly varying with index $\rho \in R$ if it has the form $c_n := n^\rho \ell_n$, $n \in N$ and $\ell_n = \ell(n)$ for some continuous $\ell \in K_0$. Then we also say that $c_n \in K_\rho$.

The theory of regular variation is well-developed and for more details see [1] and [4].

2. Proofs

We prove Theorem A in three steps.

First, we suppose that $\rho \in N$, $\ell(\cdot) := 1$, and prove the next proposition.

PROPOSITION 1. *If* $E(X_n) \to \infty$ $(n \to \infty)$, *then*

$$E(X_n^m) := \frac{1}{P_n(c)} \sum_{k \le n} k^m p_{nk} c^k \sim (E(X_n))^m \quad (n \to \infty),$$

for each $m \in N$.

Denote by A the set of all polynomials with non-positive zeros.

To prove the last assertion, we need the following lemma.

LEMMA 1. *If* $P_n(c) \in A$ *and* $E(X_n)$, $D^2(X_n)$ *are defined as above, then* $0 \le \dfrac{D^2(X_n)}{E(X_n)} < 1$ *for each* $c \in R^+$, $n \in N$.

Proof. Since $P_n(c) \in A$, it can be represented in the form

$$P_n(c) = p_{nn} \prod_{k \le n} (c + a_{nk}), \quad a_{nk} \ge 0.$$

Hence

$$E(X_n) = c\frac{d}{dc}(\log P_n(c)) = \sum_{k\le n} \frac{c}{c + a_{nk}};$$

$$D^2(X_n) = E(X_n^2) - E^2(X_n) = c\frac{d}{dc}(E(X_n)) = \sum_{k\le n} \frac{ca_{nk}}{(c + a_{nk})^2}.$$

Therefore,

$$D^2(X_n) = \sum_{k\le n} \frac{c}{c + a_{nk}} \cdot \frac{a_{nk}}{c + a_{nk}} < \sum_{k\le n} \frac{c}{c + a_{nk}} = E(X_n),$$

i.e., Lemma 1 is proved. ■

Consider now a sequence of polynomials $\{Q_m(c)\}$ generated from $P_n(c)$ by the recurrence relation

$$Q_m(c) := cQ'_{m-1}(c); \qquad Q_0(c) := P_n(c), \qquad m \in N.$$

It is easy to see that

$$Q_m(c) = \sum_{k\le n} k^m p_{nk} c^k = E(X_n^m) P_n(c) \quad m \in N.$$

Since $P_n(c) \in A$, by the classical result, its zeros are separated by the zeros of $P'_n(c)$. Hence, zeros of $Q_1(c) := cP'_n(c)$ are also non-positive.

By induction we obtain $Q_m(c) \in A, \ \ m \in N$. Therefore, we can apply Lemma 1 to the polynomial $Q = Q_m(c) \in A$ and obtain

$$0 \le T_m := \frac{D_Q^2(X_n)}{E_Q(X_n)} < 1, \quad m \in N.$$

But $T_m = E(X_n^{m+1})/E(X_n^m) - E(X_n^m)/E(X_n^{m-1})$, $m \in N$; hence

$$\frac{E(X_n^m)}{E(X_n^{m-1})} = E(X_n) + \sum_{k\le m} T_{k-1} = E(X_n) + O(m).$$

On the other hand,

$$\begin{aligned} E(X_n^m) &= \prod_{k\le m} E(X_n^k)/E(X_n^{k-1}) \\ &= \prod_{k\le m} (E(X_n) + O(k)) = E(X_n)^m + O(m^2)E(X_n)^{m-1}. \end{aligned}$$

Since $m \in N$ is fixed and $E(X_n) \to \infty$ $(n \to \infty)$, Proposition 1 is proved. ■

In the next step, we shall prove our assertion for real positive exponents i.e.,

PROPOSITION 2. *If* $E(X_n) \to \infty$ $(n \to \infty)$ *then*

$$E(X_n^\rho) \sim (E(X_n))^\rho \qquad (n \to \infty),$$

for each $\rho \in R^+$.

Proof. For this we need the well-known Lyapunov's moments inequality

LEMMA 2. *For real $r > s > t$ we have*

$$(E(X_n^s))^{r-t} \le (E(X_n^r))^{s-t} \cdot (E(X_n^t))^{r-s}.$$

Let $m < \rho < m-1$, $m \in N$. Applying Lemma 2 and Proposition 1, we get

$$E(X_n^\rho) \le (E(X_n^m))^{\rho-m+1} \cdot (E(X_n^{m-1}))^{m-\rho}$$
$$= (E(X_n))^{m(\rho-m+1)+(m-1)(m-\rho)}(1+o(1)) = (E(X_n))^\rho(1+o(1)).$$

Hence $\limsup_{n\to\infty} E(X_n^\rho)/(E(X_n))^\rho \le 1$.

Putting now in Lyapunov's inequality $r := m+1;\ s := m;\ t := \rho$ we obtain

$$E(X_n^\rho) \ge (E(X_n^m))^{m+1-\rho}/(E(X_n^{m+1}))^{m-\rho}$$
$$= (E(X_n))^{m(m+1-\rho)-(m+1)(m-\rho)}(1+o(1)) = (E(X_n))^\rho(1+o(1)),$$

i.e., $\liminf_{n\to\infty} E(X_n^\rho)/(E(X_n))^\rho \ge 1$.

Therefore, Proposition 2 is proved. ■

Now we are able to prove Theorem A. For this, we just need the following assertion which is fundamental in the Theory of Regular Variation ([1], [4]).

LEMMA 3. *For any slowly varying $\ell(\cdot)$, some $\mu \in R^+$ and $y \to \infty$, we have*

$$(i)\quad \sup_{x<y}(x^\mu \ell(x)) \sim y^\mu \ell(y); \qquad (ii)\quad \sup_{x>y} x^{-\mu}\ell(x) \sim y^{-\mu}\ell(y).$$

We shall estimate the expression T,

$$T := \frac{E(K_\rho(X_n))}{E(X_n^\rho)\ell(E(X_n))} - 1 = \frac{\sum_{k\le n} k^\rho p_{nk}(\ell(k)/\ell(E(X_n)) - 1)c^k}{\sum_{k\le n} k^\rho p_{nk} c^k}.$$

Now, for some σ, $0 < \sigma < 1$ we get

$$|T| \le \frac{\sum_{k\le n} k^\rho p_{nk}|\ell(k)/\ell(E(X_n)) - 1|c^k}{\sum_{k\le n} k^\rho p_{nk} c^k}$$
$$= \frac{1}{\sum_{k\le n} k^\rho p_{nk} c^k}\Big(\sum_{k<\sigma E(X_n)} + \sum_{\sigma E(X_n)\le k\le E(X_n)/\sigma} + \sum_{k>E(X_n)/\sigma}\Big)$$
$$= T_1 + T_2 + T_3.$$

Applying Lemma 3 (part (i)) and Proposition 2, we obtain

$$T_1 = \frac{1}{\sum_{k\le n} k^\rho p_{nk} c^k}\sum_{k<\sigma E(X_n)} k^{\rho/2} p_{nk}|k^{\rho/2}\ell(k)/\ell(E(X_n)) - k^{\rho/2}|c^k$$
$$\le \sup_{k\le\sigma E(X_n)}(k^{\rho/2}\ell(k)/\ell(E(X_n)) + k^{\rho/2})\frac{E(X_n^{\rho/2})}{E(X_n^\rho)}$$
$$\sim 2(\sigma E(X_n))^{\rho/2}\cdot(E(X_n))^{-\rho/2} \ll \sigma^{\rho/2},$$

and, analogously, using (ii) of Lemma 3,

$$T_3 \le \sup_{k > E(X_n)/\sigma} (k^{-\rho/2}\ell(k)/\ell(E(X_n)) + k^{-\rho/2}) \frac{E(X_n^{3\rho/2})}{E(X_n^\rho)}$$
$$\sim 2(E(X_n)/\sigma)^{-\rho/2} \cdot (E(X_n))^{\rho/2} \ll \sigma^{\rho/2}.$$

We also have

$$T_2 = \frac{\sum_{\sigma E(X_n) \le k \le E(X_n)/\sigma} k^\rho p_{nk} |\ell(k)/\ell(E(X_n)) - 1| c^k}{\sum_{k \le n} k^\rho p_{nk} c^k}$$
$$\le \sup_{\sigma E(X_n) \le k \le E(X_n)/\sigma} |\ell(k)/\ell(E(X_n)) - 1| = o(1) \quad (E(X_n) \to \infty),$$

by the Uniform Convergence Theorem ([1], pp. 6–11).

Therefore,

$$T \le T_1 + T_2 + T_3 = O(\sigma^{\rho/2}) + o(1) \quad (n \to \infty).$$

Since $\rho > 0$ and σ can be taken arbitrarily small, we deduce that

$$E(K_\rho(E(X_n)) \sim E(X_n^\rho)\ell(X_n) \sim (E(X_n))^\rho \ell(E(x_n)) = K_\rho(E(X_n)) \quad (n \to \infty),$$

i.e., operators E and K_ρ are asymptotically commutative, which was the content of Theorem A. Hence, the proof is done. ■

REMARK 1. In the previous proof, the sum T_3 may be empty. But then

$$T_2 \le \sup_{\sigma E(X_n) \le k \le n} |\ell(k)/\ell(E(X_n)) - 1|$$
$$\le \sup_{\sigma E(X_n) \le k \le E(X_n)/\sigma} |\ell(k)/\ell(E(X_n) - 1| = o(1) \quad (n \to \infty),$$

by Uniform Convergence Theorem again.

Finally, we give some applications of Theorem A.

EXAMPLE 1. Taking $P_n(c) := (1+c)^n$; $E(X_n) = \frac{c}{1+c}n$ $(n \to \infty)$ and putting $\frac{c}{1+c} := p$; $\frac{1}{1+c} := q$ we obtain an asymptotic formula for regularly varying moments of the Binomial Law:

$$\sum_{k \le n} k^\rho \ell_k \binom{n}{k} p^k q^{n-k} \sim p^\rho n^\rho \ell_n, \quad \rho \in R^+ \quad (n \to \infty).$$

EXAMPLE 2. Laguerre polynomials $L_n^{(a)}(c)$ of index $a > -1$ have all zeros real and positive. Hence $L_n^{(a)}(-c)$, $c > 0$, satisfy the condition of Theorem A. Using Perron's formula (cf.[3], p.197) we obtain $E(X_n) \sim \sqrt{cn}$ $(n \to \infty)$, i.e.,

$$\frac{1}{L_n^{(a)}(-c)} \sum_{k \le n} k^\rho \ell_k \binom{n+a}{n-k} \frac{c^k}{k!} \sim c^{\rho/2} n^{\rho/2} \ell(\sqrt{n}), \quad c > 0, \ \rho \in R^+ \quad (n \to \infty).$$

REMARK 2. Further considerations can show that Theorem A is also valid for negative values of exponent ρ (see [5]).

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