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GROWTH AND OSCILLATION THEORY OF SOLUTIONS OF SOME LINEAR DIFFERENTIAL EQUATIONS

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Abstract. The basic idea of this paper is to consider fixed points of solutions of the differential equation $f^{(k)} + A(z) f = 0$, $k \geq 2$, where $A(z)$ is a transcendental meromorphic function with $\rho(A) = \rho > 0$. Instead of looking at the zeros of $f(z) - z$, we proceed to a slight generalization by considering zeros of $f(z) - \varphi(z)$, where φ is a meromorphic function of finite order, while the solution of respective differential equation is of infinite order.

1. Introduction and main results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory and with basic Wiman Valiron theory as well (see [7], [8], [12], [13]). In addition, we will use $\lambda(f)$ and $\lambda(1/f)$ to denote respectively the exponents of convergence of the zero-sequence and the pole-sequence of a meromorphic function f, $\rho(f)$ to denote the order of growth of f, $\overline{\lambda}(f)$ and $\overline{\lambda}(1/f)$ to denote respectively the exponents of convergence of the sequence of distinct zeros and distinct poles of f. A meromorphic function $\varphi(z)$ is called a small function of a meromorphic function $f(z)$ if $T(r,\varphi) = o(T(r,f))$ as $r \to +\infty$, where $T(r, f)$ is the Nevanlinna characteristic function of f. In order to express the rate of growth of meromorphic solutions of infinite order, we recall the following definition.

DEFINITION 1.1. [4], [11], [14] Let f be a meromorphic function. Then the hyper-order $\rho_2(f)$ of $f(z)$ is defined by

$$\rho_2(f) = \overline{\lim_{r\to+\infty}} \frac{\log\log T(r,f)}{\log r}. \tag{1.1}$$

To give some estimates of fixed points, we recall the following definitions.

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Definition 1.2. [4], [10], [11] Let f be a meromorphic function. Then the exponent of convergence of the sequence of distinct fixed points of $f(z)$ is defined by

$$\overline{\tau}(f) = \overline{\lambda}(f - z) = \overline{\lim_{r\to+\infty}} \frac{\log \overline{N}\left(r, \frac{1}{f-z}\right)}{\log r}, \tag{1.2}$$

where $\overline{N}\left(r, \frac{1}{f}\right)$ is the counting function of distinct zeros of $f(z)$ in $\{|z| < r\}$.

Definition 1.3. [9], [10], [11] Let f be a meromorphic function. Then the hyper exponent of convergence of the sequence of distinct zeros of $f(z)$ is defined by

$$\overline{\lambda}_2(f) = \overline{\lim_{r\to+\infty}} \frac{\log\log \overline{N}\left(r, \frac{1}{f}\right)}{\log r}. \tag{1.3}$$

Definition 1.4. [10], [11], [14] Let f be a meromorphic function. Then the hyper exponent of convergence of the sequence of distinct fixed points of $f(z)$ is defined by

$$\overline{\tau}_2(f) = \overline{\lambda}_2(f - z) = \overline{\lim_{r\to+\infty}} \frac{\log\log \overline{N}\left(r, \frac{1}{f-z}\right)}{\log r}. \tag{1.4}$$

Thus $\overline{\tau}_2(f) = \overline{\lambda}_2(f - z)$ is an indication of oscillation of distinct fixed points of $f(z)$.

For $k \geq 2$, we consider the linear differential equation

$$f^{(k)} + A(z) f = 0, \tag{1.5}$$

where $A(z)$ is a transcendental meromorphic function with finite order $\rho(A) = \rho > 0$. Many important results have been obtained on the fixed points of general transcendental meromorphic functions for almost four decades (see [16]). However, there are a few studies on the fixed points of solutions of differential equations. It was in year 2000 that Z. X. Chen first pointed out the relation between the exponent of convergence of distinct fixed points and the rate of growth of solutions of second order linear differential equations with entire coefficients (see [4]). In [14], Wang and Yi investigated fixed points and hyper order of differential polynomials generated by solutions of second order linear differential equations with meromorphic coefficients. In [10], Laine and Rieppo gave improvement of the results of [14] by considering fixed points and iterated order. In [15], Wang and Lü have investigated the fixed points and hyper order of solutions of second order linear differential equations with meromorphic coefficients and their derivatives and have obtained the following result:

Theorem A. [15] *Suppose that $A(z)$ is a transcendental meromorphic function satisfying $\delta(\infty, A) = \underline{\lim}_{r\to+\infty} \frac{m(r,A)}{T(r,A)} = \delta > 0$, $\rho(A) = \rho < +\infty$. Then every meromorphic solution $f(z) \not\equiv 0$ of the equation*

$$f'' + A(z) f = 0, \tag{1.6}$$

satisfies that f and f', f'' all have infinitely many fixed points and

$$\overline{\tau}(f) = \overline{\tau}(f^{'}) = \overline{\tau}(f^{''}) = \rho(f) = +\infty, \tag{1.7}$$

$$\overline{\tau}_2(f) = \overline{\tau}_2(f^{'}) = \overline{\tau}_2(f^{''}) = \rho_2(f) = \rho. \tag{1.8}$$

Recently, Theorem A has been generalized to higher order differential equations by Liu Ming-Sheng and Zhang Xiao-Mei as follows [11]:

Theorem B. [11] *Suppose that $k \geq 2$ and $A(z)$ is a transcendental meromorphic function satisfying $\delta(\infty, A) = \overline{\lim}_{r\to+\infty} \frac{m(r,A)}{T(r,A)} = \delta > 0$, $\rho(A) = \rho < +\infty$. Then every meromorphic solution $f(z) \not\equiv 0$ of (1.5), satisfies that f and f', f'', $\dots$, $f^{(k)}$ all have infinitely many fixed points and*

$$\overline{\tau}(f) = \overline{\tau}(f') = \cdots = \overline{\tau}(f^{(k)}) = \rho(f) = +\infty, \tag{1.9}$$

$$\overline{\tau}_2(f) = \overline{\tau}_2(f') = \cdots = \overline{\tau}_2(f^{(k)}) = \rho_2(f) = \rho. \tag{1.10}$$

The main purpose of this paper is to study the relation between solutions and their derivatives of the differential equation (1.5) and small meromorphic functions. We obtain an extension of Theorems A–B. In fact, we prove the following results:

Theorem 1.1. *Let $A(z)$ be a transcendental meromorphic function of finite order $\rho(A) = \rho > 0$ such that $\delta(\infty, A) = \delta > 0$. Suppose, moreover, that either:*

(i) all poles of f are of uniformly bounded multiplicity or that

(ii) $\delta(\infty, f) > 0$.

Let d_j $(j = 0, 1, 2)$ be polynomials that are not all equal to zero, $\varphi(z)$ $(\not\equiv 0)$ be a meromorphic function of finite order. If $f(z) \not\equiv 0$ is a meromorphic solution of (1.6) with $\lambda(1/f) < +\infty$, then the differential polynomial $g(z) = d_2 f'' + d_1 f' + d_0 f$ satisfies $\overline{\lambda}(g - \varphi) = +\infty$.

Theorem 1.2. *Let $k \geq 2$ and $A(z)$ be a transcendental meromorphic function of finite order $\rho(A) = \rho > 0$ such that $\delta(\infty, A) = \delta > 0$. Suppose, moreover, that either:*

(i) all poles of f are of uniformly bounded multiplicity or that

(ii) $\delta(\infty, f) > 0$.

If $\varphi(z) \not\equiv 0$ is a meromorphic function with finite order $\rho(\varphi) < +\infty$, then every meromorphic solution $f(z) \not\equiv 0$ of (1.5), satisfies

$$\overline{\lambda}(f - \varphi) = \overline{\lambda}(f' - \varphi) = \cdots = \overline{\lambda}(f^{(k)} - \varphi) = \rho(f) = +\infty, \tag{1.11}$$

$$\overline{\lambda}_2(f - \varphi) = \overline{\lambda}_2(f' - \varphi) = \cdots = \overline{\lambda}_2(f^{(k)} - \varphi) = \rho_2(f) = \rho. \tag{1.12}$$

Setting $\varphi(z) = z$ in Theorem 1.1 and Theorem 1.2, we obtain the following corollaries:

Corollary 1.1. *Let $A(z)$ be a transcendental meromorphic function of finite order $\rho(A) = \rho > 0$ such that $\delta(\infty, A) = \delta > 0$. Suppose, moreover, that either:*

(i) all poles of f are of uniformly bounded multiplicity or that

(ii) $\delta(\infty, f) > 0$.

Let d_j $(j = 0, 1, 2)$ be polynomials that are not all equal to zero. If $f(z) \not\equiv 0$ is a meromorphic solution of (1.6) *with $\lambda\left(\frac{1}{f}\right) < +\infty$, then the differential polynomial $g(z) = d_2 f'' + d_1 f' + d_0 f$ has infinitely many fixed points and satisfies $\overline{\lambda}(g - z) = +\infty$.*

Corollary 1.2. *Let $k \geq 2$ and $A(z)$ be a transcendental meromorphic function of finite order $\rho(A) = \rho > 0$ such that $\delta(\infty, A) = \delta > 0$. Suppose, moreover, that either:*

(i) all poles of f are of uniformly bounded multiplicity or that

(ii) $\delta(\infty, f) > 0$.

Then every meromorphic solution $f(z) \not\equiv 0$ of (1.5)*, satisfies that f and f', f'', $\ldots$, $f^{(k)}$ all have infinitely many fixed points and*

$$\overline{\tau}(f) = \overline{\tau}(f') = \overline{\tau}(f'') = \cdots = \overline{\tau}(f^{(k)}) = \rho(f) = +\infty, \tag{1.13}$$

$$\overline{\tau}_2(f) = \overline{\tau}_2(f') = \overline{\tau}_2(f'') = \cdots = \overline{\tau}_2(f^{(k)}) = \rho_2(f) = \rho. \tag{1.14}$$

2. Some Lemmas

We need the following lemmas in the proofs of our theorems.

Lemma 2.1. [6] *Let f be a transcendental meromorphic function of finite order ρ, let $\Gamma = \{(k_1, j_1), (k_2, j_2), \ldots, (k_m, j_m)\}$ denote a finite set of distinct pairs of integers that satisfy $k_i > j_i \geq 0$ for $i = 1, \ldots, m$ and let $\varepsilon > 0$ be a given constant. Then the following estimations hold:*

(i) There exists a set $E_1 \subset [0, 2\pi)$ that has linear measure zero, such that if $\psi \in [0, 2\pi) - E_1$, then there is a constant $R_1 = R_1(\psi) > 1$ such that for all z satisfying $\arg z = \psi$ and $|z| \geq R_1$ and for all $(k, j) \in \Gamma$, we have

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq |z|^{(k-j)(\rho-1+\varepsilon)}. \tag{2.1}$$

(ii) There exists a set $E_2 \subset (1, \infty)$ that has finite logarithmic measure $lm(E_2) = \int_1^{+\infty} \frac{\chi_{E_2}(t)}{t} dt < +\infty$, where χ_{E_2} is the characteristic function of E_2, such that for all z satisfying $|z| \notin E_2 \cup [0, 1]$ and for all $(k, j) \in \Gamma$, we have

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq |z|^{(k-j)(\rho-1+\varepsilon)}. \tag{2.2}$$

We recall a special case of an important result due to Chiang and Hayman in [5]:

Lemma 2.2. [5] *Let f be a meromorphic solution of the differential equation*

$$f^{(n)} + A_{n-1}(z) f^{(n-1)} + \cdots + A_1(z) f' + A_0(z) f = 0 \tag{2.3}$$

in the complex plane, assuming that not all coefficients $A_j(z)$ are constants. Given a real constant $\sigma > 1$, and denoting $T(r) := \sum_{j=0}^{n-1} T(r, A_j)$, we have

$$\log m(r, f) < r^{1+\sigma} T(r) \{\log T(r)\}^{\sigma} \tag{2.4}$$

outside of an exceptional set E_3 of finite linear measure.

To avoid some problems caused by the exceptional set we recall the following Lemma.

Lemma 2.3. ([1, p. 68]). *Let $g\colon [0, +\infty) \to \mathbf{R}$ and $h\colon [0, +\infty) \to \mathbf{R}$ be monotone non-decreasing functions such that $g(r) \leq h(r)$ outside of an exceptional set E_4 of finite linear measure. Then for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \leq h(\alpha r)$ for all $r > r_0$.*

Lemma 2.4. *Let $k \geq 2$ and $A(z)$ be a transcendental meromorphic function of finite order $\rho(A) = \rho > 0$ such that $\delta(\infty, A) = \delta > 0$. Suppose, moreover, that either:*

(i) all poles of f are of uniformly bounded multiplicity or that

(ii) $\delta(\infty, f) > 0$.

Then every meromorphic solution $f(z) \not\equiv 0$ of (1.5) satisfies $\rho(f) = +\infty$ and $\rho_2(f) = \rho(A) = \rho$.

Proof. First, we prove that $\rho(f) = +\infty$. We suppose that $\rho(f) = \beta < +\infty$ and then we obtain a contradiction. Rewrite (1.5) as

$$A = -\frac{f^{(k)}}{f}. \tag{2.5}$$

By $\rho(f) = \beta < +\infty$, we have

$$m\left(r, \frac{f^{(k)}}{f}\right) = O(\log r). \tag{2.6}$$

It follows from the definition of deficiency $\delta(\infty, A)$ that for sufficiently large r, we have

$$m(r, A) \geq \frac{\delta}{2} T(r, A). \tag{2.7}$$

So when r sufficiently large, we have by (2.5)–(2.7)

$$T(r, A) \leq \frac{2}{\delta} m(r, A) = \frac{2}{\delta} m\left(r, \frac{f^{(k)}}{f}\right) = O(\log r). \tag{2.8}$$

Therefore, by (2.8), we obtain a contradiction since A is a transcendental meromorphic function with $\rho(A) = \rho > 0$. Hence $\rho(f) = +\infty$.

Now, we prove that $\gamma = \rho_2(f) = \rho(A) = \rho$. By $\rho(f) = +\infty$, there exists a set E with finite linear measure such that

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\log\left(rT\left(r,f\right)\right)\right) \tag{2.9}$$

for $r \notin E$. It follows from the definition of deficiency $\delta(\infty, A)$ and (2.9) that for sufficiently large $r \notin E$, we have

$$T(r,A) \leq \frac{2}{\delta} m(r,A) = \frac{2}{\delta} m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\log\left(rT\left(r,f\right)\right)\right). \tag{2.10}$$

By Lemma 2.3, we have for any $\alpha > 1$

$$T(r,A) \leq O\left(\log\left(\alpha r T\left(\alpha r, f\right)\right)\right). \tag{2.11}$$

for sufficiently large r. Hence by the definition of hyper-order, we obtain that

$$\gamma = \rho_2(f) = \overline{\lim_{r\to+\infty}} \frac{\log\log T(r,f)}{\log r} \geq \rho(A) = \rho. \tag{2.12}$$

To prove the converse inequality, we observe that all poles of f occur at the poles of A only. If the multiplicity of the poles of f is uniformly bounded, we have

$$T(r,f) = m(r,f) + O\left(\overline{N}(r,A)\right) = m(r,f) + O\left(T(r,A)\right), \tag{2.13}$$

while if $\delta(\infty, f) > 0$, then

$$T(r,f) = O\left(m\left(r,f\right)\right). \tag{2.14}$$

Applying now either (2.13) or (2.14) with Lemma 2.2 and Lemma 2.3, we immediately get the required reversed inequality. ■

LEMMA 2.5. [3] *Let $A_0,\ A_1, \ldots, A_{k-1},\ F \not\equiv 0$ be finite order meromorphic functions. If f is a meromorphic solution with $\rho(f) = +\infty$ of the equation*

$$f^{(k)} + A_{k-1} f^{(k-1)} + \cdots + A_1 f' + A_0 f = F, \tag{2.15}$$

then $\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty$.

LEMMA 2.6 *Let $A_0,\ A_1, \ldots, A_{k-1},\ F \not\equiv 0$ be finite order meromorphic functions. If f is a meromorphic solution of the equation (2.15) with $\rho(f) = +\infty$ and $\rho_2(f) = \rho$, then f satisfies $\overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) = \rho$.*

Proof. By equation (2.15), we can write

$$\frac{1}{f} = \frac{1}{F}\left(\frac{f^{(k)}}{f} + A_{k-1}\frac{f^{(k-1)}}{f} + \cdots + A_1\frac{f'}{f} + A_0\right). \tag{2.16}$$

If f has a zero at z_0 of order $\alpha\,(>k)$ and if $A_0,\ A_1,\ \ldots\ ,\ A_{k-1}$ are all analytic at z_0, then F has a zero at z_0 of order at least $\alpha-k$. Hence,

$$n\left(r,\frac{1}{f}\right)\leq k\ \overline{n}\left(r,\frac{1}{f}\right)+n\left(r,\frac{1}{F}\right)+\sum_{j=0}^{k-1}n\left(r,A_j\right) \tag{2.17}$$

and

$$N\left(r,\frac{1}{f}\right)\leq k\ \overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{F}\right)+\sum_{j=0}^{k-1}N\left(r,A_j\right). \tag{2.18}$$

By (2.16), we have

$$m\left(r,\frac{1}{f}\right)\leq \sum_{j=1}^{k}m\left(r,\frac{f^{(j)}}{f}\right)+\sum_{j=0}^{k-1}m\left(r,A_j\right)+m\left(r,\frac{1}{F}\right)+O\left(1\right). \tag{2.19}$$

Applying the Lemma of the logarithmic derivative (see [8]), we have

$$m\left(r,\frac{f^{(j)}}{f}\right)=O\left(\log T\left(r,f\right)+\log r\right)\quad\left(j=1,\ldots,k\right), \tag{2.20}$$

holds for all r outside a set $E\subset(0,+\infty)$ with a finite linear measure $m\left(E\right)<+\infty$. By (2.18), (2.19) and (2.20), we get

$$\begin{aligned}T\left(r,f\right)&=T\left(r,\frac{1}{f}\right)+O\left(1\right)\\&\leq k\overline{N}\left(r,\frac{1}{f}\right)+\sum_{j=0}^{k-1}T\left(r,A_j\right)+T\left(r,F\right)+O\left(\log\left(rT\left(r,f\right)\right)\right)\ \ \left(|z|=r\notin E\right).\end{aligned} \tag{2.21}$$

Since $\rho\left(f\right)=+\infty$, then there exists $\{r_n'\}$ $(r_n'\to+\infty)$ such that

$$\lim_{r_n'\to+\infty}\frac{\log T\left(r_n',f\right)}{\log r_n'}=+\infty. \tag{2.22}$$

Set the linear measure of E, $m\left(E\right)=\gamma<+\infty$, then there exists a point $r_n\in[r_n',r_n'+\gamma+1]-E$. From

$$\frac{\log T\left(r_n,f\right)}{\log r_n}\geq\frac{\log T\left(r_n',f\right)}{\log\left(r_n'+\gamma+1\right)}=\frac{\log T\left(r_n',f\right)}{\log r_n'+\log\left(1+\left(\gamma+1\right)/r_n'\right)}, \tag{2.23}$$

it follows that

$$\lim_{r_n\to+\infty}\frac{\log T\left(r_n,f\right)}{\log r_n}=+\infty. \tag{2.24}$$

Set $\sigma=\max\{\,\rho(A_j)\mid j=0,\ldots,k-1,\ \rho(F)\,\}$. Then for a given arbitrary large $\beta>\sigma$,

$$T\left(r_n,f\right)\geq r_n^{\beta} \tag{2.25}$$

holds for sufficiently large r_n. On the other hand, for any given ε with $0<2\varepsilon<\beta-\sigma$, we have

$$T\left(r_n,A_j\right)\leq\ r_n^{\sigma+\varepsilon}\quad\left(j=0,\ldots,k-1\right),\ \ T\left(r_n,F\right)\leq\ r_n^{\sigma+\varepsilon} \tag{2.26}$$

for sufficiently large r_n. Hence, we have

$$\max\left\{\frac{T(r_n,F)}{T(r_n,f)},\frac{T(r_n,A_j)}{T(r_n,f)}\ \ (j=0,\dots,k-1)\right\}\leq\frac{r_n^{\sigma+\varepsilon}}{r_n^{\beta}}\to 0,\ r_n\to+\infty. \quad (2.27)$$

Therefore,

$$T(r_n,F)\leq\frac{1}{k+3}T(r_n,f),\quad T(r_n,A_j)\leq\frac{1}{k+3}T(r_n,f)\quad (j=0,\dots,k-1) \quad (2.28)$$

holds for sufficiently large r_n. From

$$O(\log r_n+\log T(r_n,f))=o(T(r_n,f)), \quad (2.29)$$

we obtain that

$$O(\log r_n+\log T(r_n,f))\leq\frac{1}{k+3}T(r_n,f) \quad (2.30)$$

also holds for sufficiently large r_n. Thus, by (2.21), (2.28), (2.30), we have

$$T(r_n,f)\leq k(k+3)\ \overline{N}\left(r_n,\frac{1}{f}\right). \quad (2.31)$$

It yields $\overline{\lambda}_2(f)=\lambda_2(f)=\rho_2(f)=\rho$. ■

LEMMA 2.7. [7,p. 344] *Let $f(z)=\sum_{n=0}^{\infty}a_n z^n$ be an entire function, $\mu(r)$ be the maximum term, i.e., $\mu(r)=\max\{|a_n|r^n \mid n=0,1,\dots\}$ and let $\nu_f(r)$ be the central index of f, i.e., $\nu_f(r)=\max\{m,\ \mu(r)=|a_m|r^m\}$. Then*

$$\nu_f(r)=r\frac{d}{dr}\log\mu(r)<[\log\mu(r)]^2\leq[\log M(r,f)]^2 \quad (2.32)$$

holds outside a set $E_5\subset(1,+\infty)$ of r of finite logarithmic measure.

LEMMA 2.8. (Wiman-Valiron, [7], [13]) *Let $f(z)$ be a transcendental entire function, and let z be a point with $|z|=r$ at which $|f(z)|=M(r,f)$. Then the estimation*

$$\frac{f^{(k)}(z)}{f(z)}=\left(\frac{\nu_f(r)}{z}\right)^k(1+o(1))\,(k\geqslant 1\ \textit{is an integer}), \quad (2.33)$$

holds for all $|z|$ outside a set E_6 of r of finite logarithmic measure.

LEMMA 2.9. [2] *Suppose that $f(z)$ is a meromorphic function with $\rho(f)=\beta<+\infty$. Then for any given $\varepsilon>0$, there is a set $E_7\subset(1,+\infty)$ of finite logarithmic measure, such that*

$$|f(z)|\leq\exp\left\{r^{\beta+\varepsilon}\right\} \quad (2.34)$$

holds for $|z|=r\notin[0,1]\cup E_7$, $r\to+\infty$.

LEMMA 2.10. *Let $f(z)$ be a meromorphic function with $\rho(f)=+\infty$ and the exponent of convergence of the poles of $f(z)$, $\lambda(1/f)<+\infty$. Let $d_j(z)$ $(j=0,1,2)$ be polynomials that are not all equal to zero. Then*

$$g(z)=d_2(z)f''+d_1(z)f'+d_0(z)f \quad (2.35)$$

satisfies $\rho(g)=+\infty$.

Proof. We suppose that $\rho(g) = \rho < +\infty$ and then we obtain a contradiction. First, we suppose that $d_2(z) \not\equiv 0$. Set $f(z) = w(z)/h(z)$, where $h(z)$ is canonical product (or polynomial) formed with the non-zero poles of $f(z)$, $\lambda(h) = \rho(h) = \lambda(1/f) = \rho_1 < +\infty$, $w(z)$ is an entire function with $\rho(w) = \rho(f) = +\infty$. We have

$$f'(z) = \frac{w'h - h'w}{h^2} \text{ and } f''(z) = \frac{w''}{h} - \frac{wh''}{h^2} - 2\frac{w'h'}{h^2} + 2\frac{(h')^2 w}{h^3}. \tag{2.36}$$

Hence

$$\frac{f''(z)}{f(z)} = \frac{w''}{w} - \frac{h''}{h} - 2\frac{w'h'}{wh} + 2\frac{(h')^2}{h^2}, \tag{2.37}$$

$$\frac{f'(z)}{f(z)} = \frac{w'}{w} - \frac{h'}{h}. \tag{2.38}$$

By Lemma 2.1 (ii), there exists a set $E_1 \subset (1, +\infty)$ that has finite logarithmic measure, such that for all z satisfying $|z| \notin E_1 \cup [0,1]$, we have

$$\left|\frac{h^{(j)}(z)}{h(z)}\right| \le |z|^{j(\rho_1 - 1 + \varepsilon)} \quad (j = 1, 2). \tag{2.39}$$

Substituting (2.39) into (2.37) and (2.38), we obtain for all z satisfying $|z| \notin E_1 \cup [0,1]$

$$\frac{f''(z)}{f(z)} = \frac{w''}{w} + O(z^\alpha)\frac{w'}{w} + O(z^\alpha), \tag{2.40}$$

$$\frac{f'(z)}{f(z)} = \frac{w'}{w} + O(z^\alpha), \tag{2.41}$$

where α $(0 < \alpha < +\infty)$ is a constant and may be different at different places. Substituting (2.40) and (2.41) into (2.35), we obtain

$$d_2(z)\left(\frac{w''}{w} + O(z^\alpha)\frac{w'}{w} + O(z^\alpha)\right) + d_1(z)\left(\frac{w'}{w} + O(z^\alpha)\right) + d_0(z) = \frac{g(z)h(z)}{w(z)}. \tag{2.42}$$

It follows that

$$d_2(z)\frac{w''}{w} + (O(z^\alpha)d_2(z) + d_1(z))\frac{w'}{w} + O(z^\alpha)d_2(z) + O(z^\alpha)d_1(z) + d_0(z) = \frac{g(z)h(z)}{w(z)}. \tag{2.43}$$

Hence,

$$d_2(z)\frac{w''}{w} + O(z^m)\frac{w'}{w} + O(z^m) = \frac{g(z)h(z)}{w(z)}, \tag{2.44}$$

where m $(0 < m < +\infty)$ is some constant. By Lemma 2.8, there exists a set $E_2 \subset (1, +\infty)$ with logarithmic measure $lm(E_2) < +\infty$ and we can choose z satisfying $|z| = r \notin [0,1] \cup E_2$, and $|w(z)| = M(r, w)$, such that

$$\frac{w^{(j)}(z)}{w(z)} = \left(\frac{\nu_w(r)}{z}\right)^j (1 + o(1)) \quad (j = 1, 2). \tag{2.45}$$

Since $\rho(g) = \rho < +\infty$ and $\rho(h) = \lambda(1/f) = \rho_1 < +\infty$ by Lemma 2.9, there exists a set E_3 that has finite logarithmic measure, such that for all z satisfying $|z| = r \notin [0,1] \cup E_3$, $r \to +\infty$ we have

$$|g(z)| \leq \exp\left\{r^{\rho+1}\right\}, \quad |h(z)| \leq \exp\left\{r^{\rho_1+1}\right\}. \tag{2.46}$$

By Lemma 2.7, there is a set $E_4 \subset (1, +\infty)$ that has finite logarithmic measure, such that for all z satisfying $|z| = r \notin [0,1] \cup E_4$, we have

$$|\nu_w(r)| < (\log M(r, w))^2. \tag{2.47}$$

Since $\rho(w) = +\infty$, there exists $\{r'_n\}$ $(r'_n \to +\infty)$ such that

$$\lim_{r'_n \to +\infty} \frac{\log \nu_w(r'_n)}{\log r'_n} = +\infty. \tag{2.48}$$

Set the logarithmic measure of $E_1 \cup E_2 \cup E_3 \cup E_4$,

$$lm(E_1 \cup E_2 \cup E_3 \cup E_4) = \gamma < +\infty,$$

then there exists a point $r_n \in [r'_n, (\gamma+1) r'_n] - E_1 \cup E_2 \cup E_3 \cup E_4$. From

$$\frac{\log \nu_w(r_n)}{\log r_n} \geq \frac{\log \nu_w(r'_n)}{\log((\gamma+1) r'_n)} = \frac{\log \nu_w(r'_n)}{\left(1 + \frac{\log(\gamma+1)}{\log r'_n}\right) \log r'_n}, \tag{2.49}$$

it follows that

$$\lim_{r_n \to +\infty} \frac{\log \nu_w(r_n)}{\log r_n} = +\infty. \tag{2.50}$$

Then for a given arbitrary large $\beta > 2(\rho_1 + \rho + m + 3)$,

$$\nu_w(r_n) \geq r_n^{\beta} \tag{2.51}$$

holds for sufficiently large r_n. Now we take point z_n satisfying $|z_n| = r_n$ and $w(z_n) = M(r_n, w)$, by (2.44) and (2.45), we get

$$\begin{aligned} |d_2(z_n)| \left(\frac{\nu_w(r_n)}{r_n}\right)^2 |1 + o(1)| \leq \\ \leq 2L r_n^m \left(\frac{\nu_w(r_n)}{r_n}\right) |1 + o(1)| + \left|\frac{g(z_n) h(z_n)}{w(z_n)}\right|, \end{aligned} \tag{2.52}$$

where $L > 0$ is some constant. By Lemma 2.7 and (2.51), we get

$$M(r_n, w) > \exp\left(r_n^{\frac{\beta}{2}}\right). \tag{2.53}$$

Hence by (2.46), (2.53) as $r_n \to +\infty$

$$\frac{|g(z_n) h(z_n)|}{M(r_n, w)} \to 0 \tag{2.54}$$

holds. By (2.51), (2.52) , (2.54),we get

$$|d_2(z_n)| r_n^{\beta} \le |d_2(z_n)| \nu_w(r_n) \le 2LKr_n^{m+1}, \tag{2.55}$$

where $K > 0$ is some constant. This is a contradiction by $\beta > 2(\rho_1 + \rho + m + 3)$. Hence $\rho(g) = +\infty$.

Now suppose $d_2 \equiv 0$, $d_1 \not\equiv 0$. Using a similar reasoning as above, we get a contradiction. Hence $\rho(g) = +\infty$.

Finally, if $d_2 \equiv 0$, $d_1 \equiv 0$, $d_0 \not\equiv 0$, then we have $g(z) = d_0(z) f(z)$ and by d_0 is a polynomial, then we get $\rho(g) = +\infty$. ■

3. Proof of Theorem 1.1

First, we suppose $d_2 \not\equiv 0$. Suppose that $f \not\equiv 0$ is a solution of equation (1.6) with $\lambda\left(\frac{1}{f}\right) < +\infty$. Then by Lemma 2.4, we have $\rho(f) = +\infty$. Set $w = g - \varphi = d_2 f'' + d_1 f' + d_0 f - \varphi$, then by Lemma 2.10, we have $\rho(w) = \rho(g) = \rho(f) = +\infty$. In order to the prove $\overline{\lambda}(g - \varphi) = +\infty$, we need to prove only $\overline{\lambda}(w) = +\infty$. By substituting $f'' = -Af$ into w, we get

$$w = d_1 f' + (d_0 - d_2 A) f - \varphi. \tag{3.1}$$

Differentiating both a sides of equation (3.1), we obtain

$$w' = (-d_2 A + d_0 + d_1')f' + (-d_1 A - (d_2 A)' + d_0')f - \varphi'. \tag{3.2}$$

Set $\alpha_1 = d_1$, $\alpha_0 = d_0 - d_2 A$, $\beta_1 = -d_2 A + d_0 + d_1'$, $\beta_0 = -d_1 A - (d_2 A)' + d_0'$. Then we have

$$\alpha_1 f' + \alpha_0 f = w + \varphi, \tag{3.3}$$

$$\beta_1 f' + \beta_0 f = w' + \varphi'. \tag{3.4}$$

Set

$$h = \alpha_1 \beta_0 - \beta_1 \alpha_0. \tag{3.5}$$

We divide it into two cases to prove.

Case 1. If $h \equiv 0$, then by (3.3)–(3.5), we get

$$\alpha_1 w' - \beta_1 w = -(\alpha_1 \varphi' - \beta_1 \varphi) = F. \tag{3.6}$$

Now we prove that $\alpha_1 \varphi' - \beta_1 \varphi \not\equiv 0$, i.e., $\frac{d_1}{d_2}\frac{\varphi'}{\varphi} - \frac{d_0 + d_1'}{d_2} + A \not\equiv 0$. Suppose that $A = \frac{d_0 + d_1'}{d_2} - \frac{d_1}{d_2}\frac{\varphi'}{\varphi}$. Then we obtain

$$m(r, A) \le m\left(r, \frac{d_0 + d_1'}{d_2}\right) + m\left(r, \frac{\varphi'}{\varphi}\right) + m\left(r, \frac{d_1}{d_2}\right) + O(1). \tag{3.7}$$

Since $\frac{d_1}{d_2}$ and $\frac{d_0 + d_1'}{d_2}$ are rational functions, then

$$m\left(r, \frac{d_1}{d_2}\right) = O(\ln r), \ m\left(r, \frac{d_0 + d_1'}{d_2}\right) = O(\ln r). \tag{3.8}$$

By $\rho(\varphi) < +\infty$, we have

$$m\left(r, \frac{\varphi'}{\varphi}\right) = O(\ln r). \tag{3.9}$$

It follows from the definition of deficiency $\delta(\infty, A)$ that for sufficiently large r, we get

$$m(r, A) \geq \frac{\delta}{2} T(r, A). \tag{3.10}$$

So when r sufficiently large, we have by (3.7)–(3.10)

$$T(r, A) \leq \frac{2}{\delta} m(r, A) \leq O(\ln r), \tag{3.11}$$

and this contradicts that A is a transcendental meromorphic function with $\rho(A) = \rho > 0$. Hence $\alpha_1\varphi' - \beta_1\varphi \not\equiv 0$ and then $F \not\equiv 0$. By $F \not\equiv 0$, and Lemma 2.5, we obtain $\overline{\lambda}(w) = \lambda(w) = \rho(w) = +\infty$, i.e., $\overline{\lambda}(g - \varphi) = +\infty$.

Case 2. If $h \not\equiv 0$, then by (3.3)-(3.5), we get

$$f = \frac{\alpha_1(w' + \varphi') - \beta_1(w + \varphi)}{h}. \tag{3.12}$$

Substituting (3.12) into equation (1.6), we obtain

$$\frac{\alpha_1}{h} w''' + \phi_2 w'' + \phi_1 w' + \phi_0 w = -\left(\left(\frac{\alpha_1\varphi' - \beta_1\varphi}{h}\right)'' + A\left(\frac{\alpha_1\varphi' - \beta_1\varphi}{h}\right)\right) = F, \tag{3.13}$$

where ϕ_j $(j = 0, 1, 2)$ are meromorphic functions with $\rho(\phi_j) < +\infty$ $(j = 0, 1, 2)$. By $\alpha_1\varphi' - \beta_1\varphi \not\equiv 0$, $\rho\left(\frac{\alpha_1\varphi' - \beta_1\varphi}{h}\right) < +\infty$ and Lemma 2.4, we know that $F \not\equiv 0$. By Lemma 2.5, we obtain $\overline{\lambda}(w) = \lambda(w) = \rho(w) = +\infty$, i.e., $\overline{\lambda}(g - \varphi) = +\infty$.

Now suppose $d_2 \equiv 0$, $d_1 \not\equiv 0$ or $d_2 \equiv 0$, $d_1 \equiv 0$ and $d_0 \not\equiv 0$. Using a similar reasoning to that above we get $\overline{\lambda}(w) = \lambda(w) = \rho(w) = +\infty$, i.e., $\overline{\lambda}(g - \varphi) = +\infty$. ■

4. Proof of Theorem 1.2

Suppose that $f(z) \not\equiv 0$ is a meromorphic solution of equation (1.5). Then by Lemma 2.4, we have $\rho(f) = +\infty$ and $\rho_2(f) = \rho(A)$. Set $w_j = f^{(j)} - \varphi$ $(j = 0, 1, \ldots, k)$, then $\rho(w_j) = \rho(f) = +\infty$, $\rho_2(w_j) = \rho_2(f) = \rho(A)$ $(j = 0, 1, \ldots, k)$, $\overline{\lambda}(w_j) = \overline{\lambda}\left(f^{(j)} - \varphi\right)$ $(j = 0, 1, \ldots, k)$. Differentiating both sides of $w_j = f^{(j)} - \varphi$ and replacing $f^{(k)}$ with $f^{(k)} = -Af$, we obtain

$$w_j^{(k-j)} = -Af - \varphi^{(k-j)} \quad (j = 0, 1, \ldots, k). \tag{4.1}$$

Then we have

$$f = -\frac{w_j^{(k-j)} + \varphi^{(k-j)}}{A}. \tag{4.2}$$

Substituting (4.2) into equation (1.5), we get

$$\left(\frac{w_j^{(k-j)}}{A}\right)^{(k)} + w_j^{(k-j)} = -\left(\left(\frac{\varphi^{(k-j)}}{A}\right)^{(k)} + \varphi^{(k-j)}\right). \tag{4.3}$$

By (4.3) we can write

$$w_j^{(2k-j)} + \Phi_{2k-j-1} w_j^{(2k-j-1)} + \cdots + \Phi_{k-j} w_j^{(k-j)} = -A\left(\left(\frac{\varphi^{(k-j)}}{A}\right)^{(k)} + A\left(\frac{\varphi^{(k-j)}}{A}\right)\right), \quad (4.4)$$

where $\Phi_{k-j}(z), \ldots, \Phi_{2k-j-1}(z)$ $(j = 0, 1, \ldots, k)$ are meromorphic functions with $\rho(\Phi_{k-j}) \le \rho, \ldots, \rho(\Phi_{2k-j-1}) \le \rho$ $(j = 0, 1, \ldots, k)$. By $A \not\equiv 0$ and $\rho\left(\frac{\varphi^{(k-j)}}{A}\right) < +\infty$, then by Lemma 2.4, we have

$$-A\left(\left(\frac{\varphi^{(k-j)}}{A}\right)^{(k)} + A\left(\frac{\varphi^{(k-j)}}{A}\right)\right) \not\equiv 0. \quad (4.5)$$

Hence, by Lemma 2.5, we have $\overline{\lambda}(w_j) = \rho(w_j) = +\infty$ and by Lemma 2.6 $\overline{\lambda}_2(w_j) = \rho_2(w_j) = \rho(A)$. Thus

$$\overline{\lambda}(f^{(j)} - \varphi) = \rho(f) = +\infty, \quad \overline{\lambda}_2(f^{(j)} - \varphi) = \rho_2(f) = \rho(A) = \rho \quad (4.6)$$

$(j = 0, 1, \ldots, k)$. ■

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