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# ON THE UNIQUENESS OF BOUNDED WEAK SOLUTIONS TO THE NAVIER-STOKES CAUCHY PROBLEM

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**Abstract.** In this note we give a uniqueness theorem for solutions $(u, \pi)$ to the Navier-Stokes Cauchy problem, assuming that $u$ belongs to $L^\infty((0,T) \times \mathbf{R}^n)$ and $(1+|x|)^{-n-1}\pi \in L^1(0,T;L^1(\mathbf{R}^n))$, $n \geq 2$. The interest to our theorem is motivated by the fact that a possible pressure field $\widetilde{\pi}$, belonging to $L^1(0,T;\mathrm{BMO})$, satisfies in a suitable sense our assumption on the pressure, and by the fact that the proof is very simple.

## 1. Introduction

In the recent paper [5], for the solutions $(u, \pi)$ of the Navier-Stokes Cauchy problem, the uniqueness is stated assuming that the kinetic field $u$ belongs to $L^\infty((0,T) \times \mathbb{R}^n)$, $n \geq 2$, and that the pressure field $\pi$ belongs to $L^1(0,T;\mathrm{BMO})$. The quoted paper is a continuation of paper [3] where existence and uniqueness results are stated under slightly different assumptions. The quoted papers concern a special problem related to Navier-Stokes equations, which has been studied during the last century by several authors with different aims. We mention just [1], [2], [9], [10], [16]. The first contributions on existence and uniqueness of solutions with a bounded kinetic field were given in [9], [10], [16]. The Cauchy problem was studied in [9], [10], and the initial boundary value problem in exterior domains was studied in [16], the existence and uniqueness being stated in the Hölder class. In [1], [2] the problem is slightly different and is related to a problem of uniqueness. The question is to furnish uniqueness of solutions in a sufficient wide *set*, whose *boundary* is recognized by physically reasonable conditions for the dynamical variables or by mathematical counterexamples for the uniqueness of the solutions. In the paper [2], the uniqueness is achieved for smooth solutions $(u, \pi)$ assuming that $u$ belongs to $C(0,T;C(\mathbb{R}^n))$ and $\nabla\pi$ belongs to $L^p(0,T;L^p(\mathbb{R}^n))$, $p \in (1,\infty)$. In the paper [1], the uniqueness is achieved for smooth solutions $(u, \pi)$ assuming that $u$ belongs

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to $C(0,T;C(\mathbb{R}^n))$ and, for any $t>0$, $\pi(x,t)=O(|x|^{1-\varepsilon})$, $\varepsilon>0$. We stress that in the papers [1], [2] the assumption that solutions are smooth is just an exemplifying condition, and also that in [1] the problem is related with an initial boundary value problem for the Navier-Stokes system. In [1], [2], [9] the assumptions for the uniqueness are seen as *sharpness* by virtue of the following counterexample to the uniqueness: $u\equiv\sin t(1,0,0)$ and $\pi=-\cos t\, x_1$. The counterexample is exhibited in [11].

In this note we are just interested to give a uniqueness theorem for solutions $(u,\pi)$ to the Navier-Stokes Cauchy problem, assuming that $u$ belongs to $L^\infty((0,T)\times\mathbb{R}^n)$ and $(1+|x|)^{-n-1}\pi\in L^1(0,T;L^1(\mathbb{R}^n))$, $n\geq 2$. The interest to our theorem is motivated by the fact that a possible pressure field $\widetilde{\pi}$ belonging to $L^1(0,T;\mathrm{BMO})$ satisfies in a suitable sense our assumption on the pressure, and by the fact that our proof is very simple.

The plan of the paper is the following. In section 2 we give the statement of our result; in section 3 we prove some preliminary results and in section 4 we give the proof of the uniqueness theorem.

## 2. Statement of the problem

We study the uniqueness of solutions for the following Cauchy problem:

$$\begin{aligned} &u_t-\Delta u+u\cdot\nabla u=-\nabla\pi,\quad \nabla\cdot u=0,\ \text{in } (0,T)\times\mathbb{R}^n,\\ &u(0,x)=u_0(x),\ \text{in } \mathbb{R}^n,\end{aligned}\tag{2.1}$$

where $u(t,x)$ is the kinetic field and $\pi(t,x)$ is the pressure field; $u\cdot\nabla u_i=u_k\frac{\partial u_i}{\partial x_k}$, $i=1,\dots,n$; $u_0(x)$ is the initial data. As it is usual, by $(f,\psi)$ we mean $\int_\Omega f(x)\cdot\psi(x)\,dx$. In order to state our uniqueness result in a weak form, we give the following

Definition 2.1. A pair $(u,\pi)$ is said to be a *bounded weak solution* to the Navier-Stokes Cauchy problem in $(0,T)\times\mathbb{R}^n$, $T>0$, corresponding to $u_0\in L^\infty(\mathbb{R}^n)$ with divergence free in weak sense, if the following hold:

a) $u(t,x)\in L^\infty((0,T)\times\mathbb{R}^n)$ and almost everywhere $u$ has weakly divergence free; $(1+|x|)^{-n-1}\pi(t,x)\in L^1(0,T;L^1(\mathbb{R}^n))$;

b) $(u,\pi)$ satisfies the following equation:

$$\int_0^T[(u,\varphi_\tau)+(u,\Delta\varphi)+(u\otimes u,\nabla\varphi)+(\pi,\nabla\cdot\varphi)]\,d\tau=-(u_0,\varphi(0)),$$

provided that $\varphi\in C^1([0,T);C_0^\infty(\mathbb{R}^n))$ with $\varphi(x,t)=0$ in a neighborhood of $T$;

c) for each $\psi\in C_0^\infty(\mathbb{R}^n)$, $\lim_{t\to0}\frac{1}{t}\int_0^t[(u(\tau),\psi)-(u_0,\psi)]\,d\tau=0$.

Remark 2.1. In the item c) of the above definition we require that $t=0$ is a Lebesgue point for the function $(u(t),\psi)$, for any $\psi\in C_0^\infty(\Omega)$. As a consequence of Lemma 3.3, proving formula (3.8), each pair $(u,\pi)$ which satisfies conditions a)–b), redefined in a suitable way, enjoys the property: for each $\psi\in C_0^\infty(\mathbb{R}^n)$, $\lim_{t\to0}(u(t),\psi)=(u_0,\psi)$; which of course implies c).

Remark 2.2. We point out that an element of $L^1(0,T;\mathrm{BMO})$ satisfies our assumption a) of integrability related to the pressure $\pi(x,t)$ and, as a pressure field, it can be possibly a pressure field for a bounded weak solution (for the definition and properties related to the BMO space see [17]).

The aim of this paper is to prove

Theorem 2.1. *Let $(u,\pi)$ be a bounded weak solution corresponding to $u_0$. Then, $(u,\pi)$ is the unique bounded weak solution corresponding to $u_0$.*

Remark 2.4. The uniqueness of solutions of problem 2.1 (or, more generally, the uniqueness of solutions of an initial boundary value problem in an unbounded domain), stated with no condition at infinity ($|x|\to\infty$), establishes that the behavior at infinity is just recognized from one of the initial data, that is by its behavior in a neighborhood of *infinity*. In other words, given an initial data $u_0(x)$ in some functional space $X$, denoted by $(u,\pi)$, the uniqueness theorem proves that $(u,\pi)$ is unique in a set of solutions *a priori* having behavior at infinity not comparable with the one of the existence of $(u,\pi)$. In the case of the Cauchy problem the *boundary* of the wider *set* of uniqueness of solutions is established by the following counterexample to the uniqueness $(u,\pi)\equiv(\sin t(1,0,\dots,0),\cos tx_1)$. But, our theorem is not able to furnish uniqueness in the set of solutions delimited by two families of *weak solutions*, locally bounded, $(u,\pi)\equiv(\jmath_1(t)(1,0,\dots,0),-\jmath_1'(t)x_1)$ and $(\bar u,\bar\pi)\equiv(\jmath_2(t)(x_1,0,\dots,0,-x_n),\ -\frac{1}{2}\jmath_2'(t)(x_1^2-x_n^2)-\frac{1}{2}\jmath_2^2(t)(x_1^2+x_n^2))$, where $\jmath_1(t)$ and $\jmath_2(t)$ are two arbitrary Lipschitzian functions on $(0,T)$ with $\jmath_1(0)=\jmath_2(0)=0$. It is worth noting that counterexamples to the uniqueness are solutions of potential type, as well as to the Euler equation. It is also worth stressing that the last question achieves a special interest in relation with the recent results of existence given in [6], [7], [14].

## 3. Some preliminary results

In this section we state some results concerning a bounded weak solution. Lemmas 3.2 and 3.3 are concerned with qualitative properties of $u\in L^\infty(0,T;L^2_{loc}(\Omega))$ analogous to the ones of weak solutions of the $L^2$-theory due to Prodi [12] and Serrin [15]. Lemma 3.4 is concerned with a weight energy inequality of the same kind as the ones proved in [1], [2]. We start with the following

Lemma 3.1. *Let $u(t,x)\in L^\infty((0,T)\times\mathbb{R}^n)$. Then, there exists a set $T_u\subseteq(0,T)$ such that for $t\in T_u$ and for $\psi\in C_0^\infty(\mathbb{R}^n)$*

$$\lim_{h\to0}\frac{1}{h}\int_t^{t+h}(u(\tau),\psi)\,d\tau=(u(t),\psi),\tag{3.1}$$

*and* $\operatorname{meas}((0,T)-T_u)=0$.

*Proof.* Let $\{\Omega(k)\}$ be a sequence of bounded domains in $\mathbb{R}^n$ such that $\Omega(k)\subset\Omega(k+1)$ and $\lim_k\Omega(k)=\mathbb{R}^n$. Since $u(t,x)\in L^\infty((0,T)\times\mathbb{R}^n)$, then, for each $k\in\mathbf{N}$

and $p,q\geq 1$, $u(t,x)\in L^q(0,T;L^p(\Omega(k))$. Hence, by virtue of Corollary 2 (p. 88) of [8], there exists a set $T_u(k)$ such that $\operatorname{meas}((0,T)-T_u(k))=0$ and for

$$t\in T_u(k),\quad \lim_{h\to 0}\frac{1}{h}\int_t^{t+h}|u(\tau)-u(t)|_{L^p(\Omega(k))}\,d\tau=0. \tag{3.2}$$

Set $T'_u(k)=(0,T)-T_u(k)$; then $T'_u(k)\subseteq T'_u(k+1)$ and $\operatorname{meas}\bigcup_k T'_u(k)=\lim_k\operatorname{meas}T'_u(k)=0$. We set $T'_u=\bigcup_{k\in\mathbf{N}}T'_u(k)$ and $T_u=(0,T)-T'_u$. If $t\in T_u$, then $t\in T_u(k)$ uniformly in $k\in\mathbf{N}$; if $\psi\in C_0^\infty(\mathbb{R}^n))$, then $\operatorname{supp}\psi\subseteq\Omega(k)$ for any $k$ greater than some $k_0$. We have

$$\left|\frac{1}{h}\int_t^{t+h}(u(\tau),\psi)\,d\tau-(u(t),\psi)\right|=\left|\frac{1}{h}\int_t^{t+h}(u(\tau)-u(t),\psi)\,d\tau\right|.$$

Thus, from (3.2) the limit property (3.1). ∎

Lemma 3.2. *Let $(u,\pi)$ be a bounded weak solution. Then, there exists a Lebesgue measurable set $T_u\subseteq(0,T)$ such that* $\operatorname{meas}((0,T)-T_u)=0$ *and for each $t\in T_u$*

$$\int_0^t[(u,\varphi_\tau)+(u,\Delta\varphi)+(u\otimes u,\nabla\varphi)+(\pi,\nabla\cdot\varphi)]\,d\tau=(u(t),\varphi(t))-(u_0,\varphi(0)), \tag{3.3}$$

*for any $\varphi\in C^1([0,T);C_0^\infty(\mathbb{R}^n))$ with $\varphi(t,x)=0$ in a neighborhood of $T$.*

*Proof.* The proof is analogous to the one given by Prodi and Serrin. Let $T_u$ be the subset determined in Lemma 3.1. Let $t\in T_u$ and let $\theta_\varepsilon(s)$ be a smooth cut-off function such that $\theta_\varepsilon(s)=1$ for $s\in[0,t]$, $\theta_\varepsilon(s)=0$ for $s\geq t+\varepsilon$ with $-\theta'_\varepsilon\leq c/\varepsilon$ and $\int_0^T\theta'_\varepsilon(s)\,ds=-1$. We consider the relation b) of Definition 2.1 with $\phi(x,\tau)=\theta_\varepsilon(\tau)\psi(x,\tau)$, $\psi\in C^1([0,T);C_0^\infty(\mathbb{R}^n))$:

$$\int_0^{t+\varepsilon}[((u,\psi_\tau)\theta_\varepsilon+(u,\psi)\theta'_\varepsilon+(u,\Delta\psi)\theta_\varepsilon+(u\otimes u,\nabla\psi)\theta_\varepsilon+(\pi,\nabla\cdot\psi)\theta_\varepsilon]\,d\tau$$
$$=-(u_0,\psi(0)). \tag{3.4}$$

By virtue of condition a) concerning the definition of a bounded weak solution $(u,\pi)$, applying the dominate convergence theorem, each integral term with $\theta_\varepsilon$ is convergent for $\varepsilon\to 0$ and the limit is

$$\int_0^t[(u,\psi_\tau)+(u,\Delta\psi)+(u\otimes u,\nabla\psi)+(\pi,\nabla\cdot\psi)]\,d\tau. \tag{3.5}$$

Now we evaluate the convergence of the integral term with $\theta'_\varepsilon$. We have

$$\int_t^{t+\varepsilon}(u,\psi)\theta'_\varepsilon\,d\tau$$
$$=\int_t^{t+\varepsilon}[(u(\tau),\psi(\tau)-\psi(t))+(u(\tau)-(u(t),\psi(t))]\,\theta'_\varepsilon\,d\tau-(u(t),\psi(t)). \tag{3.6}$$

Employing the properties of $\theta_\varepsilon$ and Lemma 3.1, from (3.6) we deduce

$$\lim_{\varepsilon\to 0}\int_t^{t+\varepsilon}(u,\psi)\theta'_\varepsilon\,d\tau = -(u(t),\psi(t)). \tag{3.7}$$

Passing to the limit when $\varepsilon\to 0$ in (3.4), via (3.5) and (3.7) we deduce (3.1). ■

Lemma 3.3. *Let $(u,\pi)$ be a bounded weak solution. Then, it is possible to redefine by $U(t,x)$ the field $u(t,x)$ on $(0,T)$ in such a way that $U(t,x)=u(t,x)$ for any $t\in T_u$ and $U(t,x)\in L^2_{loc}(\mathbb{R}^n)$ for any $t\in(0,T)-T_u$, $U(t,x)$ has divergence free in a weak sense. Moreover, equation (3.3) holds with $U(t,x)$ for any $t\in(0,T)$:*

$$\int_0^t[(U,\varphi_\tau)+(U,\Delta\varphi)+(U\otimes U,\nabla\varphi)+(\pi,\nabla\cdot\varphi)]\,d\tau$$
$$=(U(t),\varphi(t))-(u_0,\varphi(0)), \tag{3.8}$$

*for any $t\in(0,T)$ and for any $\varphi\in C^1([0,T);C_0^\infty(\mathbb{R}^n))$ with $\varphi(x,t)=0$ in a neighborhood of $T$. Finally, for any $\psi\in C_0^\infty(\mathbb{R}^n)$, $(U(t),\psi)$ is a continuous function on $(0,T)$.*

*Proof.* Let $t\in(0,T)$ and let $\{t_i^+\}\subset T_u$ and $\{t_i^-\}\subset T_u$ be two sequences converging to $t$. Then, equation (3.3) implies that, for $\varphi(t,x)=\theta(t)\psi(x)\in C_0^\infty(\mathbb{R}^n)$, where $\theta(t)$ $C^1$-smooth function with $\theta(\tau)=1$ for $\tau\in[0,t+\delta]$,

$$(u(t_i^-)-u(t_i^+),\psi)=\int_{t_i^-}^{t_i^+}[(u,\Delta\psi)+(u\otimes u,\nabla\psi)+(\pi,\nabla\cdot\psi)]\,d\tau\to 0 \ \text{ for } \ i\to\infty. \tag{3.9}$$

Since $u(t,x)\in L^\infty(0,T;L^2_{loc}(\mathbb{R}^n))$, then, denoting by $\{t_i\}\subset T_u$ a sequence converging to $t\in(0,T)$, taking (3.9) into account, $\{u(t_i,x)\}$ weakly converges to a function $U(t,x)$ belonging to $L^2_{loc}(\mathbb{R}^n)$. Moreover, $U(t,x)$ has divergence free in weak sense. Hence, $U(t,x)$ redefines $u(t,x)$ on $(0,T)$ and the weak convergence of $u(t_i,x)\to U(t,x)$ in $L^2_{loc}(\mathbb{R}^n)$, for any $t\in(0,T)$, ensures the validity of (3.3) for any $t\in(0,T)$. Finally, the continuity of the function $(U(t),\psi)$ is a consequence of (3.8), which implies, for $\varphi=\theta\psi$,

$$(U(t)-U(s),\psi)=\int_s^t[(U,\Delta\psi)+((U\otimes U,\nabla\psi)+(\pi,\nabla\cdot\psi)]\,d\tau. \quad ■$$

In the following we are interested to employ the property (3.8). Since there is no confusion, we denote the field $U$ by $u$ again.

Lemma 3.4 *Let $(u,\pi)$ and $(\bar u,\bar\pi)$ be two bounded weak solutions. Then,*

$$|gw(t)|_2^2+\int_0^t|g\nabla w(\tau)|_2^2\,d\tau<+\infty, \ \ \textit{for any} \ \ t\in(0,T), \tag{3.10}$$

*where $g=(1+|x|)^{-\frac{n}{2}}$ and $w=\bar u-u$.*

*Proof.* We set $p = \overline{\pi} - \pi$. From equation (3.8) we deduce

$$\int_0^t [(w,\varphi_\tau) + (w,\Delta\varphi) + (w\otimes(w+u),\nabla\varphi) + (u\otimes w,\nabla\varphi) + (p,\nabla\cdot\varphi)]\,d\tau \\ = (w(t),\varphi(t)). \quad (3.11)$$

Let $h_R(|z|/R)$ with $h(r)\in[0,1]$ a nonnegative smooth cut-off function such that $h(r)=1$ for $r\in[0,1]$ and $h(r)=0$ for $r\geq 2$, $|h'(r)|\leq c$. By $J[\psi](x)$ we mean the convolution $\int_{\mathbb{R}^n} J(x-y)\psi(y)\,dy$. We set:

$$\begin{aligned} w_R(s,z) &= h_R(z)g(z)w(s,z),\\ w_{\varepsilon R}(s,y) &= J_\varepsilon[w_R(s)](y),\\ \varphi_{\eta\varepsilon R}(s,x) &= \int_0^t J_\eta(s-\tau)h_R(x)g(x)J_\varepsilon[w_{\varepsilon R}(\tau)](x)\,d\tau, \end{aligned}$$

where $J_\delta[\cdot](\xi)$ is the Friederichs mollifier. The following relations hold:

E$_1$) Since $J_\eta[\cdot](\tau)$ is an even function, then, for any $\eta>0$,

$$\int_0^t (w,\varphi_s)\,ds = \int_0^t\int_0^t \frac{\partial}{\partial s}J_\eta(s-\tau)(w_{\varepsilon R}(s),w_{\varepsilon R}(\tau))\,ds\,d\tau = 0;$$

E$_2$) since $(w_{\varepsilon R}(t),\psi)$, $\psi\in C_0^\infty(\mathbb{R}^n)$, is a continuous function on $(0,T)$, then,

$$\lim_{\eta\to 0}(w(t),\varphi_{\eta\varepsilon R}(t)) = \frac{1}{2}|w_{\varepsilon R}(t)|_2^2.$$

The following inequalities hold:

I$_1$) Since $w$ has free divergence in a weak sense, then,

$$|\nabla\cdot\varphi_{\eta\varepsilon R}| = \left|\int_0^t J_\eta(s-\tau)\nabla[h_R(x)g(x)]\cdot\int_{\mathbb{R}^n} J_\varepsilon(x-y)w_{\varepsilon R}(\tau,y)\,dy\,d\tau\right| \\ \leq c(1+|x|)^{-n-1},$$

uniformly with respect to $\eta,\varepsilon$ and $R>0$; thus, by virtue of condition a) of Definition 2.1, we have

$$\left|\int_0^t (p,\nabla\cdot\varphi_{\eta\varepsilon R})\,ds\right| \leq c\int_0^t\int_{\mathbb{R}^n}|p(s,x)|(1+|x|)^{-n-1}\,dx\,ds < +\infty,\ t\in(0,T);$$

I$_2$) since $w_{\varepsilon R}\in L^2(0,T;W^{1,2}(\mathbb{R}^n))$, then,

$$\begin{aligned}\lim_{\eta\to 0}\int_0^t (w,\Delta\varphi_{\eta\varepsilon R}(s))\,ds = &-\int_0^t |\nabla w_{\varepsilon R}(s)|_2^2 ds\\ &+\int_0^t\int_{\mathbb{R}^n} w(s,x)\cdot\Delta[h_R(x)g(x)]J_\varepsilon(x)[w_{\varepsilon R}]\,dx\,ds\\ &+\int_0^t\int_{\mathbb{R}^n} w(s,x)\cdot\nabla[h_r(x)g(x)]\cdot\nabla J_\varepsilon(x)[w_{\varepsilon R}]\,dx\,ds;\end{aligned}$$

applying the Hölder inequality we get

$$\lim_{\eta\to 0}\int_0^t (w,\Delta\varphi_{\eta\varepsilon R}(s))\,ds \le -\int_0^t |\nabla w_{\varepsilon R}(s)|_2^2\,ds$$
$$+\,c\int_0^t |w\Delta(h_R g)|_2|w_R|_2\,ds + c(\delta)\int_0^t |w\nabla(h_R g)|_2^2 + \delta\int_0^t |\nabla w_{\varepsilon R}|_2^2\,ds;$$

I$_3$) since $(w\otimes(w+u)+u\otimes w)\nabla^i(h_R g)\in L^2(0,T;L^2(\mathbb{R}^n))$ and $w_{\varepsilon R}\in L^2(0,T;W^{1,2}(\mathbb{R}^n))$, then

$$\lim_{\eta\to 0}\int_0^t (w\otimes(w+u)+u\otimes w,\nabla\varphi_{\eta\varepsilon R})\,ds$$
$$=\int_0^t (w\otimes(w+u)+u\otimes w,\sum_{\ell=0}^{1}\nabla^\ell(h_R g)\otimes J_\varepsilon(x)[\nabla^{1-\ell}w_{\varepsilon R}])\,ds;$$

applying the Hölder inequality, we obtain

$$\lim_{\eta\to 0}\int_0^t (w\otimes(w+u)+u\otimes w,\nabla\varphi_{\eta\varepsilon R})\,ds \le \int_0^t ||w|(|w|+2|u|)\nabla(h_R g)|_2|w_{\varepsilon R}|_2\,ds$$
$$+\,c(\delta)\int_0^t |(|w|^2+2|w||u|)h_R g|_2^2 + \delta\int_0^t |\nabla w_{\varepsilon R}|_2^2\,ds.$$

We substitute $\varphi_{\eta\varepsilon R}$ in (3.8); taking relations E$_1$)–E$_2$) and inequalities I$_1$)–I$_3$) into account, choosing a suitable $\delta$, we get

$$|w_{\varepsilon R}(t)|_2^2 + \int_0^t |\nabla w_{\varepsilon R}(s)|_2^2\,ds \le c\int_0^t |p(s)(1+|x|)^{-n-1}|_1\,ds$$
$$+\,c\int_0^t |w(s)\Delta(h_R g)|_2|w_{\varepsilon R}(s)|_2\,ds + c\int_0^t |w(s)\nabla(h_R g)|_2^2\,ds$$
$$+\,c\int_0^t \big||w|(|w|+2|u|)\big|\nabla(h_R g)|\big|_2|w_{\varepsilon R}(s)|_2\,ds + c\int_0^t \big||w(s)|(|w(s)|+2|u(s)|)h_R g\big|_2^2\,ds.$$

Since $w,u\in L^\infty(0,T;L^\infty(\mathbb{R}^n))$ and $(1+|x|)^{-n-1}p\in L^1(0,T;L^1(\mathbb{R}^n))$, uniformly in $\varepsilon>0$, the above inequality furnishes

$$|w_{\varepsilon R}(t)|_2^2 + \int_0^t |\nabla w_{\varepsilon R}(s)|_2^2\,ds \le c(T) + \int_0^t |w_R(s)|_2^2\,ds, \tag{3.12}$$

with $c(T)$ independent of $R$. Since, for each $t\in(0,T)$, $\{w_{\varepsilon R}\}$ strongly converges to $w_R$ in $L^2(\mathbb{R}^n)$, then, it weakly converges to $w_R$ in $L^2(0,T;W^{1,2}(\mathbb{R}^n))$. From (3.12), taking the limit for $\varepsilon\to 0$, we deduce

$$|w_R(t)|_2^2 + \int_0^t |\nabla w_R(s)|_2^2\,ds \le c(T) + \int_0^t |w_R(s)|_2^2\,ds.$$

By integrating, uniformly with respect to $R$, we obtain

$$|w_R(t)|_2^2 + \int_0^t |\nabla w_R(s)|_2^2\,ds \le c(T).$$

Applying the monotone convergence theorem we deduce (3.10). The lemma is completely proved. ■

## 4. Proof of Theorem 2.1

In this section, we assume the Euclidean dimension $n \geq 3$. The fundamental solution of the Stokes equations is denoted by $E$. The components of $E$ are the following:

$$E_{ij}(x-y,t-\tau) = -\Delta\phi(|x-y|,t-\tau)\delta_{ij} + D^2_{x_i x_j}\phi(|x-y|,t-\tau),$$

where

$$\phi(r,t) = \frac{1}{2}\frac{1}{\sqrt{\pi}^n}\frac{1}{r^{n-2}}\int_0^{r/2\sqrt{t}} \rho^{n-3}e^{-\rho^2}\,d\rho.$$

Denoting by $E_j(x-y,t-\tau)$ the j-th column of the tensor $E$, for $t-\tau>0$, the pair $(E_j(x-y,t-\tau),p)$, with $p=0$, in the $(x,t)$ variables is a solution of the Stokes equations and in the $(y,\tau)$ variables it is a solution of the adjoint equations:

$$\Phi_\tau + \Delta\Phi + \nabla p = 0, \quad \nabla\cdot\Phi = 0, \ \text{ on } (0,t)\times\mathbb{R}^n.$$

We recall some properties of $E$:

$$\begin{aligned}
&E_{ij}(z,s) \in C^\infty(\mathbb{R}^n\times(0,\infty));\\
&\Delta\phi(z,s) = -(4\pi s)^{-\frac{n}{2}}e^{-\frac{|z|^2}{4s}},\ |\nabla\phi(z,s)| \leq c\left(|z|^2+s\right)^{\frac{1}{2}-\frac{n}{2}};\\
&|E(z,s)| \leq c\left(|z|^2+s\right)^{-\frac{n}{2}},\ |\nabla E(z,s)| \leq c\left(|z|^2+s\right)^{-\frac{n}{2}-\frac{1}{2}}.
\end{aligned} \tag{4.1}$$

For any $x$-smooth field $m(x,\tau)$ with compact support such that, for each $x\in\mathbb{R}^n$, $\lim_{\tau\to t^-}\nabla\cdot m(x,\tau) = \nabla\cdot m(x,t)$ and $\lim_{\tau\to t^-} m(x,\tau) = m(x,t)$, we have

$$\begin{aligned}
\lim_{\tau\to t^-}\int_{\mathbb{R}^n} E_i(x-y,t-\tau)\cdot m(y,\tau)\,dy &= \lim_{\tau\to t^-}\Bigg[(4\pi(t-\tau))^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4(t-\tau)}} m_i(y,\tau)\\
&\quad + \int_{\mathbb{R}^n}\frac{\partial}{\partial x_i}\phi(x-y,t-\tau)\nabla\cdot m(y,\tau)\,dy\Bigg]\\
&= m_i(x,t) + \int_{\mathbb{R}^n}\frac{\partial}{\partial x_i}\phi(x-y,0)\nabla\cdot m(y,t)\,dy.
\end{aligned} \tag{4.2}$$

We define

$$\varphi_{\mu\varepsilon R} = \theta_\mu(\tau)h_R(y)\int_{R^n} J_\varepsilon(y-z)E_i(x-z,t-\tau)\,dz,$$

where $h_R$ is the space cut-off function introduced in section 3; $\theta_\mu(\tau)$ is a time cut-off function such that $\theta_\mu(\tau)=1$ for $\tau\leq t-\mu$ and $\theta_\mu(\tau)=0$ for $\tau\geq t-\frac{1}{4}\mu$; finally, $J_\varepsilon$ is the Friedrich mollifier. We consider (3.8) on $(0,t-\mu)$ with $\varphi_{\mu\varepsilon R}(x,\tau)$. Hence, taking properties of tensor $E$ into account, for $w=\bar{u}-u$, $p=\overline{\pi}-\pi$ we get

$$\begin{aligned}
&\int_0^{t-\mu}\big[(w\otimes(w+u),\nabla\varphi_{\mu\varepsilon R}) + (u\otimes w,\nabla\varphi_{\mu\varepsilon R}) + (p,\nabla\cdot\varphi_{\mu\varepsilon R})\big]\\
&\qquad + \int_0^{t-\mu}\big[2(w,\nabla h_R\cdot\nabla J_\varepsilon[E_i(x,t)))] + (w\Delta h_R, J_\varepsilon[E_i(x,t)])\big]\,d\tau\\
&\qquad\qquad = (w_{\varepsilon R}(t-\mu),E_i(x,\mu)),
\end{aligned} \tag{4.3}$$

where $w_{\varepsilon R}(z,\tau)=\int_{\mathbb{R}^n} J_\varepsilon(y-z)h_R(y)w(y,\tau)\,dy$. Taking the properties of $E$ and cut-off function $h_R$ into account, then, the following estimates hold:

I$_1$) for $R>2|x|$, for $\tau\in(0,t)$, uniformly with respect to $\mu$, we have

$$|\nabla\varphi_{\mu\varepsilon R}(y,\tau)|\le c|\nabla h_R|\big[(|x-y|^2+t-\tau)^{\frac{n}{2}}\big]^{-1}+c\int_{\mathbb{R}^n} J_\varepsilon(w)(|x-y+w|^2+t-\tau)^{-\frac{n}{2}-\frac{1}{2}}dw;$$

hence, uniformly with respect to $\mu$ and $\varepsilon$,

$$\begin{aligned}\big|\big[(w\otimes(w+u),\nabla\varphi_{\mu\varepsilon R})+(u\otimes w,\nabla\varphi_{\mu\varepsilon R})\big|\le c|w(\tau)|_\infty(|w(\tau)|_\infty+2|u(\tau)|_\infty)\times\\ \left[R^{-1}\int_{R\le|y|\le 2R}|y|^{-n}\,dy+\int_{\mathbb{R}^n}J_\varepsilon(w)\int_{\mathbb{R}^n}(|x-y+w|^2+t-\tau)^{-\frac{n}{2}-\frac{1}{2}}\,dy\,dw\right]\\ \le c|w(\tau)|_\infty(|w(\tau)|_\infty+2|u(\tau)|_\infty)\big[(cR^{-1}+(t-\tau)^{-\frac{1}{2}}\big];\end{aligned}$$

I$_2$) for $R>2|x|$, for $\tau\in(0,t)$, uniformly with respect to $\mu$ and $\varepsilon$, we have

$$|\nabla\cdot\varphi_{\mu\varepsilon R}(y,\tau)|\le c|\nabla h_R|\int_{\mathbb{R}^n}J_\varepsilon(w)|E(x-y+w,t-\tau)|\,dw\le c|\nabla h_R|(|x-y|^2+t-\tau)^{-\frac{n}{2}},$$

hence, uniformly with respect to $\mu,\varepsilon$ and $R>0$, almost everywhere in $\tau\in(0,t)$,

$$\big|(p(\tau),\nabla\cdot\varphi_{\mu\varepsilon R}(\tau))\big|\le\int_{R\le|y|\le 2R}\big|p(x,\tau)\big|(1+|y|)^{-n-1}\,dy;$$

I$_3$) for $R>2|x|$, for $\tau\in(0,t)$, uniformly with respect to $\mu$ and $\varepsilon$, we have

$$\begin{aligned}&\big|2(w,\nabla h_R\cdot\nabla J_\varepsilon[E_i(x,t))]+(w\Delta h_R,J_\varepsilon[E_i(x,t)])\big|\\ &\le 2c|w(\tau)|_\infty\left[R^{-1}\int_{R\le|y|\le 2R}|y|^{-n-1}\,dy+R^{-2}\int_{R\le|y|\le 2R}|y|^{-n}\,dy\right]\le 2cR^{-2}|w(\tau)|_\infty.\end{aligned}$$

By virtue of Lemma 3.3, for each $z\in\mathbb{R}^n$, $w_{\varepsilon R}(z,\tau)$ is a continuous function of $\tau$. Employing (4.2), we deduce

$$\begin{aligned}\lim_{\mu\to 0^+}(w_{\varepsilon R}(t-\mu),E_i(x,\mu))&=\lim_{\mu\to 0^+}\int_{\mathbb{R}^n}E_i(x-z,\mu)\cdot w_{\varepsilon R}(z,t-\mu)\,dz\\ &=w_{i_{\varepsilon R}}(x,t)+\int_{\mathbb{R}^n}\frac{\partial}{\partial x_i}\phi(x-z,0)\int_{\mathbb{R}^n}J_\varepsilon(y-z)\nabla h_R(y)\cdot w(y,t)\,dy\,dz.\qquad(4.4)\end{aligned}$$

Moreover, from $(4.1)_{22}$ and properties of $h_R$, since $2|x|<R$, for $R>0$ sufficiently large we get

$$\begin{aligned}&\left|\int_{\mathbb{R}^n}\frac{\partial}{\partial x_i}\phi(x-z,0)\int_{\mathbb{R}^n}J_\varepsilon(y-z)\nabla h_R(y)\cdot w(y,t)\,dy\,dz\right|\\ &\le c\int_{R-1\le|z|\le 2R+1}|z|^{n-1}\int_{\mathbb{R}^n}J_\varepsilon(y-z)|\nabla h_R(y)|\,|w(y,t)|\,dy\,dz\\ &\le c\int_{R-1\le|z|\le 2R+1}\frac{(1+|z|+\varepsilon)^{\frac{n}{2}}}{|z|^{n-1}}\int_{\mathbb{R}^n}J_\varepsilon(y-z)\frac{|\nabla h_R(y)|\,|w(y,t)|}{(1+|y|)^{\frac{n}{2}}}\,dy\,dz.\end{aligned}$$

Applying the Hölder inequality to the right hand side, we obtain

$$\left| \int_{\mathbb{R}^n} \frac{\partial}{\partial x_i} \phi(x-z,0) \int_{\mathbb{R}^n} J_\varepsilon(y-z) \nabla h_R(y) \cdot w(y,t)\, dy\, dz \right| \le cR |\nabla h_R| w(t)|g|_2,$$

where $g = (1+|y|)^{-\frac{n}{2}}$. Hence, by virtue of energy weight estimate (3.10), the last estimate implies, uniformly with respect to $\varepsilon$,

$$\left| \int_{\mathbb{R}^n} \frac{\partial}{\partial x_i} \phi(x-z,0) \int_{\mathbb{R}^n} J_\varepsilon(y-z) \nabla h_R(y) \cdot w(y,t)\, dy\, dz \right| \\ \le c|w(t)g|_{L^2(R-1\le|y|\le 2R+1)}, \quad (4.5)$$

Taking estimates $E_1$–$E_3$ into account, and also (4.4)–(4.5), passing to the limit when $\mu \to 0^+$, uniformly with respect to $\varepsilon$, we get

$$|w_{\varepsilon R}(x,t)| \le c|w(t)g|_{L^2(R-1\le|y|\le 2R+1)} + \int_0^t |p(\tau)(1+|y|)^{-n-1}|_{L^1(R\le|y|\le 2R)}\, d\tau \\ + 2cR^{-2} \int_0^t |w(\tau)|_\infty\, d\tau + c\int_0^t (|w(\tau|_{L^\infty(\mathbb{R}^n)} + 2|u(\tau)|_\infty)|w(\tau)|_\infty (t-\tau)^{-\frac{1}{2}}\, d\tau.$$

Since for each $t \in (0,T)$ $w_{\varepsilon R}(x,t)$ converges to $h_R(x)w(x,t)$ in $L^2(\mathbb{R}^n)$, then there exists a subsequence converging almost everywhere in $x$ to $h_R(x)w(x,t)$. Hence, from the above inequality, since $R > 2|x|$, along a suitable sequence of $\varepsilon \to 0$, we deduce that almost everywhere in $|x| < R/2$

$$|w(x,t)| \le c|w(t)g|_{L^2(R-1\le|y|\le 2R+1)} + \int_0^t |p(y,\tau)(1+|y|)^{-n-1}|_{L^1(R\le|y|\le 2R)}\, d\tau \\ + 2cR^{-2} \int_0^t |w(\tau)|_\infty\, d\tau + c\int_0^t |w(\tau)|_{L^\infty(\mathbb{R}^n)} + 2|u(\tau)|_\infty)|w(\tau)|_\infty (t-\tau)^{-\frac{1}{2}}\, d\tau.$$

By virtue of condition a) for the pressure field of a bounded weak solution and the weight energy estimate (3.10), applying the dominate convergence theorem, in the limit for $R \to +\infty$, almost everywhere in $x \in \mathbb{R}^n$, we get that

$$|w(x,t)| \le c\int_0^t (|w(\tau)|_{L^\infty(\mathbb{R}^n)} + 2|u(\tau)|_{L^\infty(\mathbb{R}^n)})|w(\tau)|_\infty (t-\tau)^{-\frac{1}{2}}\, d\tau,$$

which implies

$$|w(t)|_\infty \le c\int_0^t (|w(\tau)|_{L^\infty(\mathbb{R}^n)} + 2|u(\tau)|_{L^\infty(\mathbb{R}^n)})|w(\tau)|_\infty (t-\tau)^{-\frac{1}{2}}\, d\tau,$$

which implies the uniqueness. The theorem is completely proved. ∎

Remark 4.1. We conclude remarking that the proof of the uniqueness theorem is developed assuming the Euclidean dimension $n \ge 3$. In order to recover the 2-dimensional case, it is enough to modify the fundamental solution of the Cauchy problem. However, we do not study the 2-dimensional case. We just indicate that the fundamental solution of the Cauchy problem and its qualitative properties can be found in [4].

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When the paper has been completed Professor O. Sawada informed the author of the paper [13], which, among other results, contains a uniqueness theorem for weak bounded solutions.

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