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ON ε-APPROXIMATION AND FIXED POINTS OF NONEXPANSIVE MAPPINGS IN METRIC SPACES

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Abstract. Using fixed point theory, B. Brosowski [2] proved that if T is a nonexpansive linear operator on a normed linear space X, C a T-invariant subset of X and x a T-invariant point, then the set $P_C(x)$ of best C-approximant to x contains a T-invariant point if $P_C(x)$ is non-empty, compact and convex. Subsequently, many generalizations of the Brosowski's result have appeared. We also obtain some results on invariant points of a nonexpansive mapping for the set of ε-approximation in metric spaces thereby generalizing and extending some known results including that of Brosowski, on the subject.

Using fixed point theory, the theorem of Meinardus [6] on invariant approximation was generalized by Brosowski [2] who proved that if T is a nonexpansive linear operator on a normed linear space X, C a T-invariant subset of X and x a T-invariant point, then the set $P_C(x)$ of best C-approximant to x contains a T-invariant point if $P_C(x)$ is non-empty, compact and convex. Subsequently, various generalizations of Brosowski's result have appeared (see e.g. [5]). In the present work we also obtain some results on invariant points of a nonexpansive mapping T on the set of ε-approximation in metric spaces. Our results contain some of the results of [1], [2], [5], [6], [7], [8], [11] and [12].

To begin with, we recall a few definitions.

Let G be a non-empty subset of a metric space $(X,d), x \in X$ and $\varepsilon > 0$. An element $g_\circ \in G$ is said to be (s.t.b.) an ε-approximation or ε-approximant to x (respectively, ε-coapproximation or ε-coapproximant to x) if $d(x,g_\circ) \leq d(x,g)+\varepsilon$ (respectively, $d(g_\circ,g)+\varepsilon \leq d(x,g)$) for all $g \in G$, i.e. $(d(x,g_\circ) \leq d(x,G)+\varepsilon$ (respectively, $d(g_\circ,g)+\varepsilon \leq d(x,G)$). We shall denote by $P_G(x,\varepsilon)$ (respectively, $R_G(x,\varepsilon)$) the set of all ε-approximant (respectively, ε-coapproximant) to x, i.e. $P_G(x,\varepsilon) = \{g_\circ \in G : d(x,g_\circ) \leq d(x,G)+\varepsilon\}$ (respectively, $R_G(x,\varepsilon) = \{g_\circ \in G : d(g_\circ,g)+\varepsilon \leq d(x,G)\}$). For $\varepsilon = 0$, the set $P_G(x,\varepsilon)$ (respectively, $R_G(x,\varepsilon)$) is the set of best approximations (respectively, best coapproximations) of x in G.

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For $\varepsilon > 0$, the set $P_G(x, \varepsilon)$ is always a non-empty bounded set and is closed if G is closed. In normed linear spaces, the elements of ε-approximation were introduced by R.C. Buck (who used the term 'good approximation' for such elements) and subsequently, the study was taken up by others (see, e.g. [10]).

A sequence $\langle g_n \rangle$ in G is said to be ε-minimizing for x if $\lim_{n\to\infty} d(x, g_n) \leq d(x, G) + \varepsilon$. The set G is said to be ε-approximatively compact (see [7]) if for each $x \in X$, each ε-minimizing sequence has a subsequence converging to an element of G.

If a mapping $T\colon X \to X$ leaves subset G of X invariant, then the restriction of T to G is denoted by T/G.

If G is a closed subset of X then $T\colon G \to G$ is called a compact mapping [5] if for every bounded subset A of G, $\overline{T(A)}$ is compact in G.

A mapping $T\colon X \to X$ is s.t.b.

a) nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in X$,

b) contraction if there exists α, $0 \leq \alpha < 1$ such that $d(Tx, Ty) \leq \alpha d(x, y)$ for all $x, y \in X$.

A mapping $T\colon X \to X$ satisfies condition (A) (see [7]) if $d(Tx, y) \leq d(x, y)$ for all $x, y \in X$.

A family of maps $\{f_\alpha : \alpha \in G\}$ is s.t.b. a G-convex structure (see [3]), if

i. $f_\alpha\colon [0, 1] \to G$, i.e. f_α is a mapping from $[0, 1]$ into G for each $\alpha \in G$;

ii. $f_\alpha(1) = \alpha$ for each $\alpha \in G$;

iii. $f_\alpha(t)$ is jointly continuous in (α, t), i.e. $f_\alpha(t) \to f_{\alpha_\circ}(t_\circ)$ for $\alpha \to \alpha_\circ$ in G and $t \to t_\circ$ in $[0, 1]$, and

iv. $d(f_\alpha(t), f_\beta(t)) \leq \Phi(t) d(\alpha, \beta)$ where $\Phi\colon (0, 1) \to (0, 1)$.

For a metric space (X, d), a continuous mapping $W\colon X \times X \times [0, 1] \to X$ is s.t.b. convex structure on X if for all $x, y \in X$ and $\lambda \in [0, 1]$, we have

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda) d(u, y)$$

for all $u \in X$. The metric space (X, d) with convex structure is called a convex metric space [14].

A subset K of a convex metric space (X, d) is s.t.b. a convex set [14] if $W(x, y, \lambda) \in K$ for all $x, y \in K$ and $\lambda \in [0, 1]$.

The set K is said to be starshaped (or p-starshaped) [4] if there exists a $p \in K$ such that $W(x, p, \lambda) \in K$ for all $x \in K$ and $\lambda \in [0, 1]$.

Clearly, each convex set is starshaped but not conversely.

A convex metric space (X, d) is said to satisfy Property (I) [4] if for all $x, y, p \in X$ and $\lambda \in [0, 1]$,

$$d(W(x, p, \lambda), W(y, p, \lambda)) \leq \lambda d(x, y).$$

A normed linear space and each of its convex subsets are simple examples of convex metric spaces. There are many convex metric spaces which are not normed linear spaces (see [14]). Property (I) is always satisfied in a normed linear space.

A more general class of sets containing the starshaped sets is called 'contractive'.

A subset K of a metric space (X, d) is s.t.b. contractive if there exists a sequence $\langle f_n \rangle$ of contraction mappings of K into itself such that $f_n y \to y$ for each $y \in K$.

In a convex metric space (X, d) satisfying Property (I), every starshaped set is contractive can be seen as below.

Suppose K is starshaped with respect to $p \in K$. Define $f_n \colon K \to K$ as

$$f_n(y) = W(y, p, 1 - \frac{1}{n}), n = 1, 2, 3, \ldots$$

Consider, $d(y, f_n y) = d(y, W(y, p, 1 - \frac{1}{n})) \leq (1 - \frac{1}{n}) d(y, y) + \frac{1}{n} d(y, p) \to 0$ as $n \to \infty$. Thus $f_n y \to y$ for all $y \in K$. Moreover,

$$d(f_n x, f_n y) = d(W(x, p, 1 - \frac{1}{n}), W(y, p, 1 - \frac{1}{n})) \leq (1 - \frac{1}{n}) d(x, y)$$

for all $x, y \in K$, i.e. $\langle f_n \rangle$ is a sequence of contraction mappings.

The following result dealing with the structure of the set $P_G(x, \varepsilon)$ will be used in the sequel.

Lemma. *If G is an ε-approximatively compact set in a metric space (X, d) then $P_G(x, \varepsilon)$ is a non-empty compact set.*

Proof. By the definition of $d(x, G)$, we can find $g_\circ \in G$ such that $d(x, g_\circ) \leq d(x, G) + \varepsilon$ and so $P_G(x, \varepsilon)$ is non-empty.

Let $\langle g_n \rangle$ be a sequence in $P_G(x, \varepsilon)$, i.e. $d(x, g_n) \leq d(x, G) + \varepsilon$ for all $n = 1, 2 \ldots$ and so

$$\lim_{n \to \infty} d(x, g_n) \leq d(x, G) + \varepsilon \tag{1}$$

i.e. $\langle g_n \rangle$ is ε-minimizing sequence in G. Since G is ε-approximatively compact, $\langle g_n \rangle$ has a subsequence $\langle g_{n_i} \rangle \to g_\circ \in G$. So (1) implies $d(x, g_\circ) \leq d(x, G) + \varepsilon$, i.e. $g_\circ \in P_G(x, \varepsilon)$ and so $P_G(x, \varepsilon)$ is compact. ∎

The following result which deals with invariance of ε-approximations for nonexpansive mappings improves and generalizes Theorem 2.1 of [8].

Theorem 1. *Let T be a self mapping on a metric space (X, d), G a T-invariant subset of X and x a T-invariant point. If the set D of ε-approximant to x is a compact set with D-convex structure and T is nonexpansive on $D \cup \{x\}$, then D contains a T-invariant point.*

Proof. Since $D = \{y \in G : d(x, y) \leq d(x, G) + \varepsilon\}$, $T \colon D \to D$. In fact if $y \in D$, then

$$d(x, Ty) = d(Tx, Ty) \leq d(x, y) \leq d(x, G) + \varepsilon$$

and so $Ty \in D$.

Let $\langle k_n \rangle$, $0 \le k_n < 1$ be a sequence of real numbers such that $k_n \to 1$ as $n \to \infty$. Define T_n as $T_n z = f_{Tz}(k_n), z \in D$. Since $T(D) \subseteq D$ and $0 \le k_n < 1$, we have that each T_n is a well defined and maps D into D. Moreover, for all $y, z \in D$

$$\begin{aligned} d(T_n y, T_n z) &= d(f_{Ty}(k_n), f_{Tz}(k_n)) \\ &\le \Phi(k_n) d(Ty, Tz) \le \Phi(k_n) d(y, z), \end{aligned}$$

and so each T_n is a contraction mapping on D. Since D is compact, it follows from Banach Contraction Principle that each T_n has a unique fixed point $x_n \in D$, i.e. $T_n x_n = x_n$ for each n. Since D is compact, $\langle x_n \rangle$ has a subsequence $x_{n_i} \to \bar{x} \in D$. We claim that $T\bar{x} = \bar{x}$. Consider

$$x_{n_i} = T_{n_i} x_{n_i} = f_{Tx_{n_i}}(k_{n_i}) \to f_{T\bar{x}}(1).$$

As the family $\{f_\alpha\}$ is jointly continuous and T being nonexpansive, is continuous. Thus $x_{n_i} \to T\bar{x}$. Therefore $T\bar{x} = \bar{x}$ i.e. $\bar{x} \in D$ is T-invariant. ■

For $\varepsilon = 0$, we have

Corollary 1. *Let T be mapping on a metric space (X, d), G a T-invariant subset of X and x a T-invariant point. If the set D of best G-approximant to x is compact set with D-convex structure and T is nonexpansive on $D \cup \{x\}$, then D contains a T-invariant point.*

The above corollary improves and generalizes Theorem 2 of [7].

In view of the Lemma, we have

Corollary 2. *Let T be mapping on a metric space (X, d), G an ε-approximatively compact (approximatively compact) and T-invariant subset of X and x a T-invariant point. If the set D of ε-approximant (best approximant) to x has convex structure and T is nonexpansive on $D \cup \{x\}$, then D contains a T-invariant point.*

Theorem 2. *Let T be a self mapping on a metric space (X, d), G a T-invariant subset of X and x a T-invariant point. If the set D of ε-approximant to x is compact, contractive and T is nonexpansive on $D \cup \{x\}$, then D contains a T-invariant point.*

Proof. Since $D = \{y \in G : d(x, y) \le d(x, G) + \varepsilon\}$, $T \colon D \to D$. In fact if $y \in D$, then

$$d(x, Ty) = d(Tx, Ty) \le d(x, y) \le d(x, G) + \varepsilon$$

and so $Ty \in D$. Since D is contractive, there exists a sequence $\langle f_n \rangle$ of contraction mapping of D into itself such that $f_n z \to z$ for every $z \in D$.

We claim that $z_\circ$ is a fixed point of T. Let $\varepsilon > 0$ be given. Since $z_{n_i} \to z_\circ$ and $f_n T z_\circ \to T z_\circ$, there exist a positive integer m such that for all $n_i \ge m$

$$d(z_{n_i}, z_\circ) < \frac{\varepsilon}{2} \quad \text{and} \quad d(f_{n_i} T z_\circ, T z_\circ) < \frac{\varepsilon}{2}.$$

Again,

$$d(f_{n_i}Tz_{n_i}, f_{n_i}Tz_\circ) \le d(z_{n_i}, z_\circ) < \frac{\varepsilon}{2}.$$

Hence, $d(f_{n_i}Tz_{n_i}, Tz_\circ) \le d(f_{n_i}Tz_{n_i}, f_{n_i}Tz_\circ) + d(f_{n_i}Tz_\circ, Tz_\circ) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$, i.e. $d(f_{n_i}Tz_{n_i}, Tz_\circ) < \varepsilon$ for all $n_i \ge m$ and so $f_{n_i}Tz_{n_i} \to Tz_\circ$. But $f_{n_i}Tz_{n_i} = z_{n_i} \to z_\circ$ and therefore $Tz_\circ = z_\circ$. ∎

Using the Lemma we have

Corollary 3. *Let T be a self mapping on a metric space (X, d), G an ε-approximatively compact, T-invariant subset of X and x a T-invariant point. If the set D of ε-approximant to x is contractive and T is nonexpansive on $D \cup \{x\}$, then D contains a T-invariant point.*

Corollary 4. *Let T be mapping on a convex metric space (X, d) satisfying Property (I), G a T-invariant subset of X and x a T-invariant point. If the set D of ε-approximant to x is compact, starshaped and T is nonexpansive on $D \cup \{x\}$, then D contains a T-invariant point.*

Proof. As in Theorem 2, T is a self map on D. Since D is non-empty and starshaped, there exists $p \in D$ such that $W(z, p, \lambda) \in D$ for all $z \in D$, $\lambda \in I = [0, 1]$. Let $\langle k_n \rangle$, $0 \le k_n < 1$, be a sequence of real numbers such that $k_n \to 1$ as $n \to \infty$. Define T_n as $T_n(z) = W(Tz, p, k_n)$, $z \in D$. Since T is a self map on D and D is starshaped, each T_n is a well defined and maps D into D. Moreover,

$$d(T_ny, T_nz) = d(W(Ty, p, k_n), W(Tz, p, k_n)) \le k_n d(Ty, Tz) \le k_n d(y, z),$$

i.e. each T_n is a contraction mapping on the compact set D. So by Banach Contraction Principle each T_n has a unique fixed point $x_n \in D$, i.e. $T_n x_n = x_n$ for each n. Since D is compact, $\langle x_n \rangle$ has a subsequence $x_{n_i} \to \bar{x} \in D$. We claim that $T\bar{x} = \bar{x}$. Consider,

$$\begin{aligned} d(x_{n_i}, T\bar{x}) &= d(T_{n_i}x_{n_i}, T\bar{x}) = d(W(Tx_{n_i}, p, k_{n_i}), T\bar{x}) \\ &\le k_{n_i} d(Tx_{n_i}, T\bar{x}) + (1 - k_{n_i}) d(p, T\bar{x}) \\ &\le k_{n_i} d(x_{n_i}, \bar{x}) + d(1 - k_{n_i}) d(p, T\bar{x}) \to 0, \end{aligned}$$

and so $x_{n_i} \to T\bar{x}$. Therefore $T\bar{x} = \bar{x}$, i.e. $\bar{x}$ is T-invariant. ∎

Remarks 1. (i) Since in a convex metric space (X, d) satisfying Property (I) every starshaped set is contractive, the result also follows from Theorem 2.

(ii) Corollary 3 generalizes Theorem 2 of [11] which is a generalization of Theorem of Brosowski [2] as well as of Singh [12].

(iii) Since a Banach space is a convex metric space with Property (I) and D is compact if G is ε-approximatively compact. Theorem 2.2 of [8] is a particular case of Corollary 4.

Clearly, f_nT is a contraction on the compact set D for each n and so by Banach contraction principle, each f_nT has a unique fixed point, say z_n in D. Now the compactness of D implies that the sequence $\langle z_n \rangle$ has a subsequence $z_{n_i} \to z_0 \in D$.

For $\varepsilon = 0$, we derive the following known results as corollaries.

Corollary 5. *Let T be a self mapping on a convex metric space (X, d) satisfying Property (I), G a T-invariant subset of X and x a T-invariant point. If the set D of best G-approximant to x is non-empty compact and starshaped and T is nonexpansive on $D \cup \{x\}$, then D contains a T-invariant point.*

Corollary 6. [12]. *Let T be a nonexpansive mapping on a normed linear space X. Let G be a T-invariant subset of X and x a T-invariant point in X. If D, the set of best G-approximant to x is non-empty compact and starshaped, then it contains a T-invariant point.*

Corollary 7. [13] *Let X be a normed linear space and $T: X \to X$ be a nonexpansive mapping. Let T have a fixed point, say x, and leaves a finite-dimensional subspace G of X invariant. Then T has a fixed point which is a best G-approximant to x in G.*

Since in this case the set D is non-empty and compact, the result follows from Corollary 6.

Theorem 3. *Let T be a self mapping on a convex metric space (X, d) satisfying Property (I). Suppose G is a closed T-invariant subset of X, T/G is compact and x a T-invariant point. If the set D of ε-approximant to x is starshaped and T is nonexpansive on $D \cup \{x\}$, then D contains a T-invariant point.*

Proof. As in the proof of Theorem 2, D is T-invariant. Now D is a bounded subset of G and T/G is compact so $\overline{T(D)}$ is compact. Since D is closed and starshaped, by Theorem 3 [1] T has a fixed point in D. ■

For $\varepsilon = 0$, Theorem 3 improves Theorem 10 of [1] and also generalizes Theorem 4 of [5].

Now we give a result for T-invariant points in the set of ε-coapproximations in G for a given element x of a metric space (X, d).

Theorem 4. *Let T be a self map satisfying condition (A) on a convex metric space (X, d) satisfying Property (I), G a subset of X such that $R_G(x, \varepsilon)$ is non-empty compact, starshaped and T is nonexpansive on $R_G(x, \varepsilon)$. Then there exists a $\overline{g_\circ} \in R_G(x, \varepsilon)$ such that $T\overline{g_\circ} = \overline{g_\circ}$.*

Proof. Let $g_\circ \in R_G(x, \varepsilon)$. Consider

$$d(Tg_\circ, g) + \varepsilon \leq d(g_\circ, g) + \varepsilon \leq d(x, G)$$

and so $Tg_\circ \in R_G(x, \varepsilon)$ i.e. $T : R_G(x, \varepsilon) \to R_G(x, \varepsilon)$. Now proceeding as in Corollary 4, we shall get $\overline{g_\circ} \in R_G(x, \varepsilon)$ which is a fixed point for T. ■

Remarks 2. (i) Taking $\varepsilon = 0$, we see that Theorem 4 improves and generalizes Theorem 4.1 of [8].

(ii) Proceeding as in Theorem 1, one can show that Theorem 4 holds if starshapedness of $R_G(x,\varepsilon)$ is replaced by the condition that $R_G(x,\varepsilon)$ is a set with convex structure.

(iii) Results similar to those proved in the earlier part of the paper can be proved for the set of ε-coapproximations.

(iv) Theorem 4.2 of [8] on strong best coapproximation can also be proved for convex metric space under relaxed conditions as in Theorem 4.

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