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A NEW HYPERSPACE TOPOLOGY AND THE STUDY OF THE FUNCTION SPACE θ^*-$LC(X, Y)$

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Abstract. The intent of this paper is to introduce a new hyperspace topology on the collection of all θ-closed subsets of a topological space. The space of all θ^*-lower semicontinuous functions has been studied in detail and finally we deal with some multifunctions.

1. Introduction

In the study of hyperspace topology, the first step towards topologizing a collection of subsets of a topological space X was taken by Hausdorff [5], where he defined a metric on the collection of all nonempty closed subsets of X, where X is a bounded metric space. Vietoris then introduced a new topology on the collection of all nonempty closed subsets of a topological space (X, τ), which is known as "Vietoris Topology" or "finite topology". After that, Michael in his paper [8] dealt with different types of subsets for construction of topology. Subsequently, Fell in his paper [2] constructed a compact Hausdorff topology for the space of all closed subsets of a topological space (X, τ). After that much of work has been done on hyperspace topology. In this connection we can mention the paper [6] by Di Maio and Kočinac, where the authors have investigated the covering properties of hyperspaces related to our investigation.

In this paper we first introduce a new topology on the collection of all nonempty θ-closed subsets of a topological space (X, τ). Then we study some properties of this topology and examine the restriction of this topology on the function space of θ^*-lower semicontinuous functions. In the last section of this paper some results relating multifunctions have been discussed.

2. $\theta(X)$ with a new topology

Throughout this paper X will always mean a topological space.

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Definition 2.1. [10] A point $x \in X$ is said to be a θ-contact point of a set $A \subseteq X$ if for every neighborhood U of x, we get $cl_X U \cap A \neq \emptyset$.

The set of all θ-contact points of a set A is called the θ-closure of A and we denote this set by $\overline{A}^{\theta}$. A set A is called θ-closed if $A = \overline{A}^{\theta}$. A set is called θ-open if $X \setminus A$ is θ closed.

Remark 2.2. The collection of all θ-open sets in X forms a topology.

In this section our main interest of study is $\theta(X)$ where,

$$\theta(X) = \{ A \subseteq X : A \neq \emptyset \text{ and } A \text{ is } \theta\text{-closed } \}$$

We give $\theta(X)$ a new topology τ and discuss some properties of $(\theta(X), \tau)$.

Definition 2.3. [10] A T_2-space X is called H-closed if any open cover of X has a finite proximate subcover, i.e. a finite collection whose union is dense in X.

A set $A \subseteq X$ is called an H-set if any open cover $\{U_\alpha : \alpha \in \Lambda\}$ of A by open sets in X has a finite subfamily $\{U_{\alpha_i} : i = 1, 2, \ldots, n\}$ such that $A \subseteq \bigcup_{i=1}^{n} cl_X U_{\alpha_i}$.

Theorem 2.4. [1] *In an H-closed Urysohn space every H-set is θ-closed and every θ-closed set is an H-set.*

Definition 2.5. On $\theta(X)$ we define a topology as follows. For each $W \subseteq X$, let $W^+ = \{A \in \theta(X) : A \subseteq W\}$ and $W^- = \{A \in \theta(X) : A \cap W \neq \emptyset\}$. Consider $S_\theta = \{W^- : W \text{ is open in } X\} \cup \{W^+ : W \text{ is } \theta\text{-open in } X \text{ and } X \setminus W \text{ is an } H\text{-set}\}$. Then S_θ forms a subbase for some topology on $\theta(X)$ which we denote by τ.

Proposition 2.6. *Let $V_1, V_2, \ldots, V_n$ be subsets of X. Then*

a) $V_1^+ \cap V_2^+ \cap \cdots \cap V_n^+ = (V_1 \cap V_2 \cap \cdots \cap V_n)^+$.

b) Let $V_1, V_2, \ldots, V_n$ be θ-open sets and each $X \setminus V_i$ is an H-set for $i = 1, 2, \ldots, n$. Then $(V_1 \cap V_2 \cap \cdots \cap V_n)^+ \in S_\theta$.

Proof. a) Let $A \in V_1^+ \cap V_2^+ \cap \cdots \cap V_n^+$. Then $A \in \theta(X)$ with $A \subseteq V_i$, for each $i = 1, 2, \ldots, n$. Hence $A \subseteq V_1 \cap V_2 \cap \cdots \cap V_n$, i.e., $A \in (V_1 \cap V_2 \cap \cdots \cap V_n)^+$. Therefore

$$V_1^+ \cap V_2^+ \cap \cdots \cap V_n^+ \subseteq (V_1 \cap V_2 \cap \cdots \cap V_n)^+.$$

Conversely, let $B \in \theta(X)$ be such that $B \in (V_1 \cap V_2 \cap \cdots \cap V_n)^+$, i.e., $B \subseteq V_1 \cap V_2 \cap \cdots \cap V_n$. Hence $B \subseteq V_i$ for each $i = 1, 2, \ldots, n$, i.e., $B \in V_i^+$, for each $i = 1, 2, \ldots, n$, i.e., $B \in V_1^+ \cap V_2^+ \cap \cdots \cap V_n^+$. Therefore,

$$(V_1 \cap V_2 \cap \cdots \cap V_n)^+ \subseteq V_1^+ \cap V_2^+ \cap \cdots \cap V_n^+.$$

Thus,

$$V_1^+ \cap V_2^+ \cap \cdots \cap V_n^+ = (V_1 \cap V_2 \cap \cdots \cap V_n)^+.$$

b) Since each V_i is θ-open for $i = 1, 2, \ldots, n$, $V_1 \cap V_2 \cap \cdots \cap V_n$ is also θ-open. Now $X \setminus (V_1 \cap V_2 \cap \cdots \cap V_n) = (X \setminus V_1) \cup (X \setminus V_2) \cup \cdots \cup (X \setminus V_n)$. Since each $(X \setminus V_i)$ is an H-set for $i = 1, 2, \ldots, n$ and union of finitely many H-sets is an H-set,

$X \setminus (V_1 \cap V_2 \cap \cdots \cap V_n)$ is an H-set. So $(V_1 \cap V_2 \cap \cdots \cap V_n)$ is a θ-open set such that $X \setminus (V_1 \cap V_2 \cap \cdots \cap V_n)$ is an H-set. Hence $(V_1 \cap V_2 \cap \cdots \cap V_n)^+ \in S_\theta$. ∎

Note 2.7. Using the above proposition we can say that any basic open set in the above defined topology is of the form $V_1^- \cap V_2^- \cap \cdots \cap V_n^- \cap V_0^+$ where $V_i \subseteq V_0$ for each $i = 1, 2, \ldots, n$ and $V_1, V_2, \ldots, V_n$ are open sets, V_0 is a θ-open set with $X \setminus V_0$ an H-set.

Proposition 2.8. *$(\theta(X), \tau)$ is always T_0.*

Proof. Let $A, B \in \theta(X)$ be such that $A \neq B$. Without loss of generality, let $A \not\subseteq B$. Then $A \cap (X \setminus B) \neq \emptyset$ which implies $A \in (X \setminus B)^-$. Also, $B \cap (X \setminus B) = \emptyset$ gives $B \notin (X \setminus B)^-$. Since B is θ-closed, $(X \setminus B)$ is θ-open in X. Hence $(\theta(X), \tau)$ is T_0. ∎

Proposition 2.9. [3] *X is T_2 if and only if $\{a\}$ is θ-closed for each $a \in X$.*

Proposition 2.10. *$(\theta(X), \tau)$ is T_1 if X is T_2.*

Proof. Let $A, B \in \theta(X)$ be such that $A \neq B$. Without loss of generality, let $A \not\subseteq B$. Then $A \cap (X \setminus B) \neq \emptyset$ which implies $A \in (X \setminus B)^-$ which is an open set in $(\theta(X), \tau)$ since $(X \setminus B)$ is θ-open. Also there exists $a \in A$ such that $a \notin B$. Then $B \in (X \setminus \{a\})^+$. Since X is T_2, by Proposition 2.9, $\{a\}$ is θ-closed and hence $X \setminus \{a\}$ is θ-open. Also, $\{a\}$ is an H-set for each $a \in A$. Hence $(X \setminus \{a\})^+$ is open in $(\theta(X), \tau)$. Thus $(\theta(X), \tau)$ is T_1. ∎

Proposition 2.11. *$(\theta(X), \tau)$ is T_2 if X is Urysohn and H-closed.*

Proof. Let $A, B \in \theta(X)$ be such that $A \neq B$. Without loss of generality, let $A \not\subseteq B$. Then there exists $a \in A$ such that $a \notin B$. Since $B \in \theta(X)$, $a \notin B = \overline{B}^\theta$. Thus there exists a neighborhood U of a such that $cl_X U \cap B = \emptyset$ which implies $B \subseteq X \setminus cl_X U$. Since X is Urysohn and H-closed, $cl_X U$ is θ-closed and also an H-set. Put, $V = X \setminus cl_X U$. Then V is a θ-open set in X. Thus, $A \cap U \neq \emptyset$ which implies $A \in U^-$ and $B \in V^+$. We now show that $U^- \cap (X \setminus cl_X U)^+ = \emptyset$. If possible, let $P \in U^- \cap (X \setminus cl_X U)^+$. Then $P \cap U \neq \emptyset$ and $P \subseteq X \setminus cl_X U$ which implies $(X \setminus cl_X U) \cap U \neq \emptyset$ -a contradiction. Hence $(\theta(X), \tau)$ is T_2. ∎

Proposition 2.12. *Let $V_1, V_2, \ldots, V_n$ be open in X and V_0 be θ-open in X. Then in $(\theta(X), \tau)$, $cl_{\theta(X)}(V_1^- \cap V_2^- \cap \cdots \cap V_n^- \cap V_0^+) = (cl_X V_1)^- \cap (cl_X V_2)^- \cap \cdots \cap (cl_X V_n)^- \cap (cl_X V_0)^+$ provided X is Urysohn and H-closed.*

Proof. Let $A \notin (cl_X V_1)^- \cap (cl_X V_2)^- \cap \cdots \cap (cl_X V_n)^- \cap (cl_X V_0)^+$. Then either $A \not\subseteq cl_X V_0$ or $A \cap cl_X V_i = \emptyset$, for some i. If $A \not\subseteq cl_X V_0$, then $A \cap (X \setminus cl_X V_0) \neq \emptyset$ which implies $A \in (X \setminus cl_X V_0)^-$. But $(X \setminus cl_X V_0)^- \cap L = \emptyset$, the empty set in $\theta(X)$ where $L = V_1^- \cap V_2^- \cap \cdots \cap V_n^- \cap V_0^+$. Now if $A \cap cl_X V_i = \emptyset$, for some i, then $A \subseteq X \setminus cl_X V_i$, i.e., $A \in (X \setminus cl_X V_i)^+$. Since X is Urysohn and H-closed, $cl_X V_i$ is

θ-closed and an H-set. So $(X \setminus cl_X V_i)^+$ is open in $\theta(X)$. Now, $(X \setminus cl_X V_i)^+ \cap L = \emptyset$. This shows that $A \notin cl_{\theta(X)}(V_1^- \cap V_2^- \cap \cdots \cap V_n^- \cap V_0^+)$. Therefore,

$$cl_{\theta(X)}(V_1^- \cap V_2^- \cap \cdots \cap V_n^- \cap V_0^+) \subseteq (cl_X V_1)^- \cap (cl_X V_2)^- \cap \cdots \cap (cl_X V_n)^- \cap (cl_X V_0)^+. \tag{i}$$

Now let $A \in (cl_X V_1)^- \cap (cl_X V_2)^- \cap \cdots \cap (cl_X V_n)^- \cap (cl_X V_0)^+$ and $V = W_1^- \cap W_2^- \cap \cdots \cap W_m^- \cap W_0^+$ be an open neighborhood of A in $\theta(X)$. Then W_1, W_2, $\ldots$, W_m are open and W_0 is θ-open in X with $X \setminus W_0$ an H-set such that $W_i \subseteq W_0$, $i = 1, 2, \ldots, m$. $A \cap cl_X V_j \neq \emptyset$, for all $j = 1, 2, \ldots, n$, hence there exists $a_j \in A \cap cl_X V_j$, $j = 1, 2, \ldots, n$. Also, $A \subseteq W_0$. Therefore W_0 being an open neighborhood of a_j, $W_0 \cap V_j \neq \emptyset$, $j = 1, 2, \ldots, n$, hence there exists $x_j \in W_0 \cap V_j$, $j = 1, 2, \ldots, n$. Now, $A \cap W_i \neq \emptyset$, $i = 1, 2, \ldots, m$, hence there exists $b_i \in A \cap W_i$, $i = 1, 2, \ldots, m$. Also, $A \subseteq cl_X V_0$. Therefore, as W_i is an open neighborhood of b_i, $W_i \cap V_0 \neq \emptyset$, $i = 1, 2, \ldots, m$, hence there exists $w_i \in W_i \cap V_0$, $i = 1, 2, \ldots, m$. Let $B = \{x_1, \ldots, x_n, w_1, \ldots, w_m\}$. Since X is Urysohn, B is θ-closed. Now $B \cap W_i \neq \emptyset$, $i = 1, 2, \ldots, m$ and $B \subseteq W_0$. Also, $B \cap V_j \neq \emptyset$, $j = 1, 2, \ldots, n$ and $B \subseteq V_0$. Therefore $B \in V \cap L$. Hence $A \in cl_{\theta(X)} L$. So,

$$(cl_X V_1)^- \cap (cl_X V_2)^- \cap \cdots \cap (cl_X V_n)^- \cap (cl_X V_0)^+ \subseteq cl_{\theta(X)}(V_1^- \cap V_2^- \cap \cdots \cap V_n^- \cap V_0^+). \tag{ii}$$

From (i) and (ii) we get,

$$cl_{\theta(X)}(V_1^- \cap V_2^- \cap \cdots \cap V_n^- \cap V_0^+) = (cl_X V_1)^- \cap (cl_X V_2)^- \cap \cdots \cap (cl_X V_n)^- \cap (cl_X V_0)^+.$$

Theorem 2.13. *$(\theta(X), \tau)$ is H-closed if X is Urysohn and H-closed.*

Proof. Let $\{Y_i\}$ be a universal net of elements of $\theta(X)$. Define$Z = \{x \in X :$ for each open neighborhood U of x, $\{Y_i\}$ is eventually in $(cl_X U)^-\}$. Choose $y_i \in Y_i$. Then $\{y_i\}$ is a net in X which is H-closed and T_2. Hence $\{y_i\}$ has a θ-convergent subnet $\{y_{n_i}\}$ (say) θ-converging to y (say). Then for any open neighborhood W of y, $\{y_{n_i}\}$ is eventually in $cl_X W$, i.e., $\{Y_{n_i}\}$ is eventually in $(cl_X W)^-$ and hence $\{Y_i\}$ is eventually in $(cl_X W)^-$ (because of the universality of $\{Y_i\}$). Thus $y \in Z$ and $Z \neq \emptyset$.

Next we show that $Z \in \theta(X)$. Let $\{x_\lambda\}$ be a net in Z θ-converging to $x \in X$. Let U be an arbitrary open neighborhood of x. Since X is H-closed and Urysohn, X is almost regular. Hence there exists an open neighborhood V of x such that $x \in V \subseteq cl_X V \subseteq int_X(cl_X(U))$. Since $\{x_\lambda\}$ θ-converges to x, there exists $\lambda_0 \in \Lambda$ such that $x_\lambda \in cl_X V \subseteq int_X(cl_X(U))$, for all $\lambda \geq \lambda_0$ and since $x_\lambda \in Z$, $\{Y_i\}$ is eventually in $(cl_X U)^-$. Hence $x \in Z$, i.e., $Z \in \theta(X)$.

We now show that $\{Y_i\}$ θ-converges to Z in τ. Let $B_1^- \cap B_2^- \cap \cdots \cap B_n^- \cap B_0^+$ be an arbitrary open neighborhood of Z in τ, i.e., $Z \cap B_i \neq \emptyset$, for all $i = 1, 2, \ldots, n$ and $Z \subseteq B_0$. Let $b_j \in Z \cap B_j$, for $j = 1, 2, \ldots, n$. Since B_j is an open neighborhood of b_j, so $b_j \in Z$ which implies $\{Y_i\}$ is eventually in $(cl_X B_j)^-$, for $j = 1, 2, \ldots, n$. Therefore, $\{Y_i\}$ is eventually in $(cl_X B_1)^- \cap (cl_X B_2)^- \cap \cdots \cap (cl_X B_n)^-$. Now it suffices to show that $\{Y_i\}$ is eventually in $(cl_X B_0)^+$. Since $\{Y_i\}$ is a universal net, so either $\{Y_i\}$ is eventually in B_0^+ or in $\theta(X)\setminus B_0^+$. If $\{Y_i\}$ is eventually in $\theta(X)\setminus B_0^+$, then there exists i_0 such that $Y_i \in \theta(X)\setminus B_0^+$, for all $i \geq i_0$, i.e., $Y_i \cap (X \setminus B_o) \neq \emptyset$, for all $i \geq i_0$. We choose $z_i \in Y_i \cap (X \setminus B_0)$, for $i \geq i_0$. Then $X \setminus B_0$ being an H -set, $\{z_i\}$ has a θ-convergent subnet $\{z_{n_i}\}$ (say) θ-converging to z (say). Clearly $z \in X \setminus B_0$. Then for any open neighborhood W of z, $\{z_{n_i}\}$ is eventually in $cl_X W$, i.e., $\{Y_{n_i}\}$ is eventually in $(cl_X W)^-$ and hence $\{Y_i\}$ is eventually in $(cl_X W)^-$ (by the universality of $\{Y_i\}$) which implies $z \in Z$, i.e., $z \in Z \cap (X \setminus B_0)$ which contradicts the fact that $Z \subseteq B_0$. Hence $\{Y_i\}$ is eventually in B_0^+, i.e., in $(cl_X B_0)^+$. Thus $\{Y_i\}$ is eventually in $(cl_X B_1)^- \cap (cl_X B_2)^- \cap \cdots \cap (cl_X B_n)^- \cap (cl_X B_0)^+ = cl_{\theta(X)}(B_1^- \cap B_2^- \cap \cdots \cap B_n^- \cap B_0^+)$ which implies that $\{Y_i\}$ θ-converges to Z in τ. Hence $(\theta(X),\tau)$ is H-closed. ∎

Remark 2.14. The fact that $(\theta(X),\tau)$ is H-closed does not imply that X is Urysohn. In fact, if X is infinite with the cofinite topology, then $(\theta(X),\tau)$ is compact but X is not even T_2.

Proposition 2.15. *If X is T_2 and $(\theta(X),\tau)$ is compact, then X is compact.*

Proof. Let $\{U_\lambda : \lambda \in \Lambda\}$ be an open cover of X. Let $x \in X$. Then $x \in U_\lambda$ for some $\lambda \in \Lambda$. Since X is T_2, $\{x\}$ is θ-closed, i.e., $\{x\} \in \theta(X)$ and so, $\{x\} \in U_\lambda^-$, for $\lambda \in \Lambda$. Hence $\{U_\lambda^- : \lambda \in \Lambda\}$ is a τ-open cover of $\theta(X)$. $(\theta(X),\tau)$ being compact, $\theta(X) = \bigcup_{i=1}^n U_i^-$. Let $y \in X$. Then $\{y\} \in \theta(X) = \bigcup_{i=1}^n U_i^-$, i.e., $\{y\} \cap U_m^- \neq \emptyset$, for some m where $1 \leq m \leq n$, i.e., $y \in U_m$. Hence $X = \bigcup_{i=1}^n U_i$. Thus X is compact. ∎

Proposition 2.16. *If X is T_2 and $\theta(X),\tau)$ is Urysohn, then X is Urysohn.*

Proof. Let $x, y \in X$ be such that $x \neq y$. Now, X being T_2, $\{x\}, \{y\} \in \theta(X)$ and$\{x\} \neq \{y\}$. Since $(\theta(X),\tau)$ is Urysohn, there exists a τ-open neighbourhood $U_1^- \cap U_2^- \cap \cdots \cap U_n^- \cap U_0^+$ of $\{x\}$ and a τ-open neighbourhood $V_1^- \cap V_2^- \cap \cdots \cap V_m^- \cap V_0^+$ of $\{y\}$ such that

$$cl_{\theta(X)}(U_1^- \cap U_2^- \cap \cdots \cap U_n^- \cap U_0^+) \cap cl_{\theta(X)}(V_1^- \cap V_2^- \cap \cdots \cap V_m^- \cap V_0^+) = \emptyset$$

where $U_1, U_2, \ldots, U_n, V_1, V_2, \ldots, V_m$ are open in X; U_0, V_0 are θ-open in X with $X \setminus U_0, X \setminus V_0$ H sets, $U_i \subseteq U_0$ for $i = 1, 2, \ldots, n$, $V_i \subseteq V_0$ for $i = 1, 2, \cdots$.

Now, $\{x\} \in U_1^- \cap U_2^- \cap \cdots \cap U_n^- \cap U_0^+$ implies $x \in U_1 \cap U_2 \cap \cdots \cap U_n \cap U_0 = U_1 \cap U_2 \cap \cdots \cap U_n$ and $\{y\} \in V_1^- \cap V_2^- \cap \cdots \cap V_m^- \cap V_0^+$ implies $y \in V_1 \cap V_2 \cap \cdots \cap V_m \cap V_0 = V_1 \cap V_2 \cap \cdots \cap V_m$. We want to show that $cl_X(U_1 \cap U_2 \cap \cdots \cap U_n) \cap cl_X(V_1 \cap V_2 \cap \cdots \cap V_m) = \emptyset$. If not, let $z \in cl_X(U_1 \cap U_2 \cap \cdots \cap U_n) \cap cl_X(V_1 \cap V_2 \cap \cdots \cap V_m)$. Then for each open neighbourhood W of z, $W \cap U_1 \cap U_2 \cap \cdots \cap U_n \cap U_0 \neq \emptyset$ and $W \cap V_1 \cap V_2 \cap \cdots \cap V_m \cap V_0 \neq \emptyset$. Since for $p \in X, p \in W \cap U_1 \cap U_2 \cap \cdots \cap U_n \cap U_0$ gives $\{p\} \in W^- \cap U_1^- \cap U_2^- \cap \cdots \cap U_n^- \cap U_0^+$, hence $W \cap U_1 \cap U_2 \cap \cdots \cap U_n \cap U_0 \neq \emptyset$ implies $W^- \cap U_1^- \cap U_2^- \cap \cdots \cap U_n^- \cap U_0^+ \neq \emptyset$ and $W \cap V_1 \cap V_2 \cap \cdots \cap V_m \cap V_0 \neq \emptyset$ implies $W^- \cap V_1^- \cap V_2^- \cap \cdots \cap V_m^- \cap V_0^+ \neq \emptyset$. Then $\{z\} \in cl_{\theta(X)}(U_1^- \cap U_2^- \cap \cdots \cap U_n^- \cap U_0^+) \cap cl_{\theta(X)}(V_1^- \cap V_2^- \cap \cdots \cap V_m^- \cap V_0^+)$ - a contradiction. Hence there exists an open neighbourhood $U_1 \cap U_2 \cap \cdots \cap U_n$ of x and an open neighbourhood $V_1 \cap V_2 \cap \cdots \cap V_m$

of y such that $cl_X(U_1 \cap U_2 \cap \cdots \cap U_n) \cap cl_X(V_1 \cap V_2 \cap \cdots \cap V_m) = \emptyset$. Thus X is Urysohn. ■

Definition 2.17. A space X is locally θ-H if X contains a base $\mathcal{B}$ for its topology such that for each $B \in \mathcal{B}$, $cl_X B$ is an H-set θ-closed.

Proposition 2.18. *If X is H-closed and Urysohn, then X is locally θ-H.*

Proof. Let $\mathcal{B}$ be a base for the topology of X. Then for each $x \in X$, there exists a basic open set $B \in \mathcal{B}$ such that $x \in B$. Now, B being open, $cl_X B = \overline{B}^\theta$. Also, X being H-closed, Urysohn, $cl_X B$ is θ-closed and an H-set since θ-closed subset of an H-closed space is an H-set. Hence $\mathcal{B}$ is the required base for X such that for each $B \in \mathcal{B}$, $cl_X B$ is an H-set, θ-closed. Hence X is locally θ-H.

Proposition 2.19. *If X is T_2, locally θ-H and $(\theta(X), \tau)$ is H-closed, then X is H-closed.*

Proof. Let $\mathcal{B}$ be a base of the topology of X such that for each $B \in \mathcal{B}$, $cl_X B$ is a θ-closed H-set. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be an open cover of X. Without loss of generality, we can assume that each U_α belongs to $\mathcal{B}$. We are going to prove that there is a natural number n and $\alpha_1, \ldots, \alpha_n \in \Lambda$ such that $X = cl_X(\bigcup_{i=1}^{i=n} U_{\alpha_i})$.

If $A \in \theta(X)$, then A is a subset of X and intersects a U_α; so, $A \in U_\alpha^-$. Hence, $\{U_\alpha^- : \alpha \in \Lambda\}$ is a τ-open cover of $\theta(X)$. Since $\theta(X)$ is H-closed, there exists a finite proximate subcover of $\theta(X)$, i.e.,

$$\theta(X) = cl_{\theta(X)}(\bigcup_{i=1}^{i=n} U_{\alpha_i}^-)$$

for a natural number n and some $\alpha_1, \ldots, \alpha_n \in \Lambda$. We are going to prove that $X = cl_X(\bigcup_{i=1}^{i=n} U_{\alpha_i})$. Assume that this is not the case, then there is $x \in X \setminus (\bigcup_{j=1}^{j=n} cl_X U_{\alpha_j}) = W$. Observe that W is a θ-open set and $X \setminus W$ is an H-set. Since X is T_2, $\{x\}$ is θ-closed, so $\{x\} \in W^+$. On the other hand, there is $i \in \{1, \ldots, n\}$ such that $\{x\} \in cl_{\theta(X)} U_{\alpha_i}^-$.

Therefore $W^+ \cap U_{\alpha_i}^- \neq \emptyset$. Let $F \in W^+ \cap U_{\alpha_i}^-$. Thus, $F \subseteq W$ and $F \cap U_{\alpha_i} \neq \emptyset$. But this means that $W \cap U_{\alpha_i} \neq \emptyset$ which contradicts the definition of W. So, X must be covered by $cl_X(\bigcup_{i=1}^{i=n} U_{\alpha_i})$. ■

From Theorem 2.13, Proposition 2.18 and Proposition 2.19 we thus have

Theorem 2.20. *Let X be a Urysohn topological space. Then, X is H-closed if and only if X is locally θ-H and $(\theta(X), \tau)$ is H-closed.*

Example 2.21. Every locally compact T_2 space which is not compact is an example of a locally θ-H Urysohn space which is not H-closed.

Example 2.22. Consider the space given by J. R. Porter and R. G. Woods [9; Example 4.8].

The subset $Y = \{(\frac{1}{n}, \frac{1}{m}) : n \in N, |m| \in N\} \cup \{(\frac{1}{n}, 0) : n \in N\}$ (where N is the set of all natural numbers) of R^2 (where R is the set of all real numbers) is given the subspace topology inherited from the usual topology on the plane R^2. Let $X = Y \cup \{p^+, p^-\}$. A subset $U \subseteq X$ is defined to be open if $U \cap Y$ is open in Y and if $p^+ \in U$ (respectively, $p^- \in U$) implies that there is some $r \in N$ such that $\{(\frac{1}{n}, \frac{1}{m}) : n \geq r, m \in N\} \subseteq U$ (respectively, $\{(\frac{1}{n}, \frac{1}{m}) : n \geq r, -m \in N\} \subseteq U$. The space is H-closed and T_2. We prove that X is not locally θ-H. For $r \in N$, let $B_r^+ = \{(\frac{1}{n}, \frac{1}{m}) : n \geq r, m \in N\} \cup \{p^+\}$ $B_r^- = \{(\frac{1}{n}, \frac{1}{m}) : n \geq r, -m \in N\} \cup \{p^-\}$. We show that for any basis $\mathcal{B}$ for the topology of X, there exists $B \in \mathcal{B}$ such that $cl_X B$ is either not θ-closed or not an H-set. Let $\mathcal{B}$ be any basis for the topology of X. Since X is T_2, there exists $B \in \mathcal{B}$ such that $p^+ \in B$ and $p^- \notin cl_X B$. Now, there exists $r \in N$ such that $p^+ \in B_r^+ \subseteq B$. We show that $cl_X B$ is not θ-closed. Now, $cl_X B_r^+ = B_r^+ \cup \{(\frac{1}{n}, 0) : n \geq r\}$. We claim that p^- is a θ-contact point of $cl_X B$. In fact, if U be any open neighbourhood of p^- then there exists $t \in N$ such that $B_t^- \subseteq U$. Again $cl_X B_t^- = B_t^- \cup \{(\frac{1}{n}, 0) : n \geq t\}$. So, $cl_X B_t^- \cap cl_X B_r^+ \neq \emptyset$ which implies $cl_X U \cap cl_X B \neq \emptyset$ which implies p^- is a θ-contact point of $cl_X B$. But $p^- \notin cl_X B$. So $cl_X B$ is not θ-closed. Hence X is not locally θ-H.

3. θ -partially ordered space

Definition 3.1. [7] Let X be a topological space and $\leq$ be a partial order in it. For each subset A of X, let, $\uparrow A = \{x \in X : a \leq x \text{ for some } a \in A\}$ and $\downarrow A = \{x \in X : x \leq a \text{ for some } a \in A\}$. The sets $\uparrow A$ and $\downarrow A$ are called the increasing hull of A and decreasing hull of A respectively.

It is easy to verify that, for any $A, B \subseteq X$,

(i) $A \subseteq \uparrow A$, $A \subseteq \downarrow A$.

(ii) $A \subseteq B \Rightarrow \uparrow A \subseteq \uparrow B$ and $\downarrow A \subseteq \downarrow B$.

(iii) $\uparrow (A \cup B) = \uparrow A \cup \uparrow B$, $\downarrow (A \cup B) = \downarrow A \cup \downarrow B$.

(iv) $\uparrow (A \cap B) = \uparrow A \cap \uparrow B$, $\downarrow (A \cap B) = \downarrow A \cap \downarrow B$.

Definition 3.2. [4] A partial order $\leq$ on a topological space X is a θ-closed order if its graph $\{(x, y) \in X \times X : x \leq y\}$ is a θ-closed subset of $X \times X$.

Theorem 3.3. [3] *Every topological space X equipped with a θ-closed order $\leq$ is a Urysohn space.*

Definition 3.4. A partial order $\leq$ on a topological space X is a θ-regular order if and only if for every θ-closed subset $A \subseteq X$ and $x \in X$ with $a \not\leq x$, for all $a \in A$, there exist neighborhoods V and W of A and x respectively in X such that $\uparrow cl_X V \cap \downarrow cl_X W = \emptyset$.

Definition 3.5. A θ-partially ordered space is a pair $(Y, \leq)$ where Y is a topological space and $\leq$ is a θ-closed partial order on Y such that $\downarrow V$ is θ-open for each open subset V of Y. If , in addition $\leq$ is θ-regular, then we call Y a θ-regular θ-partially ordered space.

THEOREM 3.6. [3] *The partial order $\leq$ on a topological space X is a θ-closed order if and only if for every $x, y \in X$ with $x \not\leq y$, there exists neighborhoods U and V of x and y respectively in X such that $\uparrow cl_X U \cap \downarrow cl_X V = \emptyset$.*

THEOREM 3.7. [3] *Let X be a topological space equipped with a θ-closed order $\leq$. Let $H \subseteq X$ be an H-set in X. Then both $\uparrow H$ and $\downarrow H$ are θ-closed.*

THEOREM 3.8. *If $\leq$ is a θ-closed order on a topological space X and X is H-closed, then $\leq$ is a θ-regular order.*

Proof. Let A be a θ-closed subset of X and $x \in X$ be such that $a \not\leq x$, for all $a \in A$. Then for each $a \in A$, there exists neighborhoods U_a and V_a of a and x respectively such that $\uparrow cl_X U_a \cap \downarrow cl_X V_a = \emptyset$. Since X is equipped with the θ-closed order $\leq$, X is Urysohn. Thus X is H-closed and Urysohn. Now A being a θ-closed subset of X is an H-set. Now $\{U_a : a \in A\}$ is an open cover of A and A is an H-set. Hence there exists a finite subset $A_0 \subseteq A$ such that $A \subseteq \bigcup_{a \in A_0} cl_X U_a$. Let $V = \bigcap_{a \in A_0} V_a$. Then V is an open neighborhood of x in X. Now $\downarrow cl_X V \cap A \subseteq (\bigcap_{a \in A_0} \downarrow cl_X V_a) \cap (\bigcup_{a \in A_0} \uparrow cl_X U_a) = \emptyset$ which implies $A \subseteq X \setminus \downarrow cl_X V$. Again $\downarrow cl_X V$ is θ -closed since $cl_X V$ is an H-set. So $X \setminus \downarrow cl_X V$ is an open neighborhood of A. We claim that $\uparrow (X \setminus \downarrow cl_X V) \cap \downarrow cl_X V = \emptyset$. If not, let, $z \in \uparrow (X \setminus \downarrow cl_X V) \cap \downarrow cl_X V$. So there exists $w \in (X \setminus \downarrow cl_X V)$ such that $w \leq z$, i.e., $w \in \downarrow cl_X V$ -a contradiction. Hence $\uparrow (X \setminus \downarrow cl_X V) \cap \downarrow cl_X V = \emptyset$. This completes the proof. ■

4. Spaces of θ^* -lower semicontinuous functions

This section is devoted to an examination of spaces of θ^*-lower semicontinuous functions. Here X and Y are topological spaces and $\leq$ is a partial order on Y. Using this partial order, Ganguly and Jana have built the concept of θ^*-lower semicontinuous functions in [3].

DEFINITION 4.1. [3] A function $f : X \to Y$, Y being equipped with a partial order $\leq$ is called θ^*-lower semicontinuous w.r.t. $\leq$ at $x \in X$ if and only if for every open neighborhood V of $f(x)$ in Y, there exists an open neighborhood U of x in X such that $f(cl_X U) \subseteq \uparrow V$.

f is θ^*-lower semicontinuous w.r.t. $\leq$ if and only if it is θ^*-lower semicontinuous w.r.t.$\leq$ at each point of X.

The set of all θ^*-lower semicontinuous functions w.r.t. $\leq f : X \to Y$ is denoted by $\theta^* - LC(X, Y)$.

NOTE 4.2. The operation of 'subset' of X induces a partial order on $\theta(X)$, which is denoted by $\subseteq$.

PROPOSITION 4.3. *If X is a T_2-space and $V_i \subseteq V_0$ for $i = 1, 2, \ldots, n$, then $\uparrow (V_1^- \cap V_2^- \cap \cdots \cap V_n^- \cap V_0^+) = V_1^- \cap V_2^- \cap \cdots \cap V_n^-$.*

Proof. Let $A \in V_1^- \cap V_2^- \cap \cdots \cap V_n^-$. Then $A \cap V_i \neq \emptyset$ for $i = 1, 2, \ldots, n$. Let $x_i \in A \cap V_i$, $i = 1, 2, \ldots, n$. Now for each $i = 1, 2, \ldots, n$, $V_i \subseteq V_0$ implies that $\{x_1, x_2, \ldots, x_n\} \subseteq A \cap V_0$. Since X is T_2, $\{x_1, x_2, \ldots, x_n\}$ is θ-closed in X. Hence $\{x_1, x_2, \ldots, x_n\} \in V_1^- \cap V_2^- \cap \cdots \cap V_n^- \cap V_0^+$. Thus $A \in \uparrow (V_1^- \cap V_2^- \cap \cdots \cap V_n^- \cap V_0^+)$. Thus

$$V_1^- \cap V_2^- \cap \cdots \cap V_n^- \subseteq \uparrow (V_1^- \cap V_2^- \cap \cdots \cap V_n^- \cap V_0^+). \quad \text{(i)}$$

Conversely let $A \in \uparrow (V_1^- \cap V_2^- \cap \cdots \cap V_n^- \cap V_0^+)$. Then there exists $B \in (V_1^- \cap V_2^- \cap \cdots \cap V_n^- \cap V_0^+)$ such that $A \supseteq B$. Therefore $B \cap V_i \neq \emptyset$ for $i = 1, 2, \ldots, n$. So $A \cap V_i \neq \emptyset$ for $i = 1, 2, \ldots, n$. Consequently $A \in V_1^- \cap V_2^- \cap \cdots \cap V_n^-$. Thus

$$\uparrow (V_1^- \cap V_2^- \cap \cdots \cap V_n^- \cap V_0^+) \subseteq V_1^- \cap V_2^- \cap \cdots \cap V_n^-. \quad \text{(ii)}$$

From (i) and (ii) we have, $\uparrow (V_1^- \cap V_2^- \cap \cdots \cap V_n^- \cap V_0^+) = V_1^- \cap V_2^- \cap \cdots \cap V_n^-$. ∎

Proposition 4.4. $\uparrow (V_1^- \cap V_2^- \cap \cdots \cap V_n^-) = V_1^- \cap V_2^- \cap \cdots \cap V_n^-$.

Proof. Let $A \in \uparrow (V_1^- \cap V_2^- \cap \cdots \cap V_n^-)$. Then there exists $B \in V_1^- \cap V_2^- \cap \cdots \cap V_n^-$ such that $B \subseteq A$. Since $B \cap V_i \neq \emptyset$ for $i = 1, 2, \ldots, n$, $A \cap V_i \neq \emptyset$ for $i = 1, 2, \ldots, n$. Hence $A \in V_1^- \cap V_2^- \cap \cdots \cap V_n^-$. Thus

$$\uparrow (V_1^- \cap V_2^- \cap \cdots \cap V_n^-) \subseteq V_1^- \cap V_2^- \cap \cdots \cap V_n^-. \quad \text{(i)}$$

Also,

$$V_1^- \cap V_2^- \cap \cdots \cap V_n^- \subseteq \uparrow (V_1^- \cap V_2^- \cap \cdots \cap V_n^-). \quad \text{(ii)}$$

From (i) and (ii) we have, $\uparrow (V_1^- \cap V_2^- \cap \cdots \cap V_n^-) = V_1^- \cap V_2^- \cap \cdots \cap V_n^-$. ∎

Theorem 4.5. *Let Y be a T_2-space and let $\theta(Y)$ have the topology τ. Then a function $\Phi : X \to \theta(Y)$ is θ^*-lower semicontinuous w.r.t $\subseteq$ if and only if $\Phi^{-1}(V^-)$ is θ-open in X whenever V is an open subset of Y.*

Proof. First assume that Φ is θ^*-lower semicontinuous w.r.t $\subseteq$ and let V be an open subset of Y. Let $a \in \Phi^{-1}(V^-)$. Then $\Phi(a) \in V^-$. Since Φ is θ^*-lower semicontinuous, there exists an open neighborhood U of a such that $\Phi(cl_X U) \subseteq \uparrow (V^-) = V^-$ [by Proposition 4.4]. Hence $a \in U \subseteq cl_X U \subseteq \Phi^{-1}(V^-)$. Thus $\Phi^{-1}(V^-)$ is θ-open in X.

Conversely let the given condition holds. Let $a \in X$ and let G be any open neighborhood of $\Phi(a)$ in $\theta(Y)$. Then there exist open sets $V_1, V_2, \ldots, V_n$ and θ-open set V_0 with its complement an H -set such that $\Phi(a) \in V_1^- \cap V_2^- \cap \cdots \cap V_n^- \cap V_0^+ \subseteq G$. Define $U = \Phi^{-1}(V_1^-) \cap \Phi^{-1}(V_2^-) \cap \cdots \cap \Phi^{-1}(V_n^-)$. Since by the given condition each $\Phi^{-1}(V_i^-)$ is a θ-open set for $i = 1, 2, \ldots, n$ and finite intersection of θ-open sets is θ-open, U is θ-open in X with $a \in U$. Hence there exists an open neighborhood W of a in X such that $a \in W \subseteq cl_X W \subseteq U$, i.e., $\Phi(a) \in \Phi(W) \subseteq \Phi(cl_X W) \subseteq \Phi(U) \subseteq V_1^- \cap V_2^- \cap \cdots \cap V_n^- = \uparrow (V_1^- \cap V_2^- \cap \cdots \cap V_n^- \cap V_0^+) \subseteq \uparrow G$ [by Proposition 4.3]. Hence Φ is a θ^*-lower semicontinuous function. ∎

Definition 4.6. For each $f \in \theta^* - LC(X, Y)$, the graph of f is defined by the set$E(f) = \{(x, y) \in X \times Y : f(x) \leq y\}$.

PROPOSITION 4.7. *Let $\leq$ be a θ-closed order in Y. Then, for each $f \in \theta^* - LC(X,Y)$, $E(f)$ is a θ-closed subset of $X \times Y$.*

Proof. Let $(x,y) \in (X \times Y) \setminus E(f)$. Then $f(x) \not\leq y$. Hence there exist neighborhoods U, V of $f(x)$ and y respectively such that $\uparrow (cl_Y U) \cap \downarrow (cl_Y V) = \emptyset$. Since $f \in \theta^* - LC(X,Y)$, there exists a neighborhood W of x such that $f(cl_X W) \subseteq \uparrow U \subseteq \uparrow cl_Y U$. Hence $cl_X W \times cl_Y V$ is a neighborhood of (x,y) in $X \times Y$. Now for each $(a,b) \in cl_X W \times cl_Y V$, $f(a) \in \uparrow cl_Y U$ and $b \in cl_Y V \subseteq \downarrow cl_Y V$. If $f(a) \leq b$, then $f(a) \in \downarrow cl_Y V$ contradicting the fact that $\uparrow cl_Y U \cap \downarrow cl_Y V = \emptyset$. Hence $f(a) \not\leq b$, so that $(a,b) \notin E(f)$. Hence $E(f)$ is a θ-closed subset of $X \times Y$. ∎

REMARK 4.8. From the above proposition it follows that $E : \theta^* - LC(X,Y) \to \theta(X \times Y)$ is well-defined. Also E is one-to-one. We consider $\theta^* - LC(X,Y)$ as a subset of $\theta(X \times Y)$ by identifying each $f \in \theta^* - LC(X,Y)$ with $E(f)$ in $\theta(X \times Y)$. So any topology of $\theta(X \times Y)$ induces a topology on $\theta^* - LC(X,Y)$ by taking the subspace topology. We now give $\theta(X \times Y)$ the topology τ and consider the subspace topology τ' on $\theta^* - LC(X,Y)$.

Let us now investigate the closure of $\theta^* - LC(X,Y)$ in $\theta(X \times Y)$.

DEFINITION 4.9. Define $\overline{\theta^* - LC(X,Y)}$ to be the set of all functions $\Phi : X \to \theta(Y)$ satisfying,

(1) for every $x \in X$, $\Phi(x) = \uparrow \Phi(x)$ and

(2) for every open V in Y, $\Phi^{-1}((\uparrow V)^+)$ is θ-open in X.

Also for each $\Phi \in \overline{\theta^* - LC(X,Y)}$, define $\overline{E}(\Phi) = \{(x,y) \in X \times Y : y \in \Phi(x)\}$.

PROPOSITION 4.10. *If Y is a θ-regular, θ-partially ordered space, then $\overline{E}(\Phi)$ is a θ-closed subset of $X \times Y$ for each $\Phi \in \overline{\theta^* - LC(X,Y)}$.*

Proof. Can be proved similarly as is done in Proposition 4.7. ∎

REMARK 4.11. If Y is a θ-regular, θ-partially ordered space,

$$\overline{E} : \overline{\theta^* - LC(X,Y)} \to \theta(X \times Y)$$

is well-defined. we identify each $\Phi \in \overline{\theta^* - LC(X,Y)}$ with $\overline{E}(\Phi)$ in $\theta(X \times Y)$, forming a subset of $\theta(X \times Y)$. The topology on $\overline{\theta^* - LC(X,Y)}$ is that induced from $\theta(X \times Y)$ by taking the subspace topology.

PROPOSITION 4.12. *If Y is a θ-regular, θ-partially ordered space, then θ^*-$LC(X,Y)$ is a subspace of $\overline{\theta^* - LC(X,Y)}$.*

Proof. Let $f \in \theta^* - LC(X,Y)$ and define $\Phi : X \to \theta(Y)$ by $\Phi(x) = \{y \in Y : f(x) \leq y\}$. Now, $\uparrow \Phi(x) = \{y \in Y : u \leq y \text{ for some } u \in \Phi(x)\} = \{y \in Y : f(x) \leq u \leq y \text{ for some } u \in \Phi(x)\} = \Phi(x)$.Next let V be open in Y and let $x \in \Phi^{-1}((\uparrow V)^+)$. Then $\uparrow V$ is a neighborhood of $\Phi(x)$ in Y. Hence there exists a neighborhood U of x in X such that $f(cl_X U) \subseteq \uparrow V$. If $u \in cl_X U$, then $f(u) \in \uparrow V$ and thus $\Phi(u) \in (\uparrow V)^+$. Hence $\Phi^{-1}((\uparrow V)^+)$ is θ -open in X. So

$\Phi \in \overline{\theta^* - LC(X,Y)}$. Now, $\overline{E}(\Phi) = \{(x,y) \in X \times Y : y \in \Phi(x)\} = \{(x,y) \in X \times Y : f(x) \leq y\} = E(f)$.Hence f is identified with Φ in $\theta(X \times Y)$. Thus $\theta^* - LC(X,Y)$ is a subspace of $\overline{\theta^* - LC(X,Y)}$. ■

Theorem 4.13. *Let X be an H-closed, Urysohn space and let Y be an H-closed θ -partially ordered space. Then $\overline{\theta^* - LC(X,Y)}$ is a θ-closed subspace of $\theta(X \times Y)$.*

Proof. Let $\Gamma \in \theta(X \times Y) \setminus \overline{\theta^* - LC(X,Y)}$ and define $\Phi : X \to \theta(Y)$ by $\Phi(x) = \{y \in Y : (x,y) \in \Gamma\}$ for each $x \in X$. If possible, let, Φ satisfies condition (1). Then Φ cannot satisfy condition (2), since otherwise Φ would be in $\overline{\theta^* - LC(X,Y)}$ and we could identify Φ with $\overline{E}(\Phi) = \Gamma$ in $\theta(X \times Y)$. Hence there exists an open subset V of Y such that $\Phi^{-1}((\uparrow V)^+)$ is not a θ-open neighborhood of some x in $\Phi^{-1}((\uparrow V)^+)$. Then for every neighborhood U of x in X, there exists an $x_U \in cl_X U \setminus \Phi^{-1}((\uparrow V)^+)$ so that $\Phi(x_U) \notin (\uparrow V)^+$. So there exists $y_U \in \Phi(x_U) \setminus \uparrow V$. Since Y is equipped with the θ-closed partial order, by Theorem 3.3, Y is an Urysohn space and hence T_2. also Y being H-closed, the net $\{y_U\}$ has a θ-limit point $y \in Y \setminus \uparrow V$. also $\{x_U\}$ θ-converges to some x in X. Hence (x,y) is a θ-limit point of $\{(x_U, y_U)\}$ in $X \times Y$. Since Γ is θ-closed, $(x,y) \in \Gamma$. Now $\Phi(x) \subseteq \uparrow V$ such that $y \notin \Phi(x)$, i.e., $(x,y) \notin \Gamma$—a contradiction.

Thus Φ does not satisfy condition (1). Hence there exist $x \in X$, $y, z \in Y$ such that $y \leq z$ and $(x,y) \in \Gamma$, but $(x,z) \notin \Gamma$. Thus there exist neighborhoods U of x in X and V of z in Y such that $(cl_X U \times cl_Y V) \cap \Gamma = \emptyset$. Now define $B = (X \times Y) \setminus (cl_X U \times cl_Y V)$. We first prove that $(cl_X U \times cl_Y V)$ is an H-set and a θ-closed set. Since X is H-closed and Urysohn, U being open in X, $cl_X U$ becomes a θ-closed subset of X and hence an H-set. Also Y being a θ-partially ordered space is Urysohn and it is also H-closed. Thus $cl_Y V$ is a θ-closed subset of Y and an H-set. Hence $(cl_X U \times cl_Y V)$ is a θ-closed H-set.

Let W be the neighborhood of y in Y given by $W = (\downarrow V) \cap (Y \setminus cl_Y V)$. Then the set $G = (U \times W)^- \cap B^+$ is an open set in $(\theta(X \times Y), \tau)$ containing Γ. Now it suffices to show that $G \subseteq \theta(X \times Y) \setminus \overline{\theta^* - LC(X,Y)}$. Let $\Delta \in G$. Let $(a,b) \in \Delta \cap (U \times W)$. Since $b \in \downarrow V$, there exists some $c \in V$ such that $c \geq b$. Therefore $(a,c) \in cl_X U \times cl_Y V$ and hence $(a,c) \notin \Delta$. Now if $\Delta \in \overline{\theta^* - LC(X,Y)}$, the condition (1) would be violated -a contradiction. Thus G is a neighborhood of Γ contained in $\theta(X \times Y) \setminus \overline{\theta^* - LC(X,Y)}$. ■

Corollary 4.14. *If X is an H-closed, Urysohn space and Y is an H-closed, θ-partially ordered space, then $\overline{\theta^* - LC(X,Y)}$ is an H-set in $\theta(X \times Y)$.*

5. Some results on multifunctions

Definition 5.1. [3] A multifunction $F : X \to Y$ is called lower θ^*-semicontinuous if and only if for each $x_0 \in X$ and each open set V in Y with $F(x_0) \cap V \neq \emptyset$, there exists an open neighborhood U of x_0 such that $F(x) \cap V \neq \emptyset$, for all $x \in cl_X U$.

NOTATION 5.2. $\mathcal{A}(Y) = \{E \subseteq Y : E \neq \emptyset\}$.

For a function $f : X \to Y$, we define a multifunction $(\downarrow f)$ from X to Y by the rule, $(\downarrow f)(x) = \downarrow f(x)$, for each $x \in X$.

PROPOSITION 5.3. *For $f \in \theta^* - LC(X,Y)$, $(\downarrow f)$ is a lower θ^*-semicontinuous function from X to $\mathcal{A}(Y)$.*

Proof. Let $x_0 \in X$ and V be open in Y such that $(\downarrow f)(x_0) \cap V \neq \emptyset$, i.e., $\downarrow f(x_0) \cap V \neq \emptyset$, i.e., $\{f(x_0)\} \cap \uparrow V \neq \emptyset$, which implies $f(x_0) \in \uparrow V$. Since $f \in \theta^* - LC(X,Y)$, there exists an open neighborhood U of x_0 such that $f(cl_X U) \subseteq \uparrow (\uparrow V) \subseteq \uparrow V$. Hence for any $x \in cl_X U$, $f(x) \in \uparrow V$, i.e., $\{f(x)\} \cap \uparrow V \neq \emptyset$, i.e., $\downarrow f(x) \cap V \neq \emptyset$ which implies $(\downarrow f)(x) \cap V \neq \emptyset$, for all $x \in cl_X U$. Hence $(\downarrow f)$ is a lower θ^* -semicontinuous function from X to $\mathcal{A}(Y)$. ∎

PROPOSITION 5.4. *If $(\downarrow f)$ is a lower θ^*-semicontinuous function from X to $\mathcal{A}(Y)$, then $f \in \theta^* - LC(X,Y)$.*

Proof. Let $x_0 \in X$ and V be an open neighborhood of $f(x_0)$, i.e., $f(x_0) \in V \subseteq \uparrow V$. This implies that $f(x_0) \geq v$, for some $v \in V$, i.e., $v \in \downarrow f(x_0)$ and hence $V \cap \downarrow f(x_0) \neq \emptyset$, i.e., $(\downarrow f)(x_0) \cap V \neq \emptyset$. Since $(\downarrow f)$ is a lower θ^*-semicontinuous function from X to $\mathcal{A}(Y)$, there exists an open neighborhood U of x_0 in X such that $(\downarrow f)(x) \cap V \neq \emptyset$, for all $x \in cl_X U$. Thus $\{f(x)\} \cap \uparrow V \neq \emptyset$, for all $x \in cl_X U$ which implies $f(x) \in \uparrow V$, for all $x \in cl_X U$, i.e., $f(cl_X U) \in \uparrow V$. Hence $f \in \theta^* - LC(X,Y)$. ∎

NOTE 5.5. *Thus, the relation $f \to (\downarrow f)$ is a one-to-one correspondence between the elements in $\theta^* - LC(X,Y)$ and the multifunctions from X to Y.*

PROPOSITION 5.6. *Let Y be a T_2-space and $f : X \to \theta(Y)$ be a θ^*-lower semicontinuous function. Then the multifunction $F : X \to Y$ which sends each x to $f(x)$ is lower θ^*-semicontinuous.*

Proof. Let $x_0 \in X$ and V be open in Y such that $F(x_0) \cap V \neq \emptyset$, i.e., $f(x_0) \in V^-$ which implies $x_0 \in f^{-1}(V^-)$. Since $f \in \theta^* - LC(X,Y)$, $f^{-1}(V^-)$ is θ -open in X. Hence there exists an open neighborhood U of x_0 in X such that $x_0 \in U \subseteq cl_X U \subseteq f^{-1}(V^-)$, i.e., $f(cl_X U) \subseteq V^-$, i.e., $F(x) \cap V \neq \emptyset$, for all $x \in cl_X U$. Thus F is lower θ^*-semicontinuous. ∎

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