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WEIGHTED COMPOSITION OPERATORS ON WEIGHTED BERGMAN SPACES OF INFINITE ORDER WITH THE CLOSED RANGE PROPERTY

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Abstract. We study under which conditions weighted composition operators acting on weighted Bergman spaces of infinite order are bounded from below or, equivalently, have closed range.

I. Introduction

We denote by v a strictly positive bounded continuous function (*weight*) on the open unit disk D in the complex plane. Moreover, let $H(D)$ be the space of all analytic functions on D. We will study operators defined on weighted Bergman spaces of infinite order

$$H_v^\infty := \{ f \in H(D);\ \|f\|_v := \sup_{z\in D} v(z)|f(z)| < \infty \}.$$

Endowed with norm $\|\cdot\|_v$, these spaces are Banach spaces. They appear naturally in the study of growth conditions of analytic functions and have been studied in various articles, see e.g. [1], [2], [4], [10], [13] and the references therein.

Moreover, we consider non-constant functions $\phi \in H(D)$ satisfying $\phi(D) \subset D$ as well as functions $\psi \in H(D)$ which are not identically equal to zero. These functions induce the weighted composition operator $C_{\phi,\psi}$ which is a linear map on $H(D)$ and given by $C_{\phi,\psi} f = \psi(f \circ \phi)$.

Many authors have studied composition operators and weighted composition operators under various points of view, see e.g. [3–7], [11].

In this article we are interested in weighted composition operators acting on spaces of the type defined above. In case of Bloch spaces and weighted Bergman spaces of infinite order generated by the so-called standard weights, the closed range

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property of weighted composition operators has been analyzed by Ghatage-Zheng-Zorboska in [9] and by Palmberg in [14].

Our aim is to study when weighted composition operators acting on spaces of type H_v^∞ have closed range.

II. Preliminaries

We recommend the book of Cowen and MacCluer [8] as well as the book of Shapiro [15] for an introductory and general information on the concept of composition operators.

Let us start this section with some facts on weighted spaces. When dealing with weighted spaces and operators acting on them an important tool is the so-called *associated weight*. For a weight v it is defined by

$$\tilde{v}(z) := \frac{1}{\sup\{\,|f(z)|;\ f \in H_v^\infty,\ \|f\|_v \le 1\,\}} = \frac{1}{\|\delta_z\|_{H_v^{\infty\prime}}}, \quad z \in D,$$

where δ_z denotes the point evaluation of z.

By [1] the associated weight is continuous and we have that $\tilde{v} \ge v > 0$. Moreover, for each $z \in D$ we can find $f_z \in H_v^\infty$, $\|f_z\|_v \le 1$ such that $|f_z(z)| = \frac{1}{\tilde{v}(z)}$. We say that a weight v is *essential* if there is a constant $C > 0$ such that we arrive at the following estimate

$$v(z) \le \tilde{v}(z) \le C\tilde{v}(z) \text{ for every } z \in D.$$

Examples of essential weights as well as conditions ensuring the essentiality of weights can be found in [1] and [4].

It is well-known that the spaces $H_{\tilde{v}}^\infty$ and H_v^∞ are isometrically isomorphic. We are especially interested in *radial* weights, i.e. weights which satisfy $v(z) = v(|z|)$ for every $z \in D$. Radial weights which are non-increasing with respect to $|z|$ and such that $\lim_{|z|\to 1} v(z) = 0$ are called *typical*. In the sequel we assume that every radial weight is non-increasing. The following condition $(L1)$ which was introduced by Lusky in [13] plays an important role

$$(L1) \quad \inf_{n \in \mathbf{N}} \frac{v(1 - 2^{-n-1})}{v(1 - 2^{-n})} > 0.$$

Radial weights which satisfy $(L1)$ are always essential (see [4]), see the following illustrating examples of radial weights satisfying $(L1)$:

(a) $v_p(z) := (1 - |z|^2)^p$, $p > 0$, $z \in D$,

(b) $v(z) = (1 - \log(1 - |z|^2))^q$, $q < 0$, $z \in D$,

(c) $v(z) = (1 - |z|^2)^p(1 - \log(1 - |z|^2))^q$, $p > 0$, $q < 0$, $z \in D$,

(d) $v(z) = \sin(1 - |z|^2)$, $z \in D$.

We also need some geometric data of the open unit disk. The *pseudohyperbolic metric* is given by

$$\rho(z, a) := |\sigma_a(z)|, \text{ where } \sigma_a(z) := \frac{a - z}{1 - \overline{a}z}, \quad z, a \in D,$$

is the automorphism of D which changes 0 and a. It is well-known that the pseudohyperbolic metric is Möbius-invariant, that is, $\rho(\sigma(z),\sigma(a)) = \rho(z,a)$ for every automorphism σ of D and every $z, a \in D$. Moreover, obviously we have that $\sigma_a'(z) = \frac{1-|a|^2}{(1-\overline{a}z)^2}$. Throughout the paper we will use the following well-known equality

$$1 - |\sigma_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\overline{a}z|^2}.$$

We denote by $D(a,r)$ the pseudohyperbolic disk with center $a \in D$ and radius $r \in (0,1]$, that is

$$D(a,r) := \{ z \in D;\ |\sigma_a(z)| < r \}.$$

Calculations show that $D(a,r)$ is an Euclidean disk with center $c_E(r,a) = \frac{(1-r^2)a}{1-r^2|a|^2}$ and radius $r_E(a,r) = \frac{(1-|a|^2)r}{1-r^2|a|^2}$. The pseudohyperbolic metric is a metric which even satisfies the following stronger version of the triangle inequality

$$\rho(z,a) \leq \frac{\rho(z,b)+\rho(b,a)}{1+\rho(z,b)\rho(b,a)}.$$

Let us now turn our attention to the operators we wish to study. We will always assume that the operators $C_{\phi,\psi} : H_v^\infty \to H_v^\infty$ are bounded, which is the case if and only if the following condition holds

$$\sup_{z\in D} \frac{|\psi(z)|v(z)}{\tilde{v}(\phi(z))} < \infty \quad \text{(see [7])}.$$

By the open mapping theorem for analytic functions we know that under the given assumptions the composition operator C_ϕ is one-to-one. Since ψ is not identically equal to zero, it follows that $C_{\phi,\psi}$ is one-to-one. Moreover, $C_{\phi,\psi}$ has closed range if and only if, it is bounded from below, i.e. if there exists an $\varepsilon > 0$ such that $\|C_{\phi,\psi} f\|_v \geq \varepsilon \|f\|_v$ for every $f \in H_v^\infty$.

We close this section with an explanation of the concept our criteria are based on. These are ideas taken from [9] and [14] combined with a very useful lemma we obtained in [11].

We say that a subset H of D is a *sampling set* for the space H_v^∞ if there is $k > 0$ such that $\sup_{z\in D} v(z)|f(z)| \leq k \sup_{z\in H} v(z)|f(z)|$ for every $f \in H_v^\infty$. Moreover, for $\varepsilon > 0$ we put

$$\Omega_{\phi,\psi}^{\varepsilon,v} := \{z \in D;\ |\tau_{\phi,\psi}^v(z)| \geq \varepsilon\}, \text{ where } \tau_{\phi,\psi}^v(z) := \frac{\psi(z)v(z)}{\tilde{v}(\phi(z))}.$$

We will always assume that ε is chosen so that the set $\Omega_{\phi,\psi}^{\varepsilon,v}$ is non-empty. Finally, the operator $C_{\phi,\psi}$ on H_v^∞ is bounded if and only if $T_{\phi,\psi}^v := \sup_{z\in D} |\tau_{\phi,\psi}^v(z)| < \infty$. We assume in the sequel that $C_{\phi,\psi}$ is bounded.

III. Closed range

The proof of the following theorem is very similar to the proof of Theorem 1 in [9].

THEOREM 1. *The weighted composition operator $C_{\phi,\psi}$ is bounded from below on H_v^∞ if and only if there is $\varepsilon > 0$ such that $G_{\phi,\psi}^{\varepsilon,v} := \phi(\Omega_{\phi,\psi}^{\varepsilon,v})$ is a sampling set for H_v^∞.*

Proof. First, we assume that there is $\varepsilon > 0$ such that $G_{\phi,\psi}^{\varepsilon,v}$ is a sampling set for H_v^∞. Then for every $f \in H_v^\infty$ we obtain

$$\begin{aligned}\|f\|_v &= \sup_{z\in D} v(z)|f(z)| \leq k \sup_{z\in\Omega_{\phi,\psi}^{\varepsilon,v}} v(\phi(z))|f(\phi(z))| \\ &= k \sup_{z\in\Omega_{\phi,\psi}^{\varepsilon,v}} \frac{v(\phi(z))|\psi(z)|}{v(z)|\psi(z)|} v(z)|f(\phi(z))| \\ &\leq k\varepsilon^{-1} \sup_{z\in D} v(z)|\psi(z)||f(\phi(z))| = k\varepsilon^{-1}\|C_{\phi,\psi}f\|_v\end{aligned}$$

Thus, $C_{\phi,\psi}$ is bounded from below.

Conversely, suppose that $C_{\phi,\psi}$ is bounded from below on H_v^∞. Then there is $k > 0$ such that $\sup_{z\in D} |\psi(z)|v(z)|f(\phi(z))| \geq k$, whenever $\|f\|_v = 1$. Suppose that $\|f\|_v = 1$ and choose $z_f \in D$ with $v(z_f)|\psi(z_f)||f(\phi(z_f))| \geq \frac{k}{2}$. Hence

$$\frac{v(z_f)|\psi(z_f)|}{v(\phi(z_f))} v(\phi(z_f))|f(\phi(z_f))| \geq \frac{k}{2}.$$

Since $C_{\phi,\psi}$ is a bounded operator there is $M > 0$ with $\frac{v(z_f)|\psi(z_f)|}{v(\phi(z_f))} \leq M$ and since $v(\phi(z_f))|f(\phi(z_f))| \leq 1$, each of the factors is at least as large as $\min\left\{\frac{k}{2M}, \frac{k}{2}\right\}$. Thus, if $\varepsilon = \min\left\{\frac{k}{2M}, \frac{k}{2}\right\}$, then $G_{\phi,\psi}^{\varepsilon,v}$ is a sampling set for H_v^∞. ■

Our further results are based on the following crucial lemma which turned out to be very helpful in several contexts.

LEMMA 2. (Lindström-Wolf [11]) *Let v be a radial weight with $(L1)$ such that v is continuously differentiable with respect to $|z|$. Then there is $M > 0$ such that for $f \in H_v^\infty$ we have*

$$|v(p)f(p) - v(q)f(q)| \leq M\|f\|_v\rho(p, q)$$

for all $p, q \in D$.

PROPOSITION 3. *Let v be a radial weight with $(L1)$ such that v is continuously differentiable with respect to $|z|$. Moreover, let $M > 0$ be the constant in Lemma 2. If we can find $0 < \varepsilon < 1$ with $M(1 - \varepsilon) < 1$ such that*

$$\phi(\Omega_{\phi,\psi}^{\varepsilon,v}) \cap \overline{D(a, 1 - \varepsilon)} \neq \emptyset$$

for every $a \in D$, then $\phi(\Omega_{\phi,\psi}^{\varepsilon,v})$ is a sampling set for H_v^∞.

Proof. First, recall that the assumption is equivalent to the fact that for every $a \in D$ we can find $z_a \in \Omega^{\varepsilon,v}_{\phi,\psi}$ such that $\rho(\phi(z_a), a) \leq 1-\varepsilon$. By the lemma we obtain

$$|v(a)f(a) - v(\phi(z_a))f(\phi(z_a))| \leq (1-\varepsilon)M\|f\|_v.$$

Hence

$$v(a)|f(a)| \leq (1-\varepsilon)M\|f\|_v + v(\phi(z_a))|f(\phi(z_a))|$$

and thus

$$\sup_{a\in D} v(a)|f(a)| \leq (1-\varepsilon)M\|f\|_v + \sup_{a\in D} v(\phi(z_a))|f(\phi(z_a))|.$$

Finally,

$$\|f\|_v \leq \frac{1}{1-(1-\varepsilon)M} \sup_{z\in\phi(\Omega^{\varepsilon,v}_{\phi,\psi})} v(z)|f(z)|.$$

The claim follows. ∎

Proposition 4. *Let v be a radial weight such that $\frac{v(z)}{(1-|z|^2)^p}$ is an essential weight on D for some $0 < p < \infty$. If there exists $\varepsilon > 0$ such that $\phi(\Omega^{\varepsilon,v}_{\phi,\psi})$ is a sampling set for H^∞_v then we can find $\alpha > 0$ such that $\phi(\Omega^{\alpha,v}_{\phi,\psi}) \cap \overline{D(a, 1-\alpha)} \neq \emptyset$ for all $a \in D$.*

Proof. Choose $p > 0$ such that $w(z) = \frac{v(z)}{(1-|z|^2)^p}$ is an essential weight on D. Then there is a constant $C \geq 1$ such that $\tilde{w}(z) \leq Cw(z)$ for all $z \in D$. By assumption there is $k > 0$ such that

$$\sup_{z\in D} v(z)|f(z)| \leq k \sup_{z\in\phi(\Omega^{\varepsilon,v}_{\phi,\psi})} v(z)|f(z)|$$

for every $f \in H^\infty_v$. Let $a \in D$. Then there exists $g_a \in H^\infty_w$, $\|g_a\|_w = 1$ with $g_a(a)\tilde{w}(a) = 1$. Put

$$f_a(z) := g_a(z)\left(\frac{1-|a|^2}{(1-z\overline{a})^2}\right)^p \quad \text{for every } z \in D.$$

It follows that $\frac{1}{C} \leq \|f_a\|_v \leq 1$ since $|f_a(a)|v(a) \geq \frac{1}{C}$. Thus

$$\frac{1}{C} \leq k \sup_{z\in\phi(\Omega^{\varepsilon,v}_{\phi,\psi})} v(z)|f_a(z)|$$

and hence $\frac{1}{Ck} \leq \sup_{z\in\phi(\Omega^{\varepsilon,v}_{\phi,\psi})} v(z)|f_a(z)|$. Hence, for any fixed $\varepsilon' < \frac{1}{Ck}$ we can find $z_a \in \Omega^{\varepsilon,v}_{\phi,\psi}$ such that

$$\begin{aligned}\varepsilon' &\leq v(\phi(z_a))|f_a(\phi(z_a))| \leq \frac{v(\phi(z_a))}{v(z_a)|\psi(z_a)|}|\psi(z_a)|v(z_a)|f_a(\phi(z_a))| \\ &\leq \frac{1}{\varepsilon}|\psi(z_a)|v(z_a)|f_a(\phi(z_a))|.\end{aligned}$$

It follows that $\varepsilon'\varepsilon \leq |\psi(z_a)|v(z_a)|f_a(\phi(z_a))|$. Moreover,

$$\frac{|\psi(z_a)|v(z_a)}{\tilde{v}(\phi(z_a))}(1-|\sigma_a(\phi(z_a))|^2)^p = \frac{|\psi(z_a)|v(z_a)}{\tilde{v}(\phi(z_a))}\left|\frac{1-|a|^2}{(1-\phi(z_a)\overline{a})^2}\right|^p (1-|\phi(z_a)|^2)^p.$$

So,

$$T^v_{\phi,\psi}(1-|\sigma_a(\phi(z_a))|^2)^p \geq |\psi(z_a)||f_a(\phi(z_a))|v(z_a)\frac{(1-|\phi(z_a)|^2)^p}{|g_a(\phi(z_a))|\tilde{v}(\phi(z_a))} \geq \varepsilon'\varepsilon.$$

Then $z_a \in \Omega^{\varepsilon'\varepsilon,v}_{\phi,\psi}$ and clearly $(1-|\sigma_a(\phi(z_a))|^2)^p \geq \frac{\varepsilon'\varepsilon}{T^v_{\phi,\psi}}$ gives

$$|\sigma_a(\phi(z_a))| \leq \left(1-\left(\frac{\varepsilon\varepsilon'}{T^v_{\phi,\psi}}\right)^{\frac{1}{p}}\right)^{\frac{1}{2}}.$$

By choosing $\alpha < \varepsilon'\varepsilon$ small enough, we get $z_a \in \Omega^{\alpha,v}_{\phi,\psi}$ and $\rho(\phi(z_a),a) \leq 1-\alpha$. ∎

We close this paper with some illustrating examples of weighted composition operators with and without closed range.

Example 5. Fix $a \in D$. Then $C_{\sigma_a,(\sigma'_a)^p} : H^\infty_{v_p} \to H^\infty_{v_p}$ has closed range.

Proof. Since $|\tau^{v_p}_{\sigma_a,(\sigma'_a)^p}| = 1$ for every $z \in D$, we get that $\Omega^{\varepsilon,v_p}_{\sigma_a,(\sigma'_a)^p} = D$ for every $0<\varepsilon<1$. Then by Proposition 3 we know that $C_{\sigma_a,(\sigma'_a)^p} : H^\infty_{v_p} \to H^\infty_{v_p}$ has closed range.

Remark 6. The operators in the previous example are even isometries, as was shown in [6].

Example 7. Similar calculations prove that for a fixed $a \in D$, $p>0$ and $q<0$ as well as the weight $v(z) = (1-|z|^2)^p(1-\log(1-|z|^2))^q$ the operator $C_{\sigma_a,(\sigma'_a)^k} : H^\infty_v \to H^\infty_v$ has closed range for every k.

Example 8. By [6] we know that for each thin Blaschke product B such that $B(0)=0$ the weighted composition operator $C_{B,(B')^p} : H^\infty_{v_p} \to H^\infty_{v_p}$ is an isometry for every $0<p<\infty$.

Example 9. Let $v(z) = (1-|z|^2)(1-\log(1-|z|^2))$, $\psi \in H^\infty$ and $\phi(z) = \frac{z-1}{2}$ for every $z \in D$. Notice that for every fixed radius r we have that $\lim_{a\to 1}|\overline{D(a,r)}| = 0$. Therefore, for every $\varepsilon>0$ we can always choose a close enough to 1 such that $\phi(D) \cap \overline{D(a,1-\varepsilon)} = \emptyset$. Hence by Proposition 4, $C_{\phi,\psi}$ cannot have closed range on H^∞_v.

Example 10. Let v and ψ be as in the previous example. Next, we consider $\phi(z) = \frac{2\mathrm{arctanh}\,(z)}{\pi i+2\mathrm{arctanh}\,(z)}$. With the same arguments as above we can show that $C_{\phi,\psi}$ cannot have closed range on H^∞_v.

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